Solutions to the exponential growth model and the logistic growth model

We discussed many times in class the exponential growth model and the logistic growth model. They are described by the following differential equations:

<u>exp. growth model</u>: $\frac{dN}{dt} = rN \qquad N(0) = N_0$

<u>logistic growth model</u>: $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \qquad N(0) = N_0$

We have used many times that the solutions are

$$N(t) = N_0 e^{rt} \qquad\qquad N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right)e^{-rt}}$$

respectively.

Let's see how to get those solutions by solving the differential equations. These types of differential equations are called "separable".

(1) Rewrite $\frac{dN}{dt} = rN$ as

$$\frac{1}{N}\, dN = r\,dt$$

Let's take the indefinite integral of both sides:

$$\int \frac{1}{N}\, dN = \int r\,dt$$

which leads to

$$\ln(N) + c_1 = rt + c_2 \qquad \text{or}$$

$$\ln(N) = rt + c \qquad \left(\begin{array}{l} c = c_2 - c_1 \\ \text{is a constant} \end{array}\right)$$

Take the exponential of both sides:

$$e^{\ln(N)} = e^{rt+c} \qquad \text{or}$$

$$N(t) = e^{c} e^{rt} \qquad (\text{properties of exp})$$

$$= a\, e^{rt} \qquad (a = e^{c} \text{ is a constant})$$

The initial condition: $N(0) = N_0$ says that

$$N_0 = N(0) = a \cdot \underbrace{e^{r \cdot 0}}_{=1} = a$$

In conclusion: $\boxed{N(t) = N_0 e^{rt}}$

(2) The solution to the logistic differential equation is a little bit more complicated.

Rewrite: $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$ as

$$\frac{1}{N\left(1-\frac{N}{K}\right)}\, dN = r\,dt \qquad \text{or}$$

$$\frac{K}{N(K-N)}\, dN = r\,dt$$

Observe that $\frac{K}{N(K-N)} = \frac{1}{N} + \frac{1}{K-N}$

Thus we obtain the following expression after taking the indefinite integral on both sides:

$$\int \left(\frac{1}{N} + \frac{1}{K-N}\right) dN = \int r\,dt$$

we can also rewrite it as

$$\int \frac{1}{N}\,dN - \int \frac{1}{N-K}\,dN = \int r\,dt$$

because of the properties of indefinite integrals and the fact that

$$\frac{1}{K-N} = \frac{1}{-(N-K)} = -\frac{1}{N-K}$$

We obtain :

$$\ln(N) - \ln(N-K) = rt + c$$

a constant

or :

$$\ln\left(\frac{N}{N-K}\right) = rt + c$$

take the exponential of both sides:

$$e^{\ln\left(\frac{N}{N-K}\right)} = e^{rt+c} = e^{c}e^{rt}$$

Thus :

$$\frac{N(t)}{N(t)-K} = a\,e^{rt} \qquad \left(\begin{matrix}a \text{ some}\\ \text{constant}\end{matrix}\right)$$

We can determine "a" using the initial condition $N(0) = N_0$

We obtain

$$\frac{N(0)}{N(0)-K} = \underline{\underline{\frac{N_0}{N_0-K}}} = a \underbrace{e^{r \cdot 0}}_{1} = \underline{\underline{a}}$$

Thus our solution is given by

$$\frac{N(t)}{N(t)-K} = \frac{N_0}{N_0-k} e^{rt}$$

We want to get $N(t)$ explicitly. So let's take the reciprocal of both sides:

$$\frac{N(t)-K}{N(t)} = \frac{N_0-K}{N_0 e^{rt}} \quad \text{or}$$

$$\frac{N(t)-K}{N(t)} = \left(1 - \frac{K}{N_0}\right) e^{-rt}$$

which gives:

$$N(t)-K = N(t)\left(1 - \frac{K}{N_0}\right) e^{-rt}$$

or

$$N(t) - N(t)\left(1 - \frac{K}{N_0}\right) e^{-rt} = K$$

or $$N(t)\left[1 - \left(1 - \frac{K}{N_0}\right) e^{-rt}\right] = K$$

or

$$N(t)\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right]=K$$

or

$$N(t)=\frac{K}{1+\left(\frac{K}{N_0}-1\right)e^{-rt}}$$

as we expected. ∎