MA123 — Elem. Calculus Spring 2016
Exam 3 2016-04-14

Name: SOLUTIONS **Sec.:** ________

Do not remove this answer page — you will turn in the entire exam. No books or notes may be used. You may use an ACT-approved calculator during the exam, but NO calculator with a Computer Algebra System (CAS), networking, or camera is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of two short answer questions and eighteen multiple choice questions. Answer the short answer questions on the back of this page, and record your answers to the multiple choice questions on this page. For each multiple choice question, you will need to fill in the circle corresponding to the correct answer. It is your responsibility to make it CLEAR which response has been chosen. For example, if (a) is correct, you must write

●a ○b ○c ○d ○e

You have two hours to do this exam. Please write your name on this page, and at the top of page three.

GOOD LUCK!

3. ○a ○b ○c ○d ●e
4. ●a ○b ○c ○d ○e
5. ○a ●b ○c ○d ○e
6. ○a ○b ○c ●d ○e
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19. ○a ○b ○c ●d ○e
20. ●a ○b ○c ○d ○e

For grading use:

Multiple Choice		Short Answer
(number right)	(5 points each)	(out of 10 points)

Total	
	(out of 100 points)

Spring 2016 Exam 3 Short Answer Questions

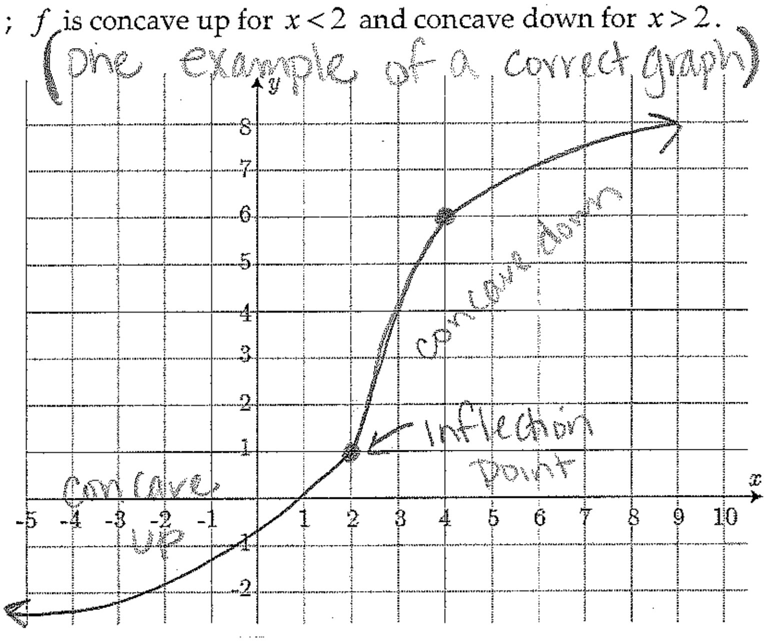
Write your answers on this page.

1. Sketch the graph of a **continuous** function $y = f(x)$ which satisfies the following: $f(4) = 6$; $f'(x) \geq 0$ for all x; f is concave up for $x < 2$ and concave down for $x > 2$.

Contains the point (4,6)

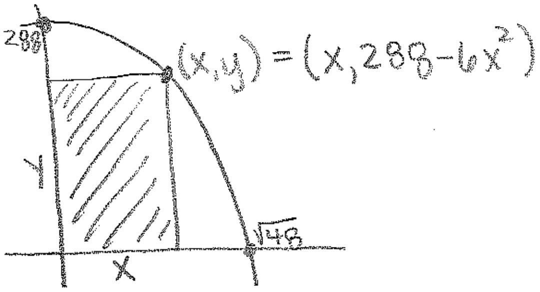
always increasing

inflection point at $x=2$



2. Find the area of the largest rectangle with one corner at the origin, the opposite corner in the first quadrant on the graph of the parabola $f(x) = 288 - 6x^2$, and sides parallel to the axes.

*You must **clearly use calculus** to find and justify your answer.*



$$A = xy$$
$$= x(288-6x^2)$$
$$= 288x - 6x^3$$
$$A' = 288 - 18x^2$$
$$288 - 18x^2 = 0$$
$$288 = 18x^2$$
$$16 = x^2$$
$$\pm 4 = x$$

$x = -4$ not in interval

$x \in [0, \sqrt{48}]$ since we're in the 1st quadrant

test $x = 0, 4, \sqrt{48}$

$A(0) = 0(288) = 0$

$A(4) = 4(192) = 768$ ← max

$A(\sqrt{48}) = \sqrt{48}(0) = 0$

Largest Possible Area: $\boxed{768 \text{ units}^2}$

Name: SOLUTIONS

Multiple Choice Questions

Show all your work on the page where the question appears.
Clearly mark your answer both on the cover page on this exam
and in the corresponding questions that follow.

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3. Where is the function $f(t) = \frac{1}{t-46}$ decreasing?

Possibilities:

(a) $f(t)$ is never decreasing
(b) $t > 46$
(c) $t < 46$
(d) $-1 < t < 46$
[✓ (e) $f(t)$ is always decreasing except at t = 46]

$f(t) = (t-46)^{-1}$

$f'(t) = -1(t-46)^{-2} = \frac{-1}{(t-46)^2}$

$f'(t)$ always negative
thus $f(t)$ always decreasing
(except where its undefined)

---

4. Where is the function $f(t) = \frac{1}{t-46}$ concave up?

Possibilities:

[✓ (a) $t > 46$]
(b) $f(t)$ is always concave up except at t = 46
(c) $t < 46$
(d) $-1 < t < 46$
(e) $f(t)$ is never concave up

$f'(t) = -1(t-46)^{-2}$

$f''(t) = (-1)(-2)(t-46)^{-3}$
$= 2(t-46)^{-3} = \frac{2}{(t-46)^3}$

$f(t)$ is concave up when
$f''(t)$ is positive

$\frac{2}{(t-46)^3} > 0$ when $(t-46) > 0$
$t > 46$

5. Suppose the derivative of $g(t)$ is $g'(t) = 2(t-9)(t-5)(t-6)$. For t in which interval(s) is g increasing?

Possibilities:

(a) $(\frac{20}{3} - \frac{1}{3}\sqrt{13}, \frac{20}{3} + \frac{1}{3}\sqrt{13})$

[✓ (b) $(5,6) \cup (9,\infty)$]

(c) $(-\infty,5) \cup (6,9)$

(d) $(2,5) \cup (6,9)$

(e) $(-\infty, \frac{20}{3} - \frac{1}{3}\sqrt{13}) \cup (\frac{20}{3} + \frac{1}{3}\sqrt{13}, \infty)$

$g'(t) = 0$ when $t = 9, 5, 6$

$g'(4) = 2(-)(-)(-) = $ "−"

$g'(\frac{11}{2}) = 2(-)(+)(-) = $ "+"

$g'(8) = 2(-)(+)(+) = $ "−"

$g'(10) = 2(+)(+)(+) = $ "+"

g is increasing when $g'(t)$ is positive → $(5,6) \cup (9,\infty)$

6. Suppose the derivative of $g(t)$ is $g'(t) = 13(t-2)(t-10)$. For t in which interval(s) is g concave up?

Possibilities:

(a) $(-\infty,6)$

(b) $(2,10)$

(c) $(-\infty,2) \cup (10,\infty)$

[✓ (d) $(6,\infty)$]

(e) $(2,6) \cup (10,13)$

Concavity → need 2nd derivative

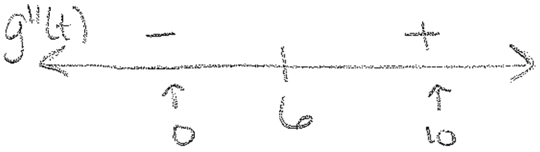
$g'(t) = 13(t-2)(t-10)$

$= 13(t^2 - 12t + 20)$

$g''(t) = 13(2t-12)$

$= 26(t-6)$

$g''(t) = 0$ when $t = 6$



$g''(0) = $ "−" $g''(10) = $ "+"

g is concave up when $g''(t)$ is positive → $(6,\infty)$

7. The following is the graph of the derivative, $f'(x)$, of the function $f(x)$. Where is the regular function $f(x)$ decreasing?

Possibilities:

(a) $(-1, \infty)$

(b) $(-\infty, \infty)$

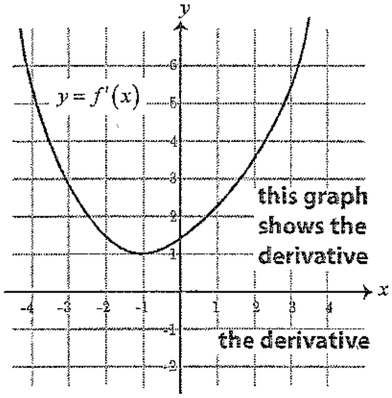
[✓ (c) nowhere]

(d) $(-\infty, -1)$

(e) $(1, \infty)$

(f decreases when f' is negative)

f' is always positive (above x-axis) thus nowhere decreasing



8. The following is the graph of the derivative, $f'(x)$, of the function $f(x)$. Where is the regular function $f(x)$ concave up?

Possibilities:

(a) nowhere

(b) $(-\infty, \infty)$

(c) $(1, \infty)$

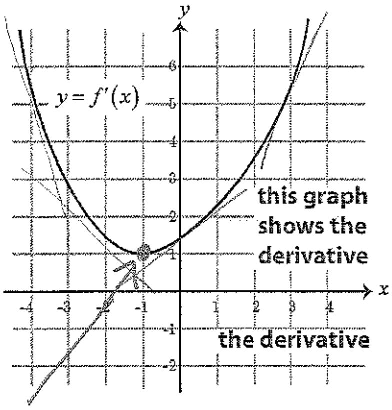
[✓ (d) $(-1, \infty)$]

(e) $(-\infty, -1)$

(f(x) concave up when f'' is positive)

look at the tangent lines

slope gives us the derivatives



slopes of tangent lines are positive after x=-1

so $f''(x) > 0$ when $x \in (-1, \infty)$

9. A landscape architect wishes to enclose a rectangular garden on one side by a brick wall costing \$70 per foot, and on the other three sides by a metal fence costing \$10 per foot. If the area of the garden is 400 square feet, find the lowest possible cost to enclose the garden.

Possibilities:

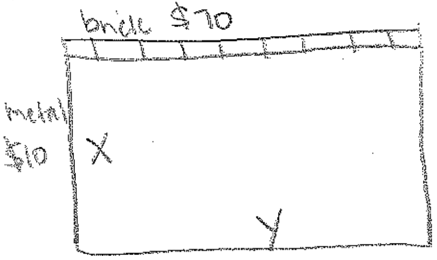
(a) \$1599.50

(b) \$1600.50

[✓ (c) \$1600.00]

(d) \$1601.00

(e) \$1601.50



$A = x \cdot y = 400$

$y = \frac{400}{x}$

$x \geq 0 \quad y = \frac{400}{x} \geq 0 \quad$ so $x \in [0, \infty)$

want to minimize cost

$C = 10(2x+y) + 70(y)$

$C = 10\left(2x+\frac{400}{x}\right) + 70\left(\frac{400}{x}\right)$

$C = 20x + \frac{4000}{x} + \frac{28000}{x} = 20x + \frac{32000}{x}$

$C = 20x + 32000x^{-1}$

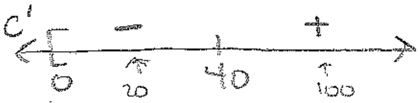
$C' = 20 - 32000x^{-2}$

$C' = 20 - \frac{32000}{x^2}$

$20 - \frac{32000}{x^2} = 0$

$20 = \frac{32000}{x^2}$

$x^2 = \frac{32000}{20} = 1600 \quad x = \pm 40 \quad \left(\begin{matrix} x = 40 \\ \text{since } -40 \\ \text{not in interval} \end{matrix}\right)$



min at $x = 40$

$C(40) = 20(40) + \frac{32000}{40} = 1600$

10. A car rental agency rents 180 cars per day at a rate of \$29 dollars per day. For each 1 dollar increase in the daily rate, 3 fewer cars are rented. At what rate should the cars be rented to produce maximum income?

Possibilities:

(a) \$43.90 per day

(b) \$44.10 per day

[✓ (c) \$44.50 per day]

(d) \$44.70 per day

(e) \$45.30 per day

let n = # cars rented

p = \$ per car

and I = income then $I = np$

let x = # of \$1 increases

$I = (180 - 3x)(29 + x)$

$= 5220 + 93x - 3x^2$

maximize income

$I' = -6x + 93$

$-6x + 93 = 0$

$93 = 6x$

$15.5 = x$

raise the cost by \$15.50

price per car \$29 + 15.50 = \$44.50

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11. Suppose the derivative of $H(s)$ is given by $H'(s) = -1/(s^2+2)$. Find the value of s in the interval $[-10, 10]$ where $H(s)$ takes on its maximum.

Possibilities:

(a) 10

[✓ (b) −10]

(c) −2

(d) $-\frac{1}{2}$

(e) 2

$$H'(s) = \frac{-1}{(s^2+2)} < 0 \text{ for all } s$$

thus H(s) is always decreasing

So max occurs at left endpoint $x = -10$

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12. Find the critical numbers of the function $f(x) = xe^{2x}$.

Possibilities:

(a) $-\frac{1}{2}, 0$

[✓ (b) $-\frac{1}{2}$]

(c) 0

(d) 2

(e) $-\frac{1}{2}, 0, e^2$

$$f'(x) = x \cdot e^{2x}(2) + (1)e^{2x}$$
$$= 2xe^{2x} + e^{2x}$$
$$= e^{2x}(2x+1)$$

$f'(x) = 0$ when $2x+1=0$

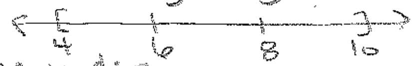
$2x = -1$

$x = \frac{-1}{2}$

---

13. Estimate the area under the graph of $-x^2 + 10x$ for x between 4 and 10, by using a partition that consists of 3 equal subintervals of $[4, 10]$ and use the right endpoint of each subinterval as a sample point.

$\Delta x = \frac{10-4}{3} = \frac{6}{3} = 2$ let $f(x) = -x^2 + 10x$



Possibilities:

(a) 128

(b) 40

(c) 108

(d) 6

[✓ (e) 80]

area under f(x) $= \Delta x \cdot f(6) + \Delta x \cdot f(8) + \Delta x \cdot f(10)$

$= 2(-6^2+10(6)) + 2(-8^2+10(8)) + 2(-10^2+10(10))$

$= 2(24) + 2(16) + 2(0)$

$= 48 + 32 = 80$

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14. A train travels in a straight westward direction along a track. The speed of the train varies, but it is measured at regular time intervals of 1/10 hour. The measurements for the first half hour are:

time	0	.1	.2	.3	.4	.5
speed	0	3	10	15	22	24

Estimate the total distance (in miles) traveled by the train during the first half hour by assuming the speed is a linear function of t on the subintervals. The speed in the table is given in miles per hour. Use all six speed measurements in your estimate.

use trapezoids to estimate

Possibilities:

[✓ (a) 6.20 miles]

(b) 12.00 miles

(c) 7.50 miles

(d) 7.40 miles

(e) 1.50 miles

distance $= \frac{(0+3)}{2}(.1) + \frac{(3+10)}{2}(.1) + \frac{(10+15)}{2}(.1) + \frac{(15+22)}{2}(.1) + \frac{(22+24)}{2}(.1)$

$= .15 + .65 + 1.25 + 1.85 + 2.3$

$= 6.2$ miles

15. One way to approximate $\int_8^A \ln(17x+1)\,dx$ is with the sum $\sum_{k=1}^{200}(\Delta x)\cdot\ln(17(8+k\Delta x)+1)$ where $\Delta x = \frac{1}{8}$. What is the best value of A to use?

Possibilities:

(a) $\frac{1}{8}$
(b) 0.01
[✓ (c) 33]
(d) 144.6627443
(e) 25

$$\Delta x = \frac{A-8}{200} = \frac{1}{8}$$

$$\frac{A-8}{200} = \frac{1}{8}$$

$$8(A-8) = 200$$

$$A-8 = 25$$

$$A = 33$$

---

16. Suppose you estimate the integral

$$\int_2^{11} x^2\,dx$$

by adding the areas of n rectangles of equal length, and using the right endpoint of each subinterval to determine the height of each rectangle. If the sum you evalute is written as

$$\sum_{k=1}^{n} \underbrace{\frac{A}{n}}_{\Delta x} \underbrace{\left(2+k\frac{A}{n}\right)^2}_{f(x_k)}$$

What value should be used for A?

Possibilities:

(a) 2
(b) 13
(c) 11
[✓ (d) 9]
(e) 441

$$\Delta x = \frac{11-2}{n} = \frac{9}{n} = \frac{A}{n}$$

$$x_k = 2 + k\Delta x$$

$$= 2 + k\frac{9}{n} = 2 + k\frac{A}{n}$$

So $A = 9$

17. Evaluate the difference of sums

$$\left(\sum_{k=1}^{30000}(6k^3+7)\right)-\left(\sum_{k=3}^{30000}(6k^3+7)\right)$$

$$= \sum_{k=1}^{2} (6k^3+7)$$

$$= \left[6(1)^3+7\right]+\left[6(2)^3+7\right]$$

$$= 13 + 55$$

$$= 68$$

Possibilities:

(a) 0

(b) ∞

(c) 162000000000007

(d) 450015000

[✓ (e) 68]

---

18. Evaluate the sum

$$\sum_{k=1}^{N}(11k^2)$$

$$= 11 \sum_{k=1}^{N} k^2$$

$$= 11\left[\frac{N(N+1)(2N+1)}{6}\right]$$

Possibilities:

[✓ (a) $11\frac{N(N+1)(2N+1)}{6}$]

(b) $11N^2-11$

(c) $11\left(\frac{N(N+1)}{2}\right)^2$

(d) $11\frac{N^2(11N+1)(22N+1)}{6}$

(e) $11\frac{N(N+1)}{2}$

19. Suppose you estimate the area under the graph of $f(x) = x^3$ from $x = 4$ to $x = 44$ by adding the areas of the rectangles as follows: partition the interval into 20 equal subintervals and use the right endpoint of each interval to determine the height of the rectangle. What is the area of the 11th rectangle?

$$\Delta x = \frac{44-4}{20} = \frac{40}{20} = 2$$

Possibilities:

- (a) 27648
- (b) 31300
- (c) 1024000
- [✓ (d) 35152]
- (e) 17576

$$x_k = 4 + k(2)$$

$$x_{11} = 4 + 11(2) = 26$$

$$\text{area of } 11^{th} \text{ rectangle} = \Delta x \cdot f(x_{11})$$

$$= 2 \cdot f(26)$$

$$= 2 \cdot (26)^3 = 35{,}152 \text{ units}^2$$

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20. Evaluate the sum $\frac{1}{101} + \frac{2}{101} + \frac{3}{101} + \frac{4}{101} + \frac{5}{101} + \frac{6}{101} + \frac{7}{101} + \frac{8}{101} + \cdots + \frac{2015}{101} + \frac{2016}{101}$.

Possibilities:

- [✓ (a) $\frac{2033136}{101}$]
- (b) $\frac{4067}{101}$
- (c) $\frac{4064256}{10201}$
- (d) 6
- (e) $\frac{2133936}{10201}$

$$= \sum_{k=1}^{2016} \frac{k}{101}$$

$$= \frac{1}{101} \sum_{k=1}^{2016} k$$

$$= \frac{1}{101} \left(\frac{2016(2016+1)}{2} \right)$$

$$= \frac{(2016)(2017)}{(101)(2)} = \frac{4066272}{202}$$

$$= \frac{2033136}{101}$$

Some Formulas

1. **Summation formulas:**

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

2. **Areas:**
   (a) Triangle $A = \frac{bh}{2}$
   (b) Circle $A = \pi r^2$
   (c) Rectangle $A = lw$
   (d) Trapezoid $A = \frac{h_1 + h_2}{2} b$

3. **Volumes:**
   (a) Rectangular Solid $V = lwh$
   (b) Sphere $V = \frac{4}{3}\pi r^3$
   (c) Cylinder $V = \pi r^2 h$
   (d) Cone $V = \frac{1}{3}\pi r^2 h$

4. **Distance:**
   (a) Distance between (x_1, y_1) and (x_2, y_2)

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$