Chapter Goals:

- Understand the relationship between the area under a curve and the definite integral.
- Understand the relationship between velocity (speed), distance and the definite integral.
- Estimate the value of a definite integral.
- Understand the summation, or Σ, notation.
- Understand the formal definition of the definite integral.

Assignments:

- Assignment 17
- Assignment 18

The basic idea: The next two problems are easy to solve as certain “problem ingredients” are constant.

Example 1 (Easy area problem):

Find the area of the region in the xy-plane bounded above by the graph of the function \( f(x) = 2 \), below by the x-axis, on the left by the line \( x = 1 \), and on the right by the line \( x = 5 \).

\[
\text{Area} = \text{base} \cdot \text{height} = 4 \cdot 2 = 8
\]

Example 2 (Easy distance traveled problem):

Suppose a car is traveling due east at a constant velocity of 55 miles per hour. How far does the car travel between noon and 2:00 pm?

\[
\frac{2}{\text{hours}} \cdot 55 \text{ mph} = 110 \text{ miles}
\]

General philosophy: By means of the integral, problems similar to the previous ones can be solved when the ingredients of the problem are no longer constant but rather changing or variable. In this Chapter, we learn how to estimate a solution to these more complex problems. The key idea is to notice that the value of the function does not vary very much over a small interval, and so it is approximatively constant over a small interval. By the end of Chapter 9 we will be able to solve these problems exactly, and by the end of Chapter 10 we will be able to solve them both exactly and easily.

Example 3:

Estimate the area under the graph of \( y = x^2 + \frac{1}{2}x \) for \( x \) between 0 and 2 in two different ways:

(a) Subdivide the interval \([0, 2]\) into four equal subintervals and use the left endpoint of each subinterval as “sample point”.

\[
\begin{array}{c|c|c|c|c|c}
\times & 0 & 0.5 & 1 & 1.5 & 2 \\
\hline
f(x) & 0 & 0.5 & 1.5 & 3 & 5 \\
\end{array}
\]

Each rectangle has base \( \frac{0.5}{4} = 0.125 \).

\[
\text{area} = 0.5 \left( 0 + 0.5 + 1.5 + 3 + 5 \right) = 2.5
\]

(b) Subdivide the interval \([0, 2]\) into four equal subintervals and use the right endpoint of each subinterval as “sample point”.

\[
\text{area} = 0.5 \left( 0.5 + 1.5 + 3 + 5 \right) = 2.5
\]

Find the difference between the two estimates (right endpoint estimate minus left endpoint estimate).

\[
\text{difference between 2 estimates} = 5 - 2.5 = 2.5
\]
Example 4:
Estimate the area under the graph of $y = 3^x$ for $x$ between 0 and 2. Use a partition that consists of four equal subintervals of $[0, 2]$ and use the left endpoint of each subinterval as a sample point.

Each subinterval has length

$$\Delta x = \frac{2}{4} = 0.5$$

Estimate of area

$$\approx 0.5 \left( 1 + 1.732 + 3 + 5.196 \right)$$

$$= 5.464$$

Note: In the previous two examples we systematically chose the value of the function at one of the endpoints of each subinterval. However, since the guiding idea is that we are assuming that the values of the function over a small subinterval do not change by very much, then we could take the value of the function at any point of the subinterval as a good sample or representative value of the function.

We could also have chosen small subintervals of different lengths. However, we are trying to establish a systematic procedure that works well in general.

Getting better estimates:

We can only expect the previous answers to be approximations of the correct answers. This is because the values of the function do change on each subinterval, even though they do not change by much.

If, however, we replace the subintervals we used by "smaller" subintervals we can reasonably expect the values of the function to vary by much less on each thinner subinterval. Thus, we can reasonably expect that the area of each thinner vertical strip under the graph of the function to be more accurately approximated by the area of these thinner rectangles. Then if we add up the areas of all these thinner rectangles, we should get a much more accurate estimate for the area of the original region.

Here is Example 3(b), revisited:

$$y = x^2 + \frac{1}{2}x$$ on $[0, 2]$

$n = 4$ equal subintervals

Area $\approx 5$

$$y = x^2 + \frac{1}{2}x$$ on $[0, 2]$

$n = 8$ equal subintervals

Area $\approx 4.3125$

We will see later that the exact value of the area under consideration in Example 3 is $\frac{11}{3} \approx 3.66$. 

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**Example 5:** Estimate the area of the ellipse given by the equation \( 4x^2 + y^2 = 49 \) as follows:

The area of the ellipse is 4 times the area of the part of the ellipse in the first quadrant \((x, y)\) positive.

Estimate the area of the ellipse in the first quadrant by solving for \(y\) in terms of \(x\). Estimate the area under the graph of \(y\) by dividing the interval \([0, 3.5]\) into four equal subintervals and using the left endpoint of each subinterval.

Each rectangle has base with length \(\Delta x = \frac{3.5}{4} = 0.875\).

\[
4x^2 + y^2 = 49 \\
\Rightarrow \quad y^2 = 49 - 4x^2 \\
\Rightarrow \quad y = \sqrt{49 - 4x^2}
\]

The area of the ellipse (using the above method) is approximately \(4 \times 2.1412 = 8.5648\).

**Trapezoids versus rectangles:**

We could use trapezoids instead of rectangles to obtain better estimates, even though the calculations get a little bit more complicated. This will occur in Examples 6 and 21. We recall that the area of a trapezoid is

\[
\text{Area of a trapezoid} = \frac{(b_1 + b_2) \cdot h}{2}.
\]

**Example 6:** A train travels in a straight westward direction along a track. The velocity of the train varies, but it is measured at regular time intervals of 1/10 hour. The measurements for the first half hour are

<table>
<thead>
<tr>
<th>time</th>
<th>0 0.1 0.2 0.3 0.4 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>velocity</td>
<td>0 10 15 18 20 25</td>
</tr>
</tbody>
</table>

We will see later that the total distance traveled by the train is equal to the area underneath the graph of the velocity function and lying above the t-axis. Compute the total distance traveled by the train during the first half hour by assuming the velocity is a linear function of \(t\) on the subintervals. (The velocity in the table is given in miles per hour.)

\[
\text{area} = 0.1 \cdot \frac{10}{2} + 0.1 \left( \frac{10 + 15}{2} \right) + 0.1 \left( \frac{15 + 18}{2} \right) \\
+ 0.1 \left( \frac{18 + 20}{2} \right) + 0.1 \left( \frac{20 + 25}{2} \right)
\]

\[= 7.55 \text{ miles}\]
Example 7: Estimate the area under the graph of \( y = \frac{1}{x} \) for \( x \) between 1 and 31 in two different ways:

(a) Subdivide the interval \([1, 31]\) into 30 equal subintervals and use the left endpoint of each subinterval as the sample point.

(b) Subdivide the interval \([1, 31]\) into 30 equal subintervals and use the right endpoint of each subinterval as the sample point.

Find the difference between the two estimates (left endpoint estimate minus right endpoint estimate).

\[
S_L = 1 \cdot \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{30}\right)
\]

\[
S_R = 1 \cdot \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{30} + \frac{1}{31}\right)
\]

\[
\text{difference} = S_L - S_R = 1 - \frac{1}{31} = \frac{31 - 1}{31} = \frac{30}{31} \approx 0.96774
\]

Example 8: Estimate the area under the graph of \( y = \frac{1}{x} \) for \( x \) between 1 and \( n \) where \( n \) is an integer. Use right endpoints of equal subintervals of length one so the actual area is larger than your estimate. Make this estimate for several large values of \( n \), such as \( n = 10, n = 20 \), etc. What is your guess for the limit of the estimate as \( n \) tends to infinity?

\[
\text{area from } 1 \text{ to } 10 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{10} = 1.92896
\]

\[
\text{area from } 1 \text{ to } 20 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{20} = 2.5977
\]

\[
\text{area from } 1 \text{ to } 30 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{30} = 2.9749
\]

etc...  

as \( n \to \infty \), \( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} \) becomes large and large \( n \to \infty \), so limit equals \( \infty \).

**Sigma (Σ) notation:** In approximating areas we have encountered sums with many terms. A convenient way of writing such sums uses the Greek letter \( \Sigma \) (which corresponds to our capital \( S \)) and is called **sigma notation.** More precisely, if \( a_1, a_2, \ldots, a_n \) are real numbers we denote the sum

\[
a_1 + a_2 + \cdots + a_n
\]

by using the notation

\[
\sum_{k=1}^{n} a_k.
\]

The integer \( k \) is called an **index** or **counter** and takes on (in this case) the values 1, 2, \ldots, \( n \).
For example,
\[
\sum_{k=1}^{6} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 1 + 4 + 9 + 16 + 25 + 36 = 91
\]
whereas
\[
\sum_{k=3}^{6} k^2 = 3^2 + 4^2 + 5^2 + 6^2 = 9 + 16 + 25 + 36 = 86.
\]

Example 9: Evaluate the sum \( \sum_{k=1}^{5} (2k - 1) \).

\[
(2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + (2 \cdot 4 - 1) + (2 \cdot 5 - 1) \\
= 2 \left( 1 + 2 + 3 + 4 + 5 \right) + (-1 - 1 - 1 - 1 - 1) \\
= 2 \cdot 15 - 5 = \text{25}
\]

Example 10: Evaluate the sum \( \sum_{k=2}^{6} (6k^3 + 3) \).

\[
(6 \cdot 2^3 + 3) + (6 \cdot 3^3 + 3) + (6 \cdot 4^3 + 3) + (6 \cdot 5^3 + 3) + (6 \cdot 6^3 + 3) \\
= 6 \left( 2^3 + 3^3 + 4^3 + 5^3 + 6^3 \right) + (3 + 3 + 3 + 3 + 3) \\
= 6 \cdot 440 + 15 = 2655
\]

Example 11: Evaluate the sum \( \sum_{k=1}^{5} (3k^2 + k) \).

\[
(3 \cdot 1^2 + 1) + (3 \cdot 2^2 + 2) + (3 \cdot 3^2 + 3) + (3 \cdot 4^2 + 4) + (3 \cdot 5^2 + 5) \\
= 3 \left( 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \right) + (1 + 2 + 3 + 4 + 5) \\
= 3 \cdot 55 + 15 = 180
\]

Example 12: Evaluate the sum \( \sum_{k=1}^{112} 75 \).

\[
= \overline{75 + 75 + 75 + \ldots + 75} = 75 \cdot 112 = 8400
\]
The idea we have used so far is to break up or subdivide the given interval \([a, b]\) into lots of little pieces, or subintervals, on each of which the variable \(x\), and thus the function \(f(x)\), does not change much. The technical phrase for doing this is "to partition" \([a, b]\).

**Definition of a partition:** A partition of an interval \([a, b]\) is a collection of points \(\{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}\), listed increasingly, on the \(x\)-axis with \(a = x_0\) and \(x_n = b\). That is: \(a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\). These points subdivide the interval \([a, b]\) into \(n\) subintervals: \([a, x_1]\), \([x_1, x_2]\), \([x_2, x_3]\), \ldots, \([x_{n-1}, b]\).

The \(k\)-th subinterval is thus of the form \([x_{k-1}, x_k]\) and it has length \(\Delta x_k = x_k - x_{k-1}\).

**Assumption:** Set \(\|P\| = \max_{1 \leq k \leq n} \{\Delta x_i\}\). We will always assume that our partition \(P\) is such that \(\|P\| \to 0\) as \(n \to \infty\). In other words, we always assume that the length of the longest (and as a consequence of all) subinterval(s) tend(s) to zero whenever the number of subintervals in our partition \(P\) becomes very large.

**The definite integral:**

Let \(f(x)\) be a function defined on an interval \([a, b]\). Partition the interval \([a, b]\) in \(n\) subintervals of lengths \(\Delta x_1, \ldots, \Delta x_n\), respectively. For \(k = 1, \ldots, n\) pick a representative point \(p_k\) in the corresponding \(k\)-th subinterval. The definite integral of the function \(f\) from \(a\) to \(b\) is defined as

\[
\lim_{n \to \infty} \sum_{k=1}^{n} f(p_k) \cdot \Delta x_k = \lim_{n \to \infty} \left( f(p_1) \cdot \Delta x_1 + f(p_2) \cdot \Delta x_2 + \cdots + f(p_n) \cdot \Delta x_n \right) = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(p_k) \cdot \Delta x_k
\]

and it is denoted by

\[
\int_{a}^{b} f(x) \, dx.
\]

The sum \(\sum_{k=1}^{n} f(p_k) \cdot \Delta x_k\) is called a Riemann sum in honor of the German mathematician Bernhard Riemann (1826-1866), who developed the above ideas in full generality. The symbol \(\int\) is called the integral sign. It is an elongated capital S, of the kind used in the 1600s and 1700s. The letter S stands for the summation performed in computing a Riemann sum. The numbers \(a\) and \(b\) are called the lower and upper limits of integration, respectively. The function \(f(x)\) is called the integrand and the symbol \(dx\) is called the differential of \(x\). You can think of the \(dx\) as representing what happens to the term \(\Delta x\) in the limit, as the size \(\Delta x\) of the subintervals gets closer and closer to zero.

**Note:** The role of \(x\) in a definite integral is the one of a dummy variable. In fact \(\int_{a}^{b} x^2 \, dx\) and \(\int_{a}^{b} t^2 \, dt\) have the same meaning. They represent the same number.

**Note:** We recall from Chapter 3 that a limit does not necessarily exist. However:

**Theorem:** Let \(f(x)\) be a continuous function on the interval \([a, b]\) then \(\int_{a}^{b} f(x) \, dx\) exists. That is, the limit used in the definition of the definite integral exists.

**Regular partitions:** As we observed earlier, it is computationally easier to partition the interval \([a, b]\) into \(n\) subintervals of equal length. Therefore each subinterval has length \(\Delta x = \frac{b - a}{n}\) (we drop the index \(k\) as it is no longer necessary). In this case, there is a simple formula for the points of the partition, namely:

\[
x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2 \cdot \Delta x, \quad \ldots \quad x_k = a + k \cdot \Delta x, \quad \ldots, \quad x_{n-1} = a + (n - 1) \cdot \Delta x, \quad x_n = b
\]

or, more concisely,

\[
x_k = a + k \cdot \frac{b - a}{n} \quad \text{for} \quad k = 0, 1, 2, \ldots, n.
\]
**Right versus left endpoint estimates:**

Observe that \( x_k \), the right endpoint of the \( k \)-th subinterval, is also the left endpoint of the \((k+1)\)-th subinterval. Thus the Riemann sum estimate for the definite integral of a function \( f \) defined over an interval \([a, b]\) can be written in either of the following two forms:

\[
\sum_{k=0}^{n-1} f(x_k) \cdot \Delta x_{k+1} \quad \text{or} \quad \sum_{k=1}^{n} f(x_k) \cdot \Delta x_k
\]

depending on whether we use left or right endpoints, respectively.

If we are dealing with a regular partition, the above sums become

\[
\sum_{k=0}^{n-1} f(a + k \cdot \Delta x) \cdot \Delta x \quad \text{or} \quad \sum_{k=1}^{n} f(a + k \cdot \Delta x) \cdot \Delta x
\]

respectively, with \( \Delta x = (b-a)/n \) and \( x_k = a + k \cdot \Delta x \) for \( k = 0, 1, 2, \ldots, n \).

**Example 13:** Suppose you estimate the integral \( \int_{1}^{7} 8^2 \, dx \) by evaluating the sum

\[
\sum_{k=1}^{n} 8^{1+k \cdot \Delta x} \cdot \Delta x.
\]

If you use \( \Delta x = 0.2 \), what value should you use for \( n \), the upper limit of the summation?

\[
\text{Length of interval} \quad 7 - 1 = 6 \quad \implies \quad \Delta x = \frac{6-1}{n} \quad \Rightarrow \quad 0.2 = \frac{6}{n} \quad \Rightarrow \quad n = \frac{6}{0.2} = 30
\]

**Example 14:** Suppose you estimate the integral \( \int_{2}^{10} x^2 \, dx \) by evaluating the sum

\[
\sum_{k=1}^{n} (2+k \cdot \Delta x)^2 \cdot \Delta x.
\]

If you use \( n = 10 \) intervals, what value should you use for \( \Delta x \), the length of each interval?

\[
\Delta x = \frac{10-2}{10} = 0.8
\]

**Example 15:** Suppose you estimate the integral \( \int_{-6}^{0} x^2 \, dx \) by the sum

\[
\sum_{k=1}^{n} [A + B(k \Delta x) + C(k \Delta x)^2] \cdot \Delta x,
\]

where \( n = 30 \) and \( \Delta x = 0.2 \). The terms in the sum equal areas of rectangles obtained by using right end points of the subintervals of length \( \Delta x \) as sample points. What is the value of \( B \)?

\[
\Delta x = \frac{b-a}{n} = \frac{0-(-6)}{30} = \frac{0.2}{30} = \frac{0.2}{6} = 0.2
\]

\[
\Delta x = 0.2 = \frac{0.2}{6} = 0.2
\]

\[
\sum_{k=1}^{30} (-6+k \cdot \Delta x)^2 \cdot \Delta x = \sum_{k=1}^{30} \left[ (26-12kx + (k \Delta x)^2) \right] \Delta x
\]

\[
\therefore \quad A = 36 \quad B = -12 \quad C = 1
\]

\[
\int_{-6}^{0} x^2 \, dx \quad \text{by} \quad \sum_{k=1}^{30} (-6+k \cdot \Delta x)^2 \cdot \Delta x
\]

\[
\text{We approximate } \int_{-6}^{0} x^2 \, dx \text{ by expanding more.}
\]

\[
\therefore \quad A = 36 \quad B = -12 \quad C = 1
\]

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Example 16: Suppose you estimate the integral \( \int_{5}^{15} x^3 \, dx \) by the sum

\[
\sum_{k=1}^{n} (a + k \Delta x)^3 \cdot \Delta x,
\]

where \( n = 50 \) and \( \Delta x = 0.2 \). The terms in the sum equal areas of rectangles obtained by using right end points of the subintervals of length \( \Delta x \) as sample points. What is the value of \( a \)?

\[
\begin{align*}
x_k &= a + k \Delta x \\
&= 5 + k \cdot 0.2 \\
\therefore \quad a &= 5
\end{align*}
\]

Example 17: Suppose you estimate the integral \( \int_{3}^{15} f(x) \, dx \) by adding the areas of \( n \) rectangles of equal base length, and you use the right endpoint of each subinterval to determine the height of each rectangle. If the sum you evaluate is written as

\[
\sum_{k=1}^{n} f(3 + k \cdot A/n) \cdot A/n,
\]

what is \( A \)?

\[
\begin{align*}
\Delta x &= \frac{b-a}{n} \\
&= \frac{15-3}{12} \\
&= \frac{12}{12}
\end{align*}
\]

\( \therefore \) it is the **ame** height of the interval

Example 18: Suppose you estimate the integral \( \int_{3}^{9} f(x) \, dx \) by evaluating a sum

\[
\sum_{k=1}^{n} f(3 + k \cdot \Delta x) \cdot \Delta x.
\]

If you use \( n = 6 \) intervals of equal length, what value should you use for \( \Delta x \)?

\[
\Delta x = \frac{b-a}{n} = \frac{9-3}{6} = \frac{6}{6} = 1
\]

Example 19: Suppose you estimate the area under the graph of \( f(x) = x^3 \) from \( x = 4 \) to \( x = 24 \) by adding the areas of rectangles as follows: partition the interval into 20 equal subintervals and use the right endpoint of each interval to determine the height of the rectangle. What is the area of the 15th rectangle?

\[
\Delta x = \frac{24-4}{20} = 1
\]
Example 20: Suppose you estimate the area under the graph of \( f(x) = \frac{1}{x} \) from \( x = 12 \) to \( x = 112 \) by adding the areas of rectangles as follows: partition the interval into 50 equal subintervals and use the left endpoint of each interval to determine the height of the rectangle. What is the area of the 24th rectangle?

\[
\begin{align*}
\text{Area of } 24^\text{th} \text{ rectangle} & = \frac{1}{58} \cdot 2 = \frac{1}{29} \\
\end{align*}
\]

Example 21: Suppose you are given the following data points for a function \( f(x) \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>12</td>
</tr>
</tbody>
</table>

If \( f \) is a linear function on each interval between the given points, find \( \int_1^4 f(x) \, dx \).

\[
\text{Area} = \frac{(2+5) \cdot 1}{2} + \frac{(5+8) \cdot 1}{2} + \frac{(8+12) \cdot 1}{2} = \frac{40}{2} = 20
\]

Example 22: Suppose \( f(x) \) is the greatest integer function, i.e., \( f(x) \) equals the greatest integer less than or equal to \( x \). So for example \( f(2.3) = 2 \), \( f(4) = 4 \), and \( f(6.9) = 6 \). Find \( \int_6^{10} f(x) \, dx \).

(Hint: Draw a picture. See also Example 19 in Chapter 3.)

\[
\int_6^{10} f(x) \, dx = \text{area under the graph (from the picture)}
\]

\[
= 6 \cdot 1 + 7 \cdot 1 + 8 \cdot 1 + 9 \cdot 1 = 30
\]