MA137 – Calculus 1 with Life Science Applications
Preliminaries and Elementary Functions
(Sections 1.1 & 1.2)

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Basic Functions

We introduce the basic functions that we will consider throughout the remainder of the semester.

- **polynomial functions**
  A polynomial function is a function of the form
  $$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$
  where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are (real) constants with $a_n \neq 0$. The coefficient a_n is called the leading coefficient, and n is called the degree of the polynomial function. The largest possible domain of f is $\mathbb{R}$.

  Examples Suppose a, b, c, and m are constants.

  - Constant functions: $f(x) = c$ (graph is a horizontal line);
  - Linear functions: $f(x) = mx + b$ (graph is a straight line);
  - Quadratic functions: $f(x) = ax^2 + bx + c$ (graph is a parabola).

- **rational functions**

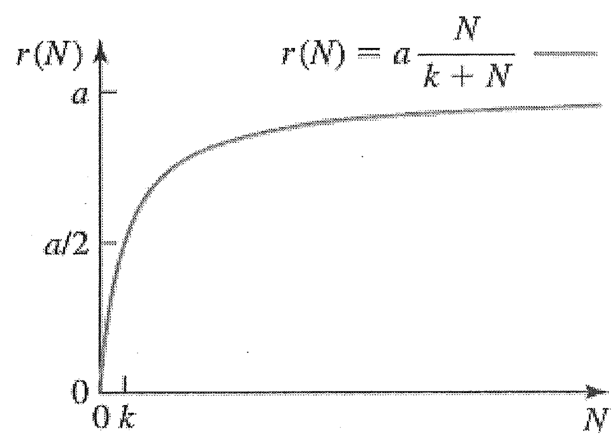
  A rational function is the quotient of two polynomial functions $p(x)$ and $q(x)$: $f(x) = \dfrac{p(x)}{q(x)}$ for $q(x) \neq 0$.

Example The **Monod growth function** is frequently used to describe the per capita growth rate of organisms when the rate depends on the concentration of some nutrient and becomes saturated for large enough nutrient concentrations.

If we denote the concentration of the nutrient by N, then the per capita growth rate $r(N)$ is given by

$$r(N) = \frac{aN}{k+N}, \qquad N \geq 0$$

where a and k are positive constants.

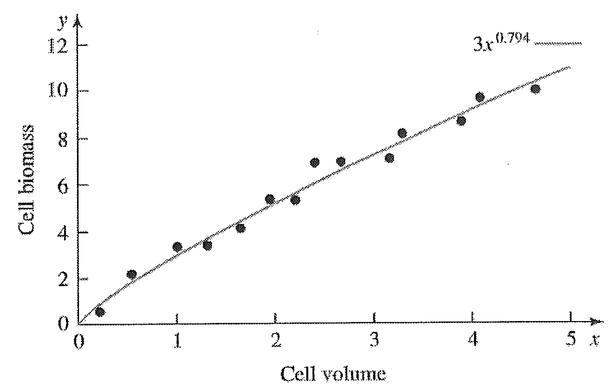


- **power functions**
  A power function is of the form $f(x) = x^r$ where r is a real number.

  Example Power functions are frequently found in "scaling relations" between biological variables (e.g., organ sizes).
  Finding such relationships is the objective of **allometry**. For example, in a study of 45 species of unicellular algae, a relationship between cell volume and cell biomass was sought. It was found [see, Niklas (1994)] that

$$\text{cell biomass} \propto (\text{cell volume})^{0.794}$$

  Most scaling relations are to be interpreted in a statistical sense; they are obtained by fitting a curve to data points. The data points are typically scattered about the fitted curve given by the scaling relation.



- **exponential and logarithmic functions**
- **trigonometric functions**

Example 1 (Problem #52, Section 1.1, p. 14):

The Celsius scale is devised so that $0°C$ is the freezing point of water (at 1 atmosphere of pressure) and $100°C$ is the boiling point of water (at 1 atmosphere of pressure).
If you are more familiar with the Fahrenheit scale, then you know that water freezes at $32°F$ and boils at $212°F$.

(a) Find a linear equation/function that relates temperature measured in degrees Celsius and temperature measured in degrees Fahrenheit.

(b) The normal body temperature in humans ranges from $97.6°F$ to $99.6°F$. Convert this temperature range into degrees Celsius.

$F = aC + b$ a linear relation

* when $C = 0$ then $F = 32$ so that

$32 = a \cdot 0 + b \implies \boxed{b = 32}$

* when $C = 100$ then $F = 212$ so that

$212 = a \cdot 100 + 32 \implies 100a = 212 - 32$

$\implies a = \frac{180}{100} = 9/5 \quad \therefore \quad \boxed{F = \frac{9}{5} C + 32}$

(alternatively, we can solve for C

$F = 9/5\, C + 32 \iff 5F = 9C + 160$

$\iff \underline{C} = \underline{\frac{5}{9} F - \frac{160}{9} \cong 5/9\, F - 17.7}$)

(b)

$$\boxed{97.6 \le F \le 99.6}$$

is the range of normal body temperature in humans

Substitute: $97.6 \le \frac{9}{5} C + 32 \le 99.6$

and write it in terms of C alone

$$\iff 97.6 - 32 \le 9/5\, C \le 99.6 - 32$$

$$\iff 65.6 \le 9/5\, C \le 67.6$$

$$\iff 5/9 \cdot 65.6 \le C \le \frac{5}{9}\, 67.6$$

$$\iff \boxed{36.44 \le C \le 37.55}$$

Example 2:

When a certain drug is taken orally, the concentration of the drug in the patient's bloodstream after t minutes is given by

$$C(t) = 0.06t - 0.0002t^2,$$

where $0 \leq t \leq 240$ and the concentration is measured in mg/L.
When is the maximum serum concentration reached?
What is that maximum concentration?

Consider $C(t) = 0.06t - 0.0002t^2$

and rewrite it as

$$C(t) = -0.0002t^2 + 0.06t$$

We want to complete the squares :

$$C(t) = -0.0002\left[t^2 - \frac{0.06}{0.0002}t\right] = -0.0002\left[t^2 - 300t\right]$$

$$= -0.0002\left[t^2 - 300t + \left(\frac{300}{2}\right)^2\right] \underline{\underline{+ 4.5}}$$

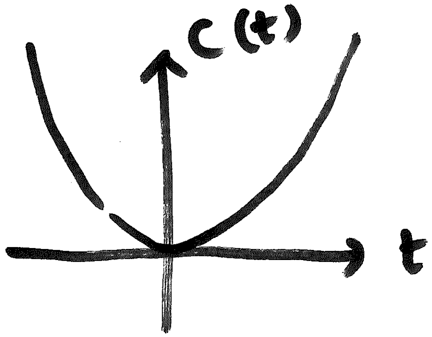
notice that $4.5 = 0.0002\left(\frac{300}{2}\right)^2$

$$\therefore \boxed{C(t) = -0.0002(t-150)^2 + 4.5}$$

$\therefore$ maximum serum concentration $\underline{\text{at } t = 150}$; $\therefore$ maximum concentration $\underline{4.5}$

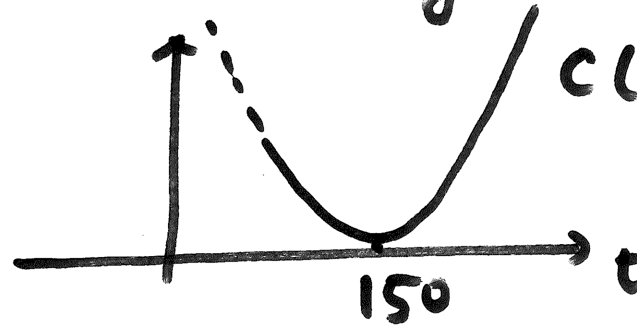
The graph of $C(t) = -0.0002\,(t-150)^2 + 4.5$

is obtained as follows via elementary transformat.

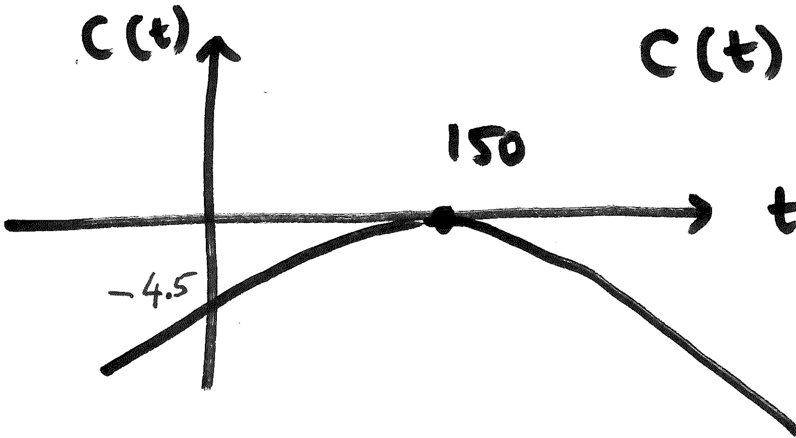


$C(t) = t^2$

$\rightsquigarrow$



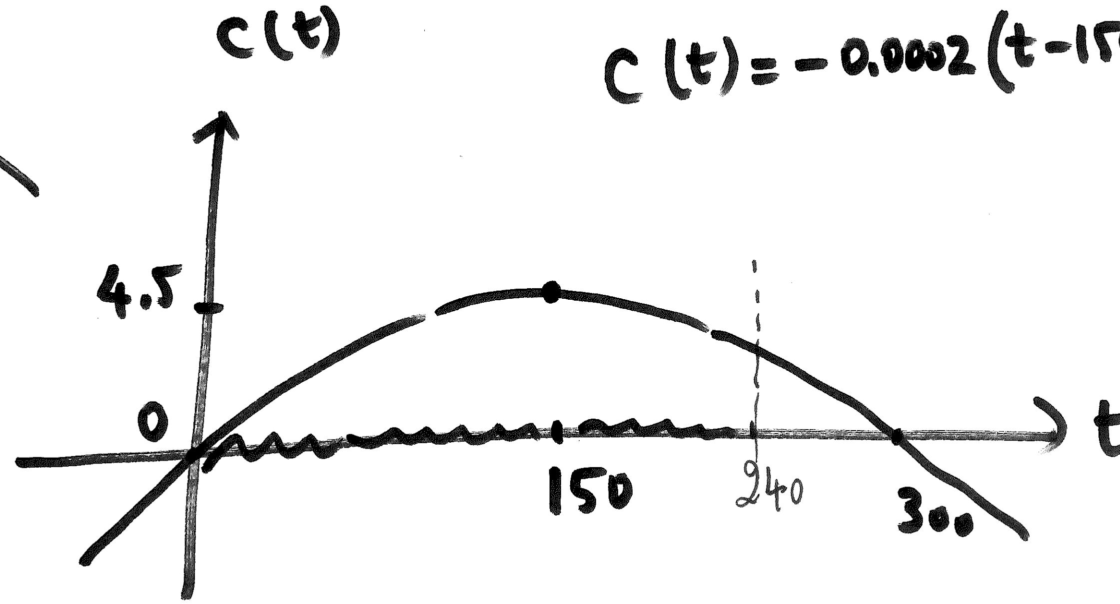
$C(t) = (t-150)^2$



$C(t) = -0.0002\,(t-150)^2$

$C(t) = -0.0002\,(t-150)^2 + 4.5$

Finally



Example 3: (Michaelis-Menten enzymatic reaction)

According to the Michaelis-Menten equation (1913) when a chemical reaction involving a substrate S is catalyzed by an enzime, the rate of reaction $V = V([\mathrm{S}])$ is given by the expression

$$V = \frac{V_{\max}[\mathrm{S}]}{K_m + [\mathrm{S}]},$$

where [S] denotes substrate concentration (for examples in moles per liter), and $V_{\max}$ and K_m are constants.

$V_{\max}$ is the maximal velocity of the reaction and K_m is the Michaelis constant.

K_m is the substrate concentration at which the reaction achieves half of the maximum velocity.

Graph V assuming that $V_{\max} = 3$ and $K_m = 2$. That is,

$$V = \frac{3[\mathrm{S}]}{2 + [\mathrm{S}]}.$$

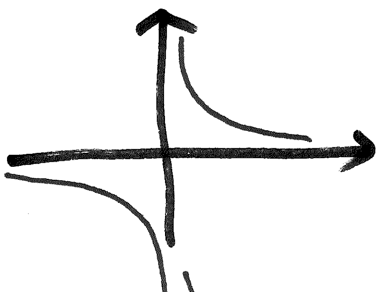
$$V = \frac{V_{max}[S]}{K_m + [S]} \rightsquigarrow V = \frac{3[S]}{2+[S]}$$ or if you prefer to change variables $$y = \frac{3x}{2+x}$$

Rewrite as: $$\boxed{y} = \frac{3x}{2+x} = \frac{3x+6-6}{x+2} = \frac{3(x+2)-6}{x+2}$$

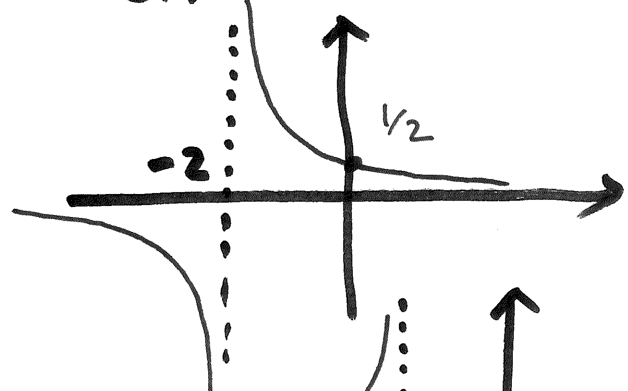
$$= \frac{3(x+2)}{x+2} - \frac{6}{x+2} = \boxed{3 - \frac{6}{x+2}}$$

Now use the elementary transformations:

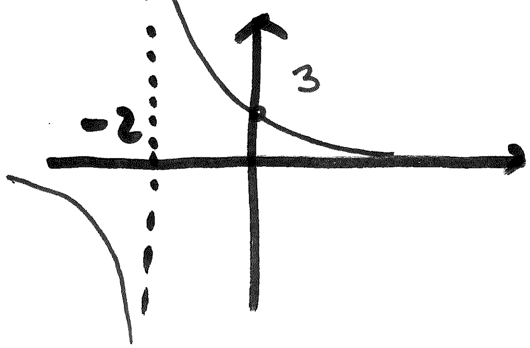
$y = \frac{1}{x}$



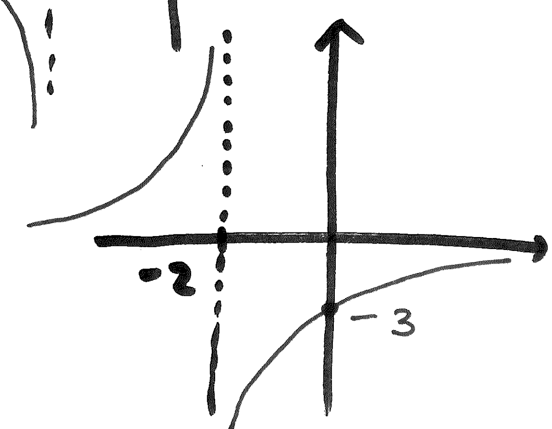
$\rightsquigarrow$ $y = \frac{1}{x+2}$



$y = \frac{6}{x+2}$

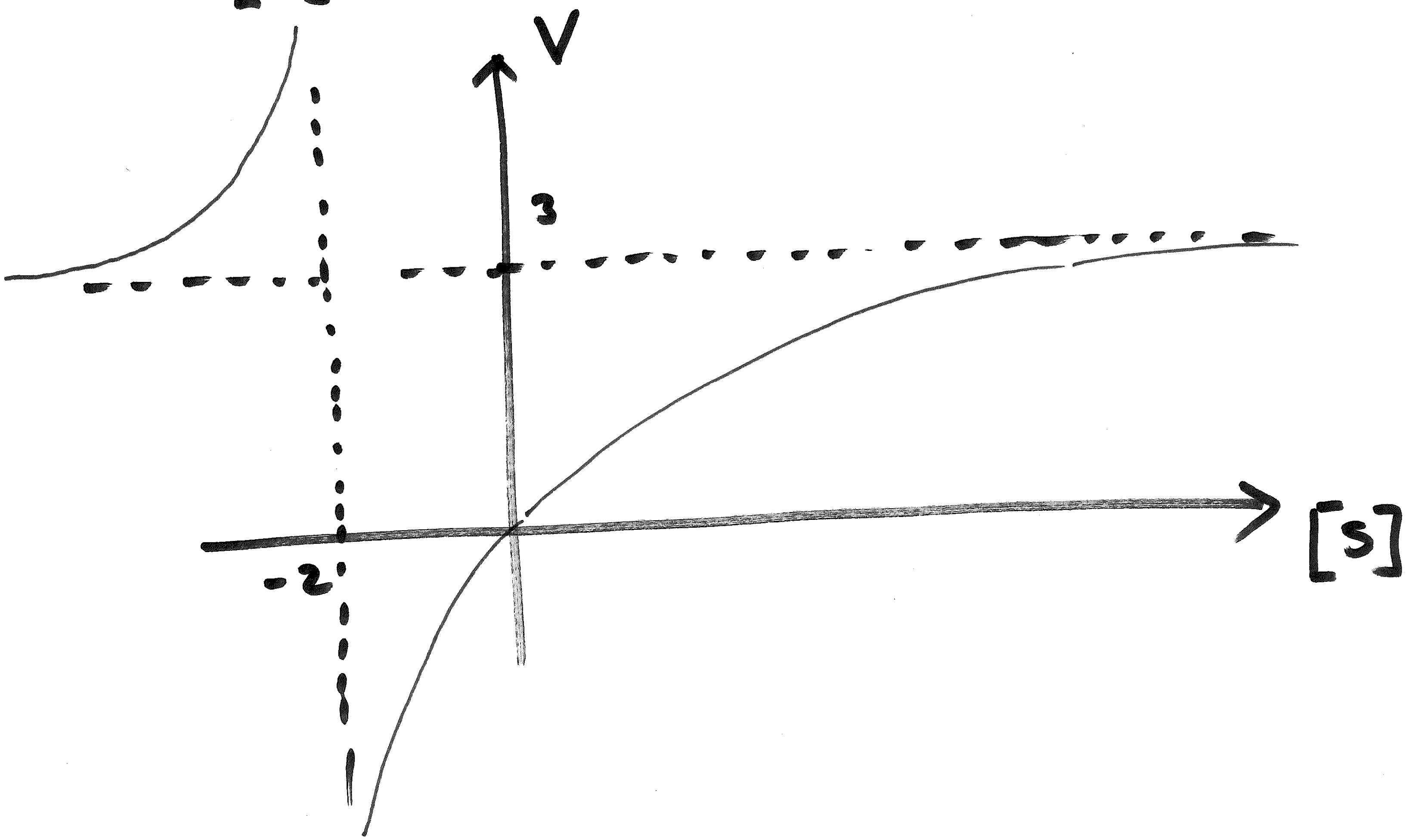


$y = -\frac{6}{x+2}$



Finally and changing back the variables

$$V = \frac{3[S]}{2+[S]} = 3 - \frac{6}{[S]+2}$$



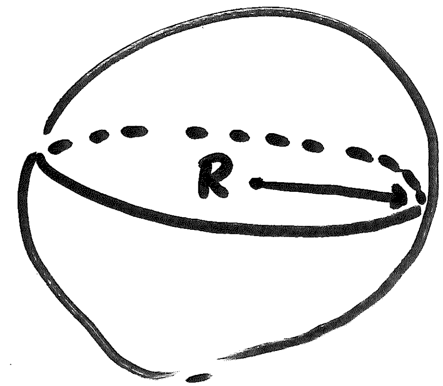
Example 4:

Find the scaling relation between the surface area S and the volume V of a sphere of radius R.

[More precisely, show that $S = (36\pi)^{1/3}\, V^{2/3}$, that is, $S \propto V^{2/3}$.]

Recall that the volume of a sphere of radius R is

$$V = \frac{4}{3}\pi R^3$$



The surface area of a sphere of radius R is: $S = 4\pi R^2$

We want to write S as a function of V.

<u>FROM</u>: $V = \frac{4}{3}\pi R^3 \longrightarrow \frac{3}{4\pi} V = R^3$

$$\longrightarrow R = \sqrt[3]{\frac{3}{4\pi} V} = \left(\frac{3}{4\pi} V\right)^{1/3}$$

Substitute in $S = 4\pi R^2$ to get

$$S = 4\pi\left[\left(\frac{3}{4\pi} V\right)^{1/3}\right]^2$$

$$\therefore \quad S = 4\pi \left(\frac{3}{4\pi}\right)^{2/3} \cdot V^{2/3}$$

$$= \left[(4\pi)^3 \left(\frac{3}{4\pi}\right)^2\right]^{1/3} \cdot V^{2/3}$$

$$= \left(64\pi^3 \cdot \frac{9}{16\pi^2}\right)^{1/3} \cdot V^{2/3}$$

$$= (36\pi)^{1/3} \cdot V^{2/3}$$

<u>i.e.</u> $\boxed{S \propto V^{2/3}}$