MA123 — Elem. Calculus   Spring 2018   **Name:** Solutions   **Sec.:** ________
Exam 3   2018-4-12

Do not remove this answer page — you will turn in the entire exam. No books or notes may be used. You may use an ACT-approved calculator during the exam, but NO calculator with a Computer Algebra System (CAS), networking, or camera is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of two short answer questions and eighteen multiple choice questions. Answer the short answer questions on the back of this page, and record your answers to the multiple choice questions on this page. For each multiple choice question, you will need to fill in the circle corresponding to the correct answer. It is your responsibility to make it CLEAR which response has been chosen. For example, if (a) is correct, you must write

(a) (b) (c) (d) (e)

You have two hours to do this exam. Please write your name and section number on this page.

GOOD LUCK!

3. (a) (b) **(c)** (d) (e)
4. (a) **(b)** (c) (d) (e)
5. (a) (b) (c) **(d)** (e)
6. **(a)** (b) (c) (d) (e)
7. (a) (b) **(c)** (d) (e)
8. (a) (b) (c) (d) **(e)**
9. (a) **(b)** (c) (d) (e)
10. **(a)** (b) (c) (d) (e)
11. (a) (b) (c) (d) **(e)**
12. (a) (b) **(c)** (d) (e)
13. **(a)** (b) (c) (d) (e)
14. (a) (b) (c) (d) **(e)**
15. (a) (b) (c) **(d)** (e)
16. (a) (b) **(c)** (d) (e)
17. (a) (b) (c) **(d)** (e)
18. (a) **(b)** (c) (d) (e)
19. (a) (b) **(c)** (d) (e)
20. **(a)** (b) (c) (d) (e)

For grading use:

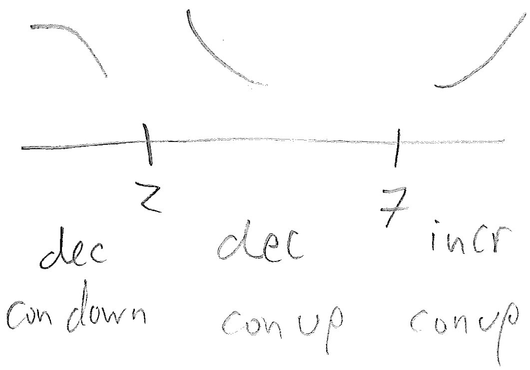
Multiple Choice		Short Answer
(number right)	(5 points each)	(out of 10 points)

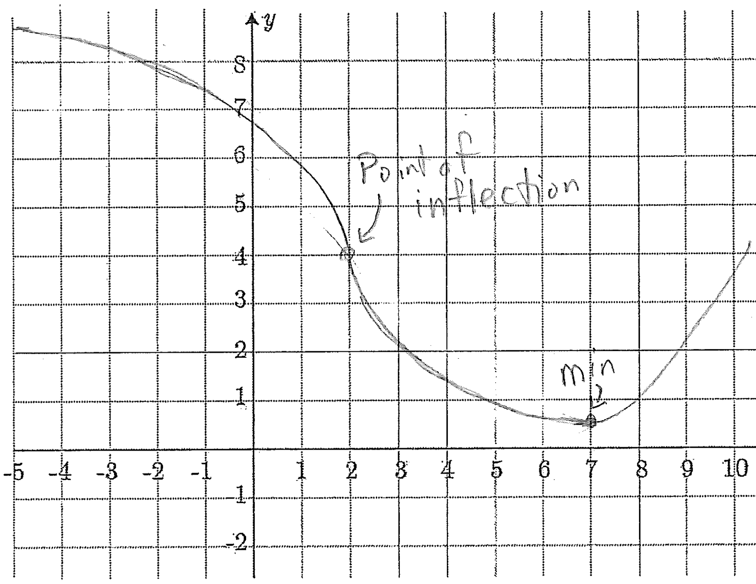
Total	
	(out of 100 points)

Spring 2018 Exam 3 Short Answer Questions

Write answers on this page. Your work must be clear and legible to be sure you will get full credit.

1. Sketch the graph of a **continuous** function $y = f(x)$ which satisfies $f'(x) < 0$ for $x < 7$; $f'(x) > 0$ for $x > 7$; $f''(x) < 0$ for $x < 2$; $f''(x) > 0$ for $x > 2$.





2. The product of two positive real numbers x and y is 21. Find the minimum value of the expression $3x + 2y$. *You must* ***clearly use calculus*** *to find* ***and justify*** *your answer. Your final answer does* ***not*** *need to be simplified.*

$xy = 21 \qquad y = \frac{21}{x}$

$f = 3x + 2y = 3x + 2\left(\frac{21}{x}\right) = 3x + \frac{42}{x}$

$f'(x) = 3 - \frac{42}{x^2}$

f': $-$ at 1, 0 at $\sqrt{14}$, $+$ at 10

test points

critical points:

$f'(x) = 0$

$3 - \frac{42}{x^2} = 0$

$3 = \frac{42}{x^2}$

$x^2 = \frac{42}{3} = 14$

$x = \sqrt{14} \quad (x > 0)$

So $f(x)$ has a minimum at $x = \sqrt{14}$ (no end points)

$f(\sqrt{14}) = 3\sqrt{14} + \frac{42}{\sqrt{14}}$

$= 3\sqrt{14} + 3\sqrt{14}$

$= 6\sqrt{14}$

Minimum possible value: $6\sqrt{14} \approx 22.45$

Name: Solutions

Multiple Choice Questions

Show all your work on the page where the question appears.
Clearly mark your answer both on the cover page on this exam
and in the corresponding questions that follow.

3. Where is the function $f(t) = \dfrac{1}{t-23}$ decreasing?

Possibilities:

(a) $t > 23$

(b) $-1 < t < 23$

(c) $f(t)$ is always decreasing except at t = 23

(d) $f(t)$ is never decreasing

(e) $t < 23$

$f(t) = (t-23)^{-1}$

$f'(t) = -(t-23)^{-2}$

$= \dfrac{-1}{(t-23)^2}$ ← numerator always negative; ↖ denominator always positive

f: decreasing, decreasing

f': $-$ | $-$ at 23

f is decreasing in $(-\infty, -23)$ and $(-23, \infty)$.

4. Where is the function $f(t) = t^4 - 12t^3 - 8$ concave up?

Possibilities:

(a) ~~$0 < t < 6$~~

(b) $t < 0$ and $t > 6$

(c) $t < 9$

(d) $f(t)$ is always concave up

(e) $t > 9$

$f'(t) = 4t^3 - 36t^2$

$f''(t) = 12t^2 - 72t$

$= 12t(t-6)$

f is concave up when $f''(t) > 0$.

	$t<0$	$0<t<6$	$t>6$
$f''(t)$	+ + +	− − − − −	+ +
$t-6$	− − −	− − − − −	+ +
$12t$	− −	+ + + +	+ +

conc up 0 conc down 6 conc up

5. Suppose the derivative of $g(t)$ is $g'(t) = 11(t-3)(t-9)^2$. Find all interval(s) of values of t in which g is **increasing**.

Possibilities:

(a) $(3, 11)$

(b) $(-\infty, 3) \cup (9, \infty)$

(c) $(-\infty, 3)$

(d) $(3, \infty)$

(e) $(-\infty, 9)$

Where is $g'(t) > 0$?

$- \; - \; - \quad | \quad + \; + \; + \quad | \quad + \; + \qquad g'(t)$

$+ \; + \; + \quad | \quad + \; + \; + \quad | \quad + \; + \qquad (t-9)^2$

$- \; - \; - \quad | \quad + \; + \; + \quad | \quad + \; + \qquad 11(t-3)$

decr 3 incr 9 incr

6. Suppose the derivative of $g(t)$ is $g'(t) = 11t^2 - 132t + 297$. For t in which interval(s) is g concave up?

Possibilities:

(a) $(6, \infty)$

(b) $(-\infty, 3) \cup (9, \infty)$

(c) $(-\infty, 6)$

(d) $(3, 6) \cup (9, 11)$

(e) $(3, 9)$

$$g''(t) = 22t - 132$$
$$= 22(t-6)$$

$- \; - \; - \quad | \quad + \; + \; + \qquad 22(t-6)$

conc down 6 conc up

7. The following is the graph of the **derivative**, $f'(x)$, of the function $f(x)$. Where is the original function $f(x)$ decreasing?

Possibilities:

(a) $(1, \infty)$

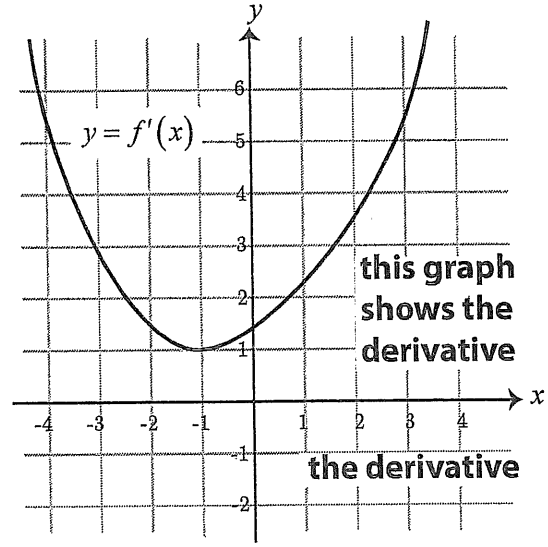
(b) $(-\infty, -1)$

(c) nowhere

(d) $(-1, \infty)$

(e) $(-\infty, \infty)$

$f'(x)$ is never < 0, so $f(x)$ is never decreasing



8. The following is the graph of the **derivative**, $f'(x)$, of the function $f(x)$. Where is the original function $f(x)$ concave up?

Possibilities:

(a) $(1, \infty)$

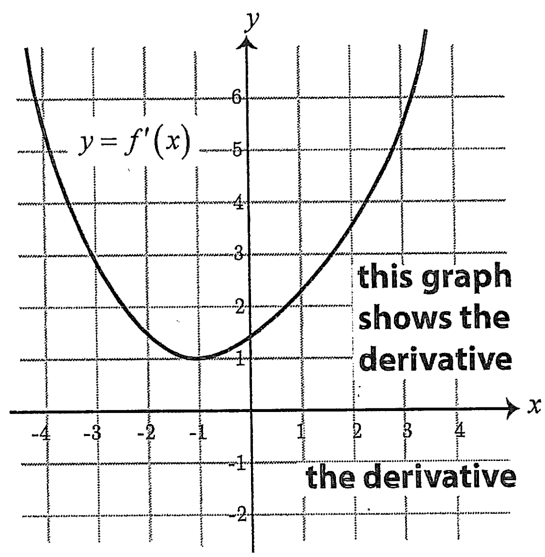
(b) $(-\infty, -1)$

(c) nowhere

(d) ~~$(-\infty, \infty)$~~

(e) $(-1, \infty)$

$f(x)$ conc up $\Updownarrow$ $f''(x) > 0$ $\Updownarrow$ $f'(x)$ increasing



9. Find the critical numbers of the function

$$f(x) = \frac{8x}{3x^2 + 12}.$$

Possibilities:

(a) $-4, 0$

(b) $-2, 2$

(c) $-\frac{8}{3}, \frac{8}{12}$

(d) $-4, 8$

(e) $-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}$

$$f'(x) = \frac{(3x^2+12)8 - 8x(6x)}{(3x^2+12)^2}$$

$$= \frac{24x^2 + 96 - 48x^2}{(3x^2+12)^2}$$

$$= \frac{-24x^2+96}{(3x^2+12)^2} = \frac{24(4-x^2)}{(3x^2+12)^2}$$

So $f'(x)=0$ when $x=2, -2$

Also $3x^2+12$ is never 0, so $f'(x)$ is defined at all x.

10. Consider the graph of the original function, $f(x)$.
For this function, what are the signs of $f'(-3)$ and $f''(-3)$?

Possibilities:

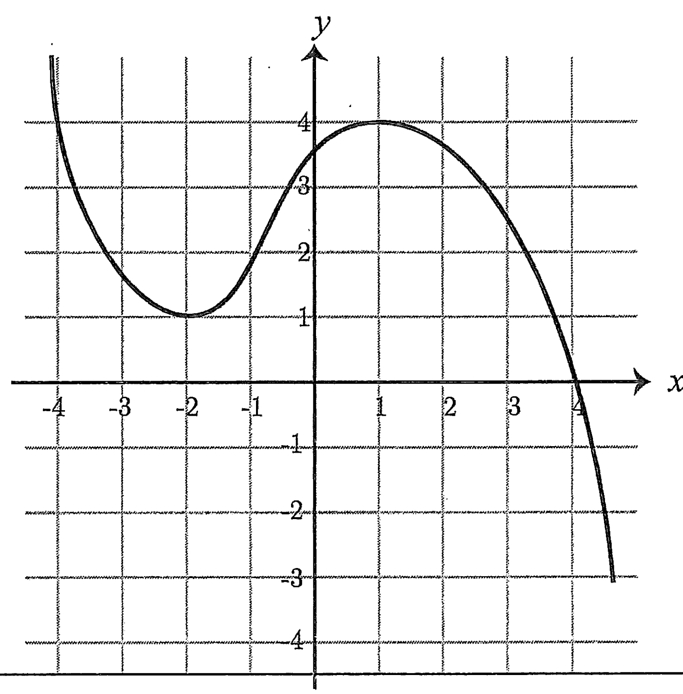
(a) $f'(-3) < 0$ and $f''(-3) > 0$

(b) $f'(-3) > 0$ and $f''(-3) > 0$

(c) $f'(-3) > 0$ and $f''(-3) < 0$

(d) $f'(-3) = 0$ and $f''(-3) < 0$

(e) $f'(-3) < 0$ and $f''(-3) < 0$



$f(x)$ is decreasing at $x=-3$, so $f'(-3)<0$

$f(x)$ is concave up at $x=-3$, so $f''(-3)>0$

11. Find the area of the largest rectangle whose sides are parallel to the coordinate axes, whose bottom-left corner is at $(0,0)$ and whose top-right corner is on the graph of $y = 21x - x^2$.

Possibilities:

(a) $\frac{21}{2}$

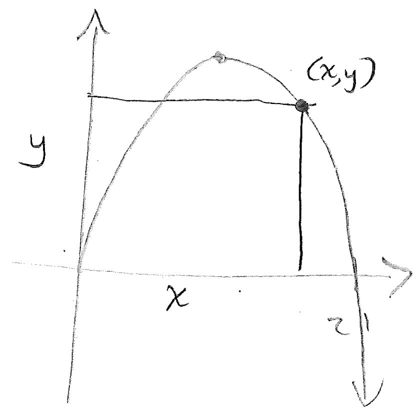
(b) $\frac{9261}{8}$

(c) 0

(d) 420

(e) 1372

width of rectangle is x, height is y



$A = xy = x(21x - x^2)$
$= 21x^2 - x^3$
$A' = 42x - 3x^2$
$= 3x(14 - x)$
critical points $x = 0, 14$

$A(0) = 0$
$A(14) = 21(14)^2 - (14)^3 = 1372 \leftarrow$ max
$x = 21$ is also an endpoint, but $A(21) = 0$.

12. An open box is to be made out of a 6-inch by 18-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. If we find the dimensions of the resulting box that has the largest volume, what is its height?

Possibilities:

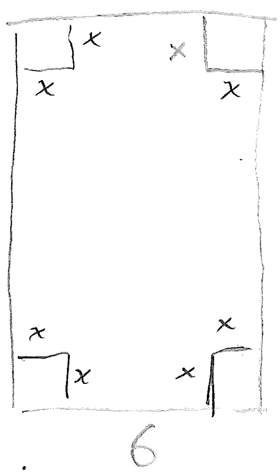
(a) 1.15 inches

(b) 1.25 inches

(c) 1.35 inches

(d) 1.45 inches

(e) 1.55 inches

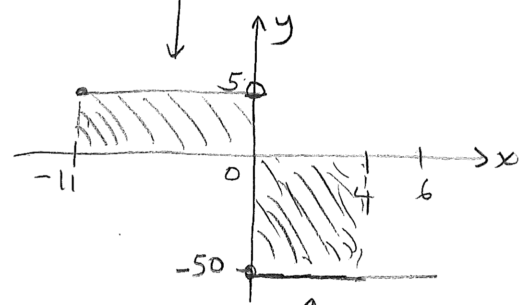


length width height
$V = (6-2x)\cdot(18-2x)\cdot x$
$= 4x^3 - 48x^2 + 108x$
$V' = 12x^2 - 96x + 108$
$= 12(x^2 - 8x + 9)$
$= 12(x - 4 - \sqrt{7})(x - 4 + \sqrt{7})$
x can be at most 3, so the only critical value is $4 - \sqrt{7} \approx 1.35$

$V'(1) = 12 - 96 + 108 = 24 > 0$
$V'(2) = 12(4) - 96(2) + 108 = -28 < 0$
So V has a max at $x = 4 - \sqrt{7}$

height $= x$, so the height of the largest box is $4 - \sqrt{7} \approx 1.35$ in area is 55 above x-axis

13. Given the function $f(x) = \begin{cases} 0 & \text{if } x < -11 \\ 5 & \text{if } -11 \le x < 0 \\ -50 & \text{if } 0 \le x < 6 \\ 0 & \text{if } x \ge 6 \end{cases}$



area is 200, below the x-axis

evaluate the definite integral

$$\int_{-11}^{4} f(x)\, dx$$

Possibilities:

(a) -145

(b) 55

(c) 255

(d) -245

(e) -255

$= \int_{-11}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx$

$= \int_{-11}^{0} 5\,dx + \int_{0}^{4} -50\,dx$

$= 5(11) + -50(4)$

$= 55 - 200 = -145$

14. The graph of $y = f(x)$ shown below includes a semicircle and a straight line. Evaluate the definite integral $\int_{-4}^{4} f(x)\,dx$.

Possibilities:

(a) $-2\pi + 6$

(b) $-4\pi + 8$

(c) $-2\pi - 8$

(d) $2\pi + 8$

(e) $-2\pi + 8$

$\int_{-4}^{0} f(x)\,dx = -\frac{1}{2}$(area of circle of radius 2)

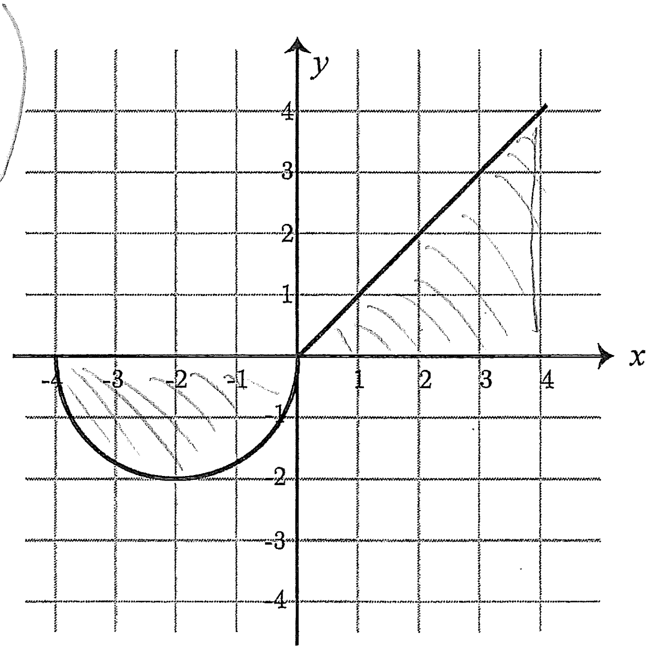
$= -\frac{1}{2}\pi 2^2 = -2\pi$

$\int_{0}^{4} f(x)\,dx$

$=$ area of triangle of base 4 and height 4

$= \frac{1}{2} 4^2 = 8$

Total area $= -2\pi + 8$



15. Suppose that $\int_2^7 f(x)\,dx = 24$, $\int_{28}^{35} f(x)\,dx = 48$, and $\int_2^{35} f(x)\,dx = 11$. Find the value of $\int_7^{28} f(x)\,dx$.

Possibilities:

(a) 83

(b) 13

(c) -83

(d) -61

(e) -93

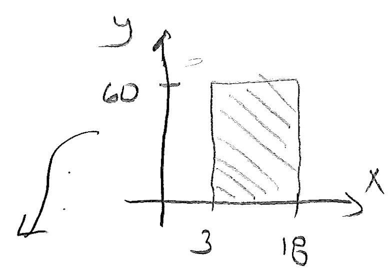
$$\underbrace{\int_2^7 f(x)\,dx}_{\text{given}} + \underbrace{\int_7^{28} f(x)\,dx}_{\text{want}} + \underbrace{\int_{28}^{35} f(x)\,dx}_{\text{given}} = \underbrace{\int_2^{35} f(x)\,dx}_{\text{given}}$$

$$24 + \int_7^{28} f(x)\,dx + 48 = 11$$

$$\int_7^{28} f(x)\,dx = 11 - 24 - 48$$

$$= -61$$

16. Suppose that $\int_3^{18} f(x)\,dx = 8$. Find the value of $\int_3^{18} (3f(x) + 60)\,dx$.



Possibilities:

(a) 84

(b) 1104

(c) 924

(d) 204

(e) 39

$$\int_3^{18} (3f(x) + 60)\,dx$$

$$= 3\int_3^{18} f(x)\,dx + \int_3^{18} 60\,dx$$

$$= 3(8) + 60(18-3)$$

$$= 24 + 60(15) = 924$$

17. The graph of $y = f(x)$ shown below consists of straight lines. Find the **average value** of $f(x)$ on the interval $[-3, 2]$.

Possibilities:

(a) 2

(b) $\frac{13}{10}$

(c) 3

(d) $\frac{13}{5}$

(e) $\frac{2}{5}$

average value of $f(x)$ on $[-3,2)$

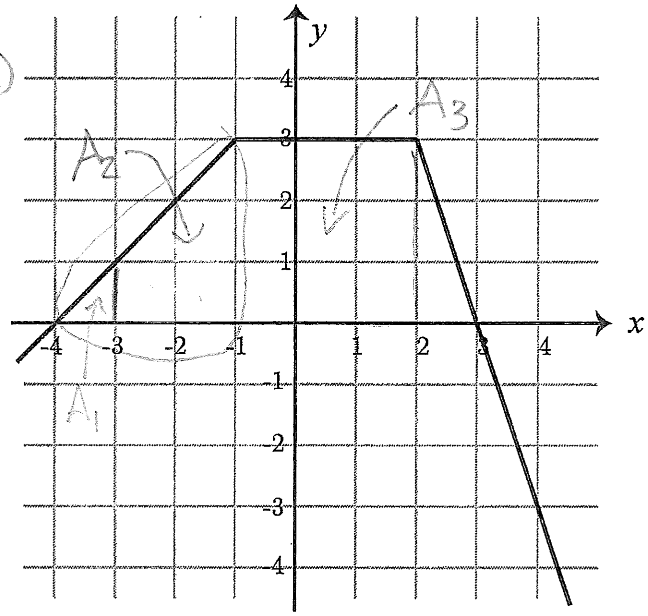
$= \frac{1}{2--3}\int_{-3}^{2} f(x)\,dx$

$= \frac{1}{5}\int_{-3}^{2} f(x)\,dx$

$= \frac{1}{5}(A_2 - A_1 + A_3)$

$= \frac{1}{5}\left[\frac{1}{2}3\cdot3 - \frac{1}{2}1\cdot1 + 3\cdot3\right]$

$= \frac{1}{5}\left(\frac{9}{2} - \frac{1}{2} + 9\right) = \frac{1}{5}(4+9) = \frac{13}{5}$



18. Estimate the area under the graph of $y = x^2 + 7$ for x between 0 and 6, by using a partition that consists of 3 equal subintervals of $[0, 6]$ and use the left endpoint of each subinterval as a sample point.

Possibilities:

(a) 168

(b) 82

(c) 114

(d) 154

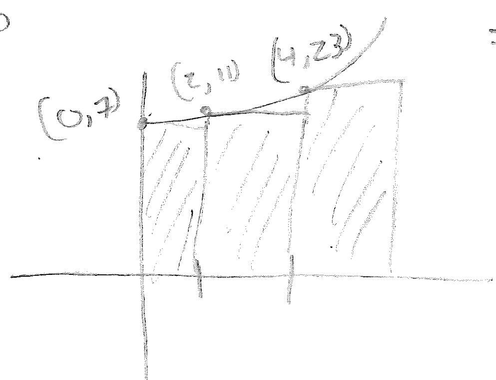
(e) 77

$\int_0^6 x^2+7\,dx \approx f(0)\cdot2 + f(2)\cdot2 + f(4)\cdot2$

$= 7\cdot2 + 11\cdot2 + 23\cdot2$

$= 14 + 22 + 46$

$= 82$



19. Suppose you estimate the area under the graph of $f(x) = \dfrac{1}{x}$ from $x = 6$ to $x = 24$ by adding the areas of the rectangles as follows: partition the interval into 6 equal subintervals and use the right endpoint of each interval to determine the height of the rectangle. What is the area of the 3rd rectangle?

Possibilities:

(a) $\frac{341}{280}$

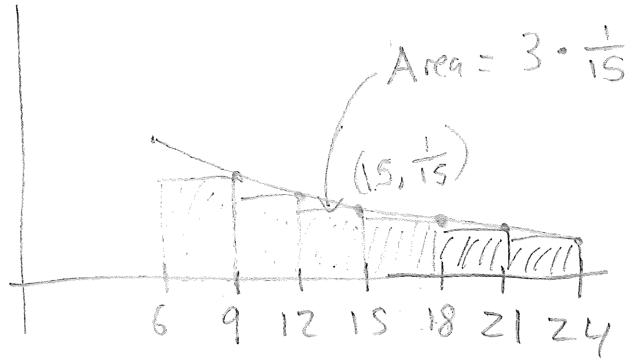
(b) $\frac{1}{15}$

(c) $\frac{1}{5}$

(d) $-2\ln(2) + \ln(5)$

(e) $\frac{1}{4}$

Area $= 3 \cdot \frac{1}{15} = \frac{1}{5}$



20. The rate (in liters per minute) at which water drains from a tank is recorded at half-minute intervals. Use the average of the left- and right-endpoint approximations to estimate the total amount of water drained during the first 2 minutes.

t min	0	.5	1	1.5	2
l/min	4	8	16	22	23

(Alternate method: use trapezoids.)

Use all five measurements in your estimate.

Possibilities:

(a) 29.75 liters

(b) 11.50 liters

(c) 25.00 liters

(d) 8.00 liters

(e) 36.50 liters

Left endpoint estimate:

$\frac{1}{2}4 + \frac{1}{2}8 + \frac{1}{2}16 + \frac{1}{2}22$

$= 2 + 4 + 8 + 11 = 25\,\ell$

Right endpoint estimate:

$\frac{1}{2}8 + \frac{1}{2}16 + \frac{1}{2}22 + \frac{1}{2}23$

$= 4 + 8 + 11 + 11.5 = 34.5\,\ell$

average $= \dfrac{25 + 34.5}{2} = \dfrac{59.5}{2} = 29.75\,\ell$

Some Formulas

1. **Areas:**
   (a) Triangle $A = \dfrac{bh}{2}$
   (b) Circle $A = \pi r^2$
   (c) Rectangle $A = lw$
   (d) Trapezoid $A = \dfrac{h_1 + h_2}{2} b$

2. **Volumes:**
   (a) Rectangular Solid $V = lwh$
   (b) Sphere $V = \frac{4}{3}\pi r^3$
   (c) Cylinder $V = \pi r^2 h$
   (d) Cone $V = \frac{1}{3}\pi r^2 h$