MA137 – Calculus 1 with Life Science Applications
Operations on Functions
Inverse of a Function and its Graph
(Sections 1.2 & 1.3)

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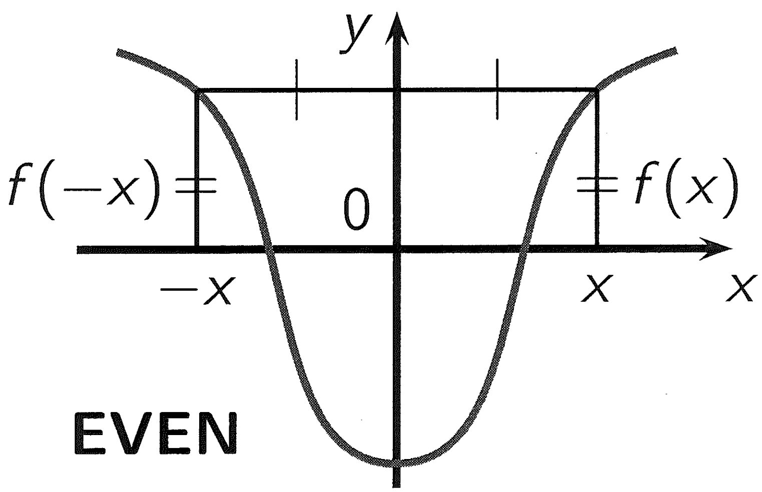
August 28, 2017

Even and Odd Functions

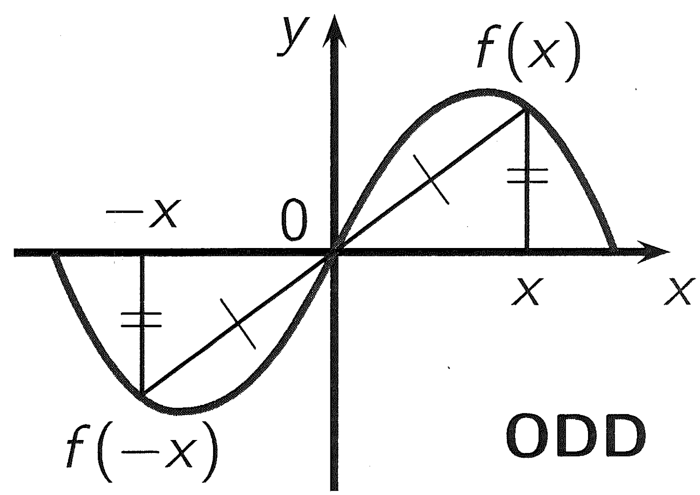
Let f be a function.

f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.



Graph symmetric wrt y-axis.



Graph symmetric wrt $(0, 0)$.

Example:

$y = \cos x$ is an **even** function; $y = \sin x$ is an **odd** function.

Example 1:

Determine whether the following functions are even or odd:

$$f(x) = x^3 + 2x^5$$

$$g(x) = x^2 - 3x^4$$

(a) $f(x) = x^3 + 2x^5$

$f(-x) = (-x)^3 + 2(-x)^5 = -x^3 - 2x^5 = -[x^3 + 2x^5]$

$= -f(x)$

thus f is an odd function

(b) $g(x) = x^2 - 3x^4$

$g(-x) = (-x)^2 - 3(-x)^4 = x^2 - 3x^4 = g(x)$

thus g is an even function

NOTE :

In (a) the polynomial only has odd exponents

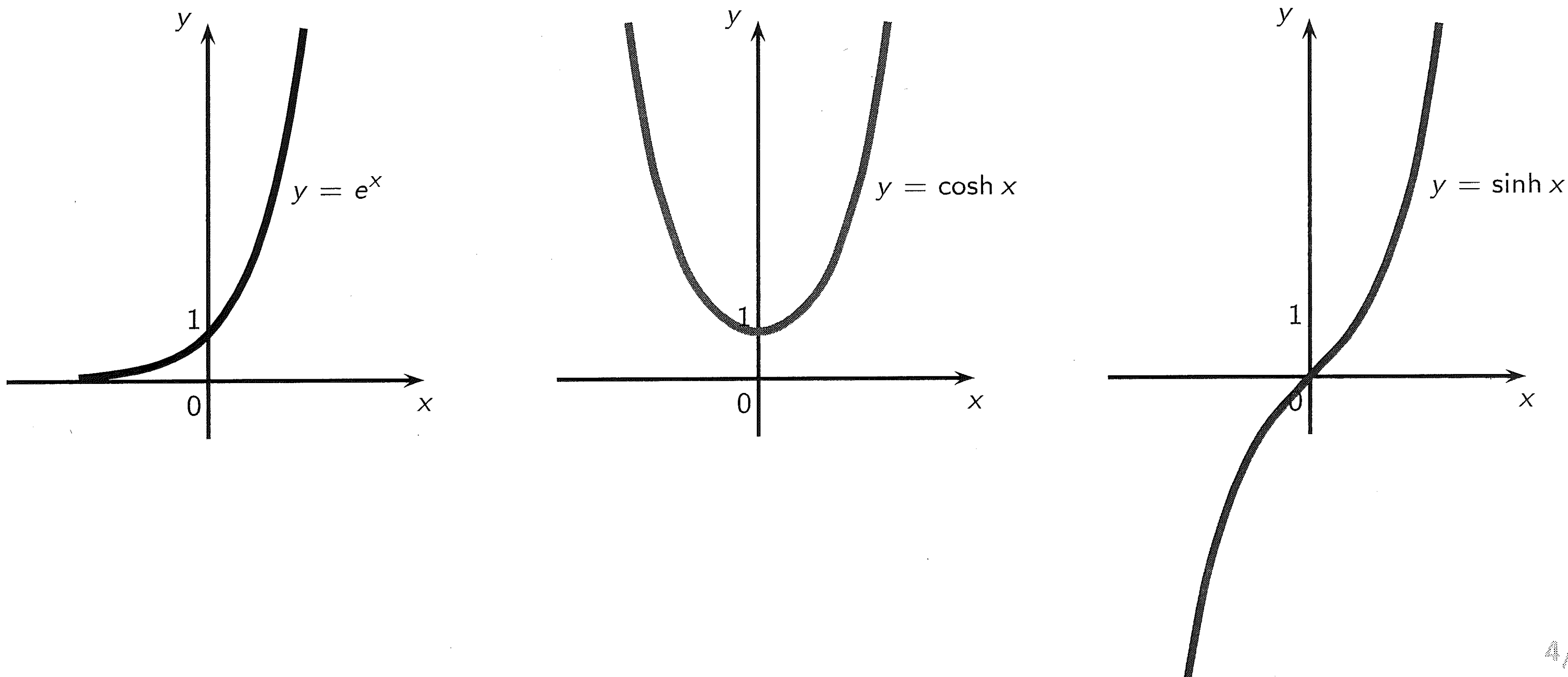
In (b) the polynomial only has even exponents

Curious/Amazing Fact!

Any function can be uniquely written as an even plus an odd function.

Example:

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\cosh x} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\sinh x}$$



Example 2: (Online Homework HW02, #11)

(Since we talked about trigonometric functions...)

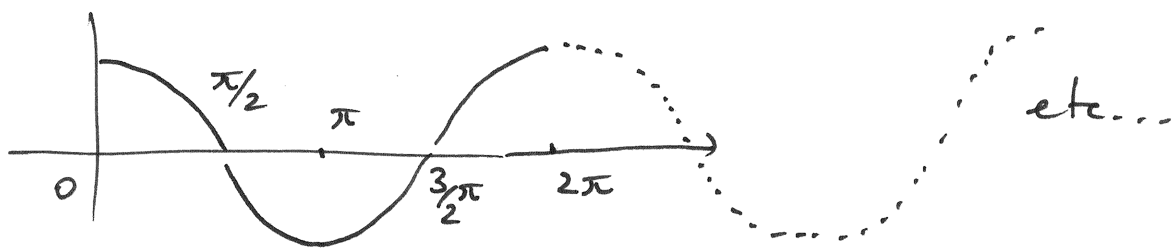
The lungs do not completely empty or completely fill in normal breathing. The volume of the lungs normally varies between 2140 ml and 2700 ml with a breathing rate of 22 breaths/min. This exchange of air is called the *tidal volume*.

One approximation for the volume of air in the lungs uses the cosine function written in the following manner:

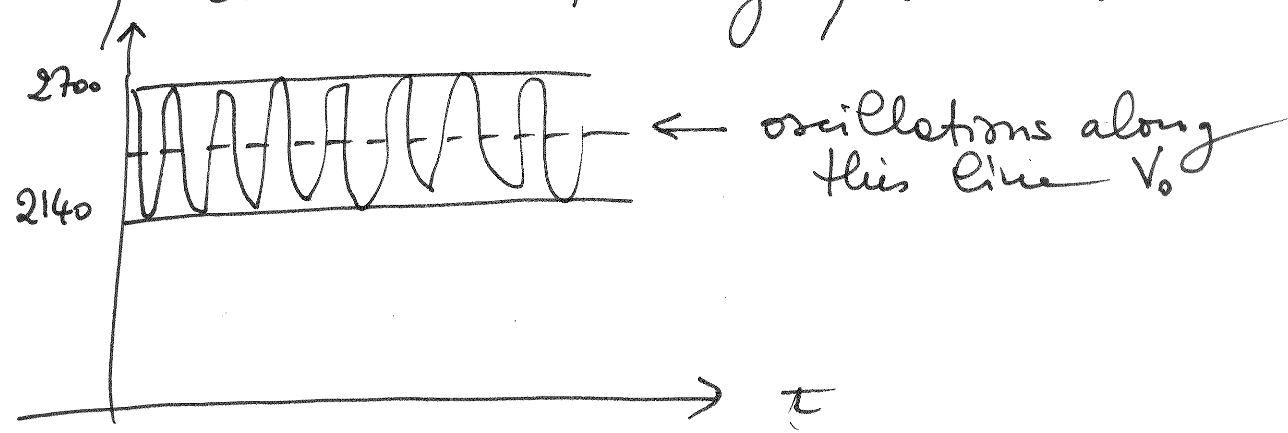
$$V(t) = A + B\cos(\omega t),$$

where A, B, and ω are constants and t is in minutes. Use the data above to create a model, finding the constants $A =$ ______, $B =$ ______, and $\omega =$ ______, that simulates the normal breathing of an individual for one minute.

Fact : $y = \cos(t)$ is a period function, of period 2π, with values between 1 and -1. That is the amplitude of the oscillation is 2 and the oscillation is along the line $y = 0$.



In our case $V(t) = A + B\cos(wt)$ represents the volume of the lungs. This volume ranges between 2140 and 2700 ml. There must be 22 full cycles in a 1 minute period. The graph must look like:



Notice that the amplitute of the oscillations is $2700 - 2140 = 560$ hence B must be half of that: $\boxed{B = 280}$. Moreover the oscillations must be about the line $V_0 = 2140 + 280 = \underline{2420}$

(Observe $2420 = \frac{2140 + 2700}{2} = 2700 - 280$) $\therefore$ $\boxed{A = 2420}$

$$V(t) = 2420 + 280 \cos(\underline{\underline{\omega}} t)$$

To determine $\underline{\underline{\omega}}$, observe that since we have 22 breaths in one minute, the volume must have the same values at time: t, $t + \frac{1}{22}$, $t + \frac{2}{22}$, $t + \frac{3}{22}$, ..., $t + \frac{22}{22} = t + 1$. That is

$$\underbrace{V(t) = V\left(t + \tfrac{1}{22}\right)} = V\left(t + \tfrac{2}{22}\right) = \text{etc...}$$

in particular, during the interval t, $t + 1/22$ we have one full breath.

This means :

$$V(t) = V(t + 1/22) \iff A + B\cos(wt) \overset{\text{MUST}}{=} A + B\cos\left(w\left(t + \frac{1}{22}\right)\right)$$

$$\iff \cos(wt) = \cos\left(wt + w \cdot \frac{1}{22}\right)$$

Because $\cos(\cdot)$ is periodic of period 2π

we must have $\quad w \cdot \frac{1}{22} = 2\pi$

$$\implies \quad w = 2\pi \cdot 22 = 44\pi = 138.23$$

Thus : $\boxed{V(t) = 2420 + 280\cos(138.23\, t)}$ //

Combining functions

Let f and g be functions with domains A and B. We define new functions $f+g$, $f-g$, fg, and f/g as follows:

$(f+g)(x) = f(x) + g(x)$ Domain $A \cap B$

$(f-g)(x) = f(x) - g(x)$ Domain $A \cap B$

$(fg)(x) = f(x)g(x)$ Domain $A \cap B$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

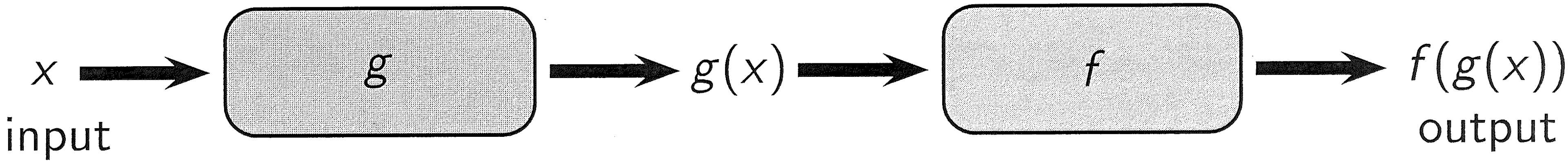
Composition of Functions

Given any two functions f and g, we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f, we can then calculate the value of $f(g(x))$.

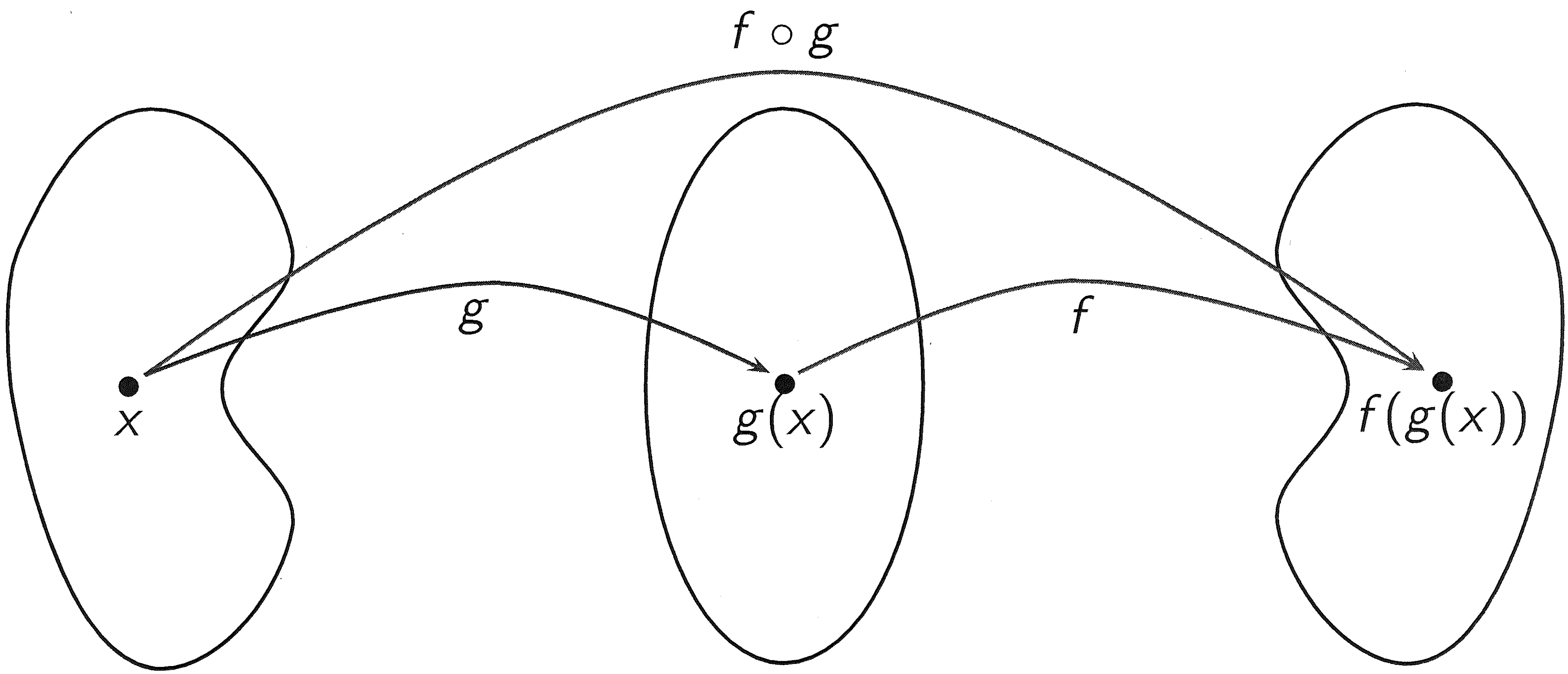
The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f. It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (read: 'f composed with g' or 'f after g')

$$\boxed{(f \circ g)(x) \stackrel{\text{def}}{=} f(g(x)).}$$

WARNING: $f \circ g \neq g \circ f$.



Machine diagram of $f \circ g$



Arrow diagram of $f \circ g$

Example 3:

Let $f(x) = \dfrac{x}{x+1}$ and $g(x) = 2x - 1$.

Find the functions $f \circ g$, $g \circ f$, and $f \circ f$ and their domains.

$$f(x) = \frac{x}{x+1} \qquad g(x) = 2x - 1$$

(*) $$(f \circ g)(x) = f(g(x)) = \frac{g(x)}{g(x)+1} = \frac{2x-1}{(2x-1)+1} = \boxed{\frac{2x-1}{2x}} \checkmark$$

$$= 1 - \frac{1}{2x}$$

(**) $$(g \circ f)(x) = g(f(x)) = 2f(x) - 1 = 2 \cdot \frac{x}{x+1} - 1$$

$$= \frac{2x}{x+1} - 1 = \frac{2x - (x+1)}{x+1} = \boxed{\frac{x-1}{x+1}} \checkmark$$

(***) $$(f \circ f)(x) = f(f(x)) = \frac{f(x)}{f(x)+1} =$$

$$= \frac{x/_{x+1}}{x/_{x+1} + 1} = \frac{\frac{x}{x+1}}{\frac{x+(x+1)}{x+1}} = \frac{x}{\cancel{x+1}} \cdot \frac{\cancel{x+1}}{2x+1}$$

$$= \frac{x}{2x+1}$$

Example 4:

Express the function $F(x) = \dfrac{x^2}{x^2 + 4}$ in the form $F(x) = f(g(x))$.

$$F(x) = \frac{x^2}{x^2+4}$$

can be thought of as the following composition

$$x \overset{g}{\longmapsto} x^2 \overset{f}{\longmapsto} \frac{x^2}{x^2+4}$$

thus: $g(x) = x^2$; $f(x) = \dfrac{x}{x+4}$

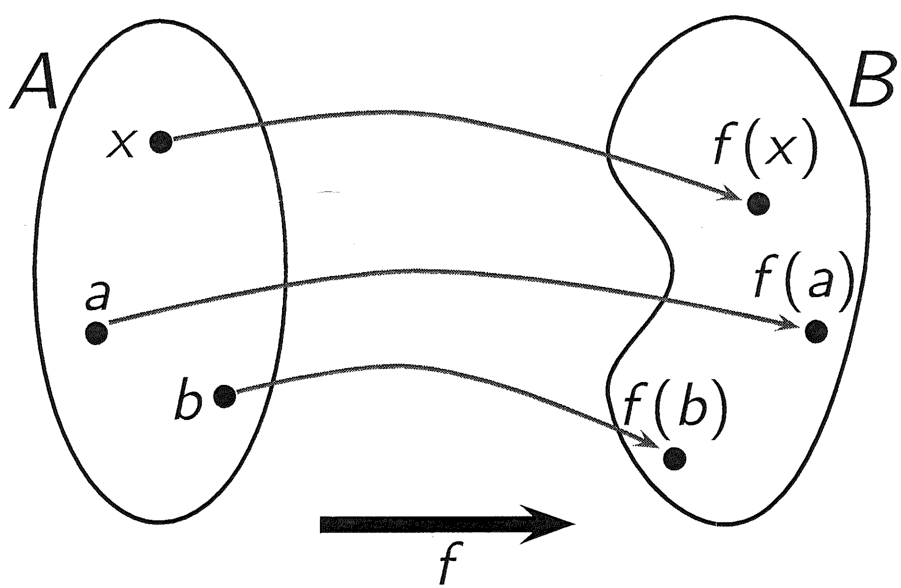
Definition of a One-One Function

A function f with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

An equivalent way of writing the above condition is:

$$\text{If } \ f(x_1) = f(x_2), \ \text{ then } \ x_1 = x_2.$$



Horizontal Line Test

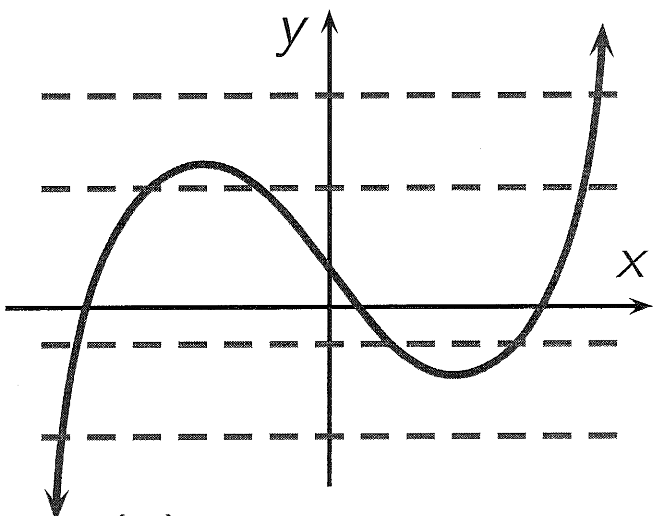
For functions that can be graphed in the coordinate plane, there is a useful criterion to determine whether a function is one-to-one or not.

Horizontal Line Test

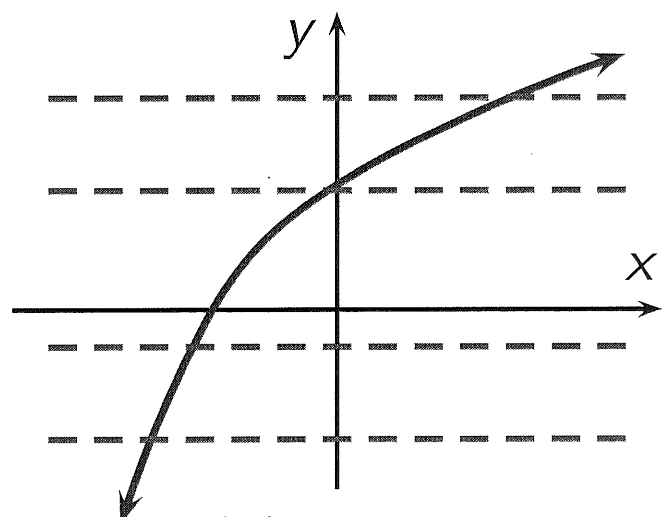
A function is one-to-one

$\Longleftrightarrow$

no horizontal line intersects its graph more than once.



$f(x)$ **is not** one-to-one



$f(x)$ **is** one-to-one

The Inverse of a Function

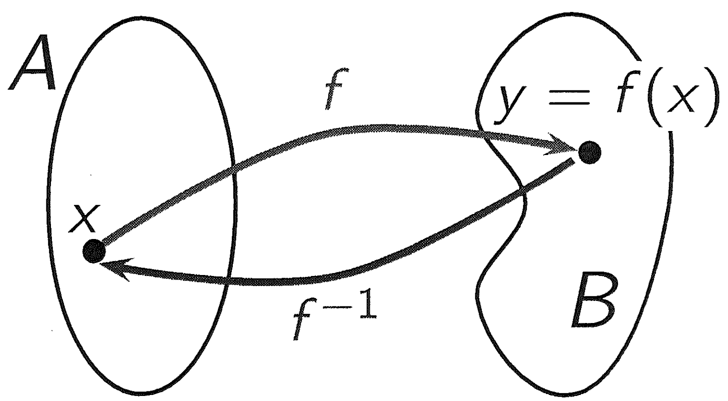
One-to-one functions are precisely those for which one can define a (unique) **inverse function** according to the following definition.

Definition of the Inverse of a Function

Let f be a one-to-one function with domain A and range B. Its **inverse function** f^{-1} has domain B and range A and is defined by

$$\boxed{f^{-1}(y) = x \quad \Longleftrightarrow \quad f(x) = y,}$$

for any $y \in B$.



If f takes x to y,
then f^{-1} takes y back to x.
I.e., f^{-1} undoes what f does.

NOTE:
f^{-1} does NOT mean $\frac{1}{f}$.

Properties of Inverse Functions

Let $f(x)$ be a one-to-one function with domain A and range B. The inverse function $f^{-1}(y)$ satisfies the following "cancellation" properties:

$$\boxed{f^{-1}(f(x)) = x \quad \text{for every} \quad x \in A}$$

$$\boxed{f(f^{-1}(y)) = y \quad \text{for every} \quad y \in B}$$

Conversely, any function $f^{-1}(y)$ satisfying the above conditions is the inverse of $f(x)$.

Remark:

Typically we write functions in terms of x.
To do this, we need to interchange x and y in $x = f^{-1}(y)$.

Example 5:

Show that the functions $f(x) = x^5$ and $g(x) = x^{1/5}$ are inverses of each other.

$f(x) = x^5 \qquad g(x) = x^{1/5}$

$(f \circ g)(x) = f(g(x)) = [g(x)]^5 = [x^{1/5}]^5 = x$

$(g \circ f)(x) = g(f(x)) = [f(x)]^{1/5} = [x^5]^{1/5} = x$

thus: $(f \circ g)(x) = x$ ✓

and

$(g \circ f)(x) = x$ ✓

How to find the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 6: (Online Homework HW02, # 12)

Find the inverse of $y = \dfrac{2-3x}{8-7x}$.

1. $y = \dfrac{2-3x}{8-7x}$

2. Solve for x in terms of y

$y(8-7x) = 2-3x \longrightarrow 8y - 7xy = 2 - 3x$

$3x - 7xy = 2 - 8y \longrightarrow x(3-7y) = 2-8y$

$\longrightarrow x = \dfrac{2-8y}{3-7y}$

3. Interchange x and y

$$\therefore \boxed{y = \dfrac{2-8x}{3-7x}}$$

Example 7: (Exam 1, Spring 15, # 4)

One of the main quantities that epidemiologists try to measure for infectious diseases is the so-called basic reproduction number, R_0. Biologically, this is the expected number of new infections that an infected individual will produce when introduced into a completely susceptible population.
We can try to modify this by introducing vaccination to control the probability of an outbreak of the disease. We want to know the fraction of the population that we have to vaccinate to achieve a target outbreak probability. If v is the vaccination fraction, then the outbreak probability as a function of v is

$$P = 1 - \frac{1}{R_0(1-v)}.$$

Find the inverse of this function to obtain v, the vaccination coverage needed, as a function of P, the given target outbreak probability.

$$P = 1 - \frac{1}{R_0(1-v)} \longrightarrow \frac{1}{R_0(1-v)} = 1 - P$$

$$\longrightarrow \frac{1}{R_0(1-P)} = 1 - v \longrightarrow \boxed{v = 1 - \frac{1}{R_0(1-P)}}$$

that is we wrote the vaccination coverage needed v in terms of the target outbreak probability P.

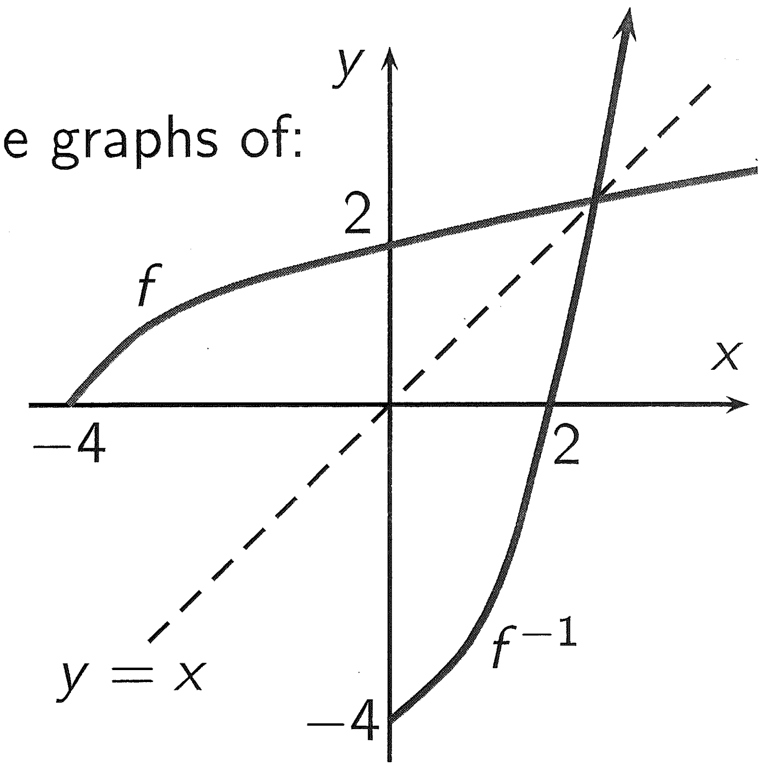
(no need to exchange $v \longleftrightarrow P$)

Graph of the Inverse Function

The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f. **The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.**

The picture on the right hand side shows the graphs of:

$$f(x) = \sqrt{x+4}$$
and
$$f^{-1}(x) = x^2 - 4, \; x \geq 0.$$



Example 8:

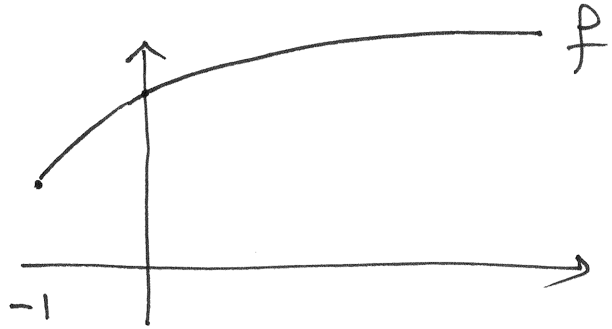
Find the inverse of the function $f(x) = 1 + \sqrt{1+x}$.

Find the domain and range of f and f^{-1}.

Graph f and f^{-1} on the same cartesian plane.

$f(x) = 1 + \sqrt{1+x}$

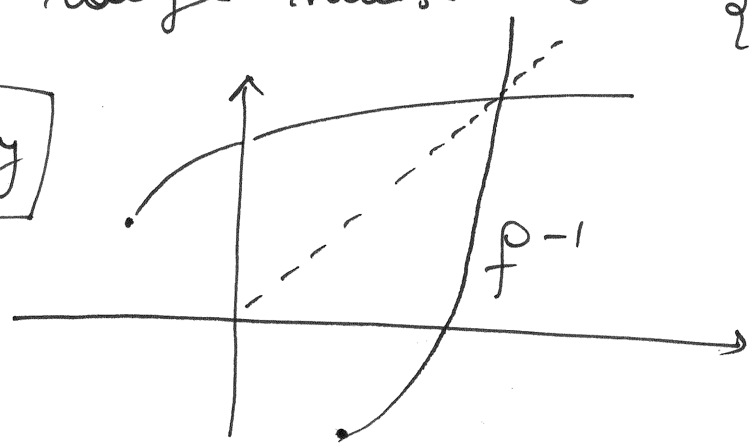
has domain $\{x \in \mathbb{R} \mid x \geq -1\}$ and range $\{y \in \mathbb{R} \mid y \geq 1\}$



the domain of the inverse f^{-1} must be $\{x \in \mathbb{R} \mid x \geq 1\}$

and the range must be $\{y \in \mathbb{R} \mid y \geq -1\}$

graphically



algebraically

$y = 1 + \sqrt{1+x}$

$\longrightarrow y - 1 = \sqrt{1+x}$

$\longrightarrow (y-1)^2 = 1 + x$

$\longrightarrow x = (y-1)^2 - 1 \longrightarrow \boxed{y = (x-1)^2 - 1}$ $\quad x \geq 1$