MA137 – Calculus 1 with Life Science Applications
Exponential and Logarithmic Functions
(Sections 1.1 and 1.2)

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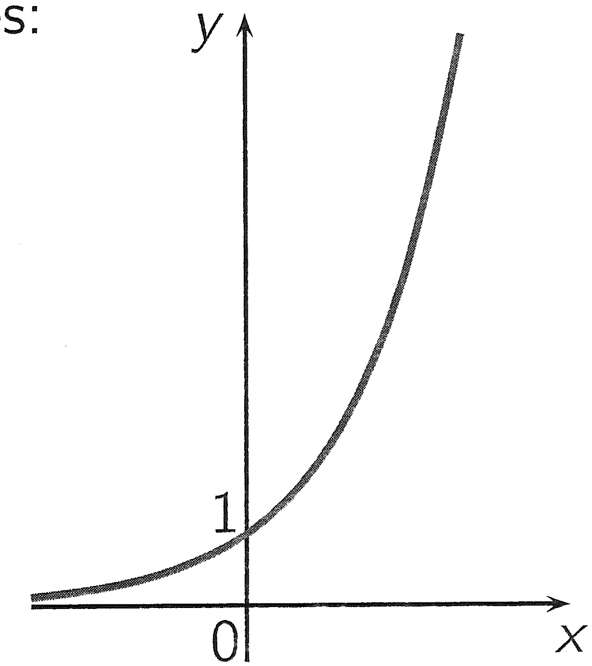
Exponential Functions

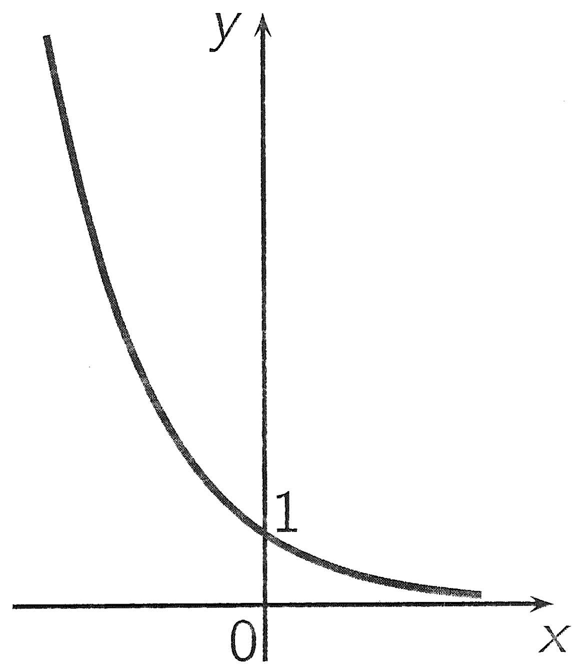
The **exponential function**

$$f(x) = a^x \qquad (a > 0,\ a \neq 1)$$

has domain $\mathbb{R}$ and range $(0, \infty)$. The graph of $f(x)$ has one of these shapes:



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$
for $0 < a < 1$

Laws of Exponents

Let a and b be real numbers so that $a, b > 0$ and $a, b \neq 1$.

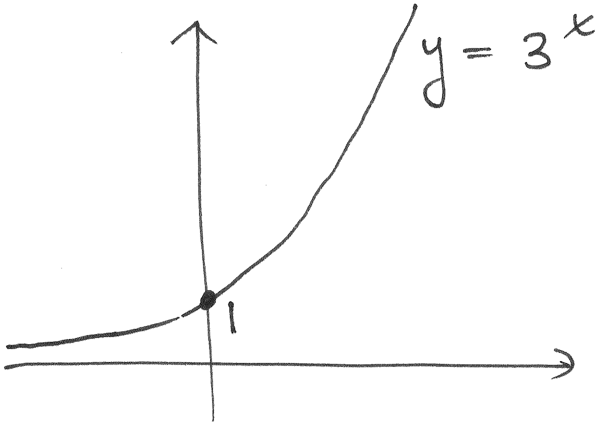
- $a^0 = 1$
- $a^u a^v = a^{u+v}$
- $\dfrac{a^u}{a^v} = a^{u-v}$ In particular, $\dfrac{1}{a^v} = \dfrac{a^0}{a^v} = a^{0-v} = a^{-v}$
- $(a^u)^v = a^{uv}$ In particular, $a^{1/n} = \sqrt[n]{a}$
- $(ab)^u = a^u b^u$ $\left(\dfrac{a}{b}\right)^u = \dfrac{a^u}{b^u}$

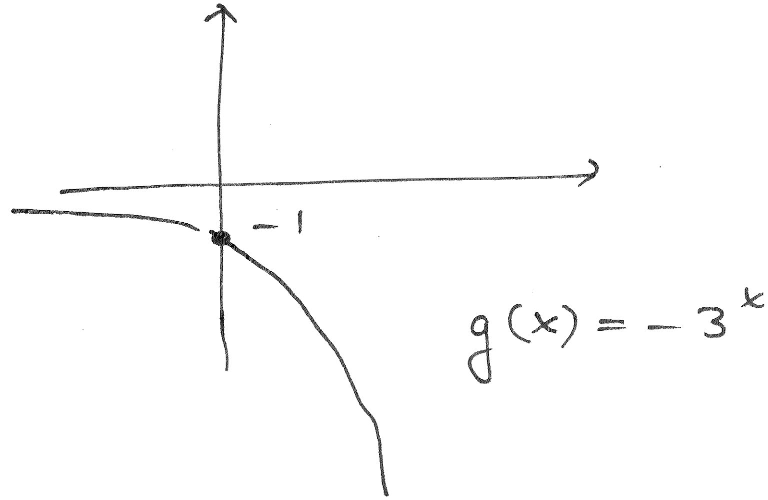
Example 1:

Use the graph of $f(x) = 3^x$ to sketch the graph of each function:

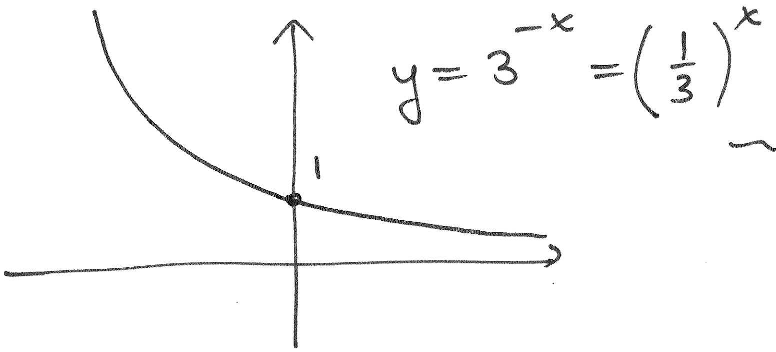
$g(x) = -3^x$

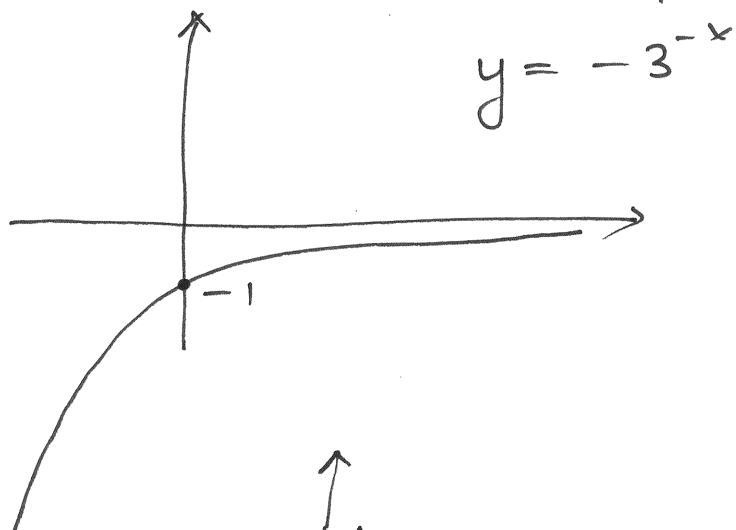
$h(x) = 1 - 3^{-x}$



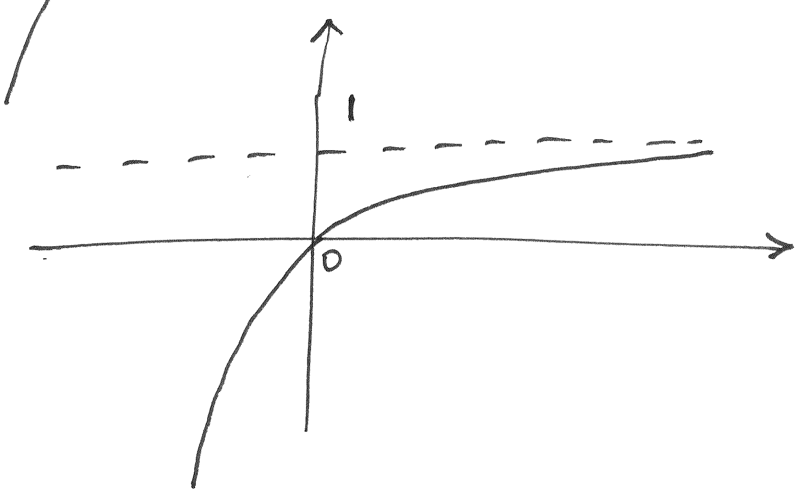


Let's construct the graph of $h(x) = 1 - 3^{-x}$ in steps:





Finally, $y = 1 - 3^{-x}$



Why are exponential functions of interest?

Suppose that we study a population of 100 individuals and suppose that it grows annually at a 3% rate. Describe the population growth at time t.

(You can also consider \$ 100 in a bank growing annually at a 3%)

$P_0 = P(0) = 100$; growth rate = 3% or $r = 0.03$

$P(1) = 100 + \underbrace{0.03 \cdot 100}_{\text{growth}} = 100 + 3 = 103 = 100\,(1.03)$

$P(2) = 103 + \underbrace{0.03 \cdot 103}_{\text{growth}} = 103\,(1 + 0.03) = 100\,(1.03)\,(1.03)$

$= 100\,(1.03)^2$

In general $\boxed{P(t) = 100\,(1.03)^t}$ OR $\boxed{P(t) = P_0\,(1+r)^t}$

The formula for compounded interest is

$$P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt}$$

(see earlier discussion with $n = 1$)

where P_0 = initial principal

r = interest rate per year

n = number of times interest is compounded per year

t = number of years.

If n becomes very large ($\equiv$ interest is compounded continuously) the above formula becomes

$$P(t) = P_0 \, e^{rt}$$

The Number 'e' (Euler's constant)

The most important base is the number denoted by the letter e.

The number e is defined as the value that $\left(1+\frac{1}{n}\right)^n$ approaches as n becomes very large.

Correct to five decimal places (note that e is an irrational number), $e \approx 2.71828$.

n	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

The Natural Exponential Function

The Natural Exponential Function

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

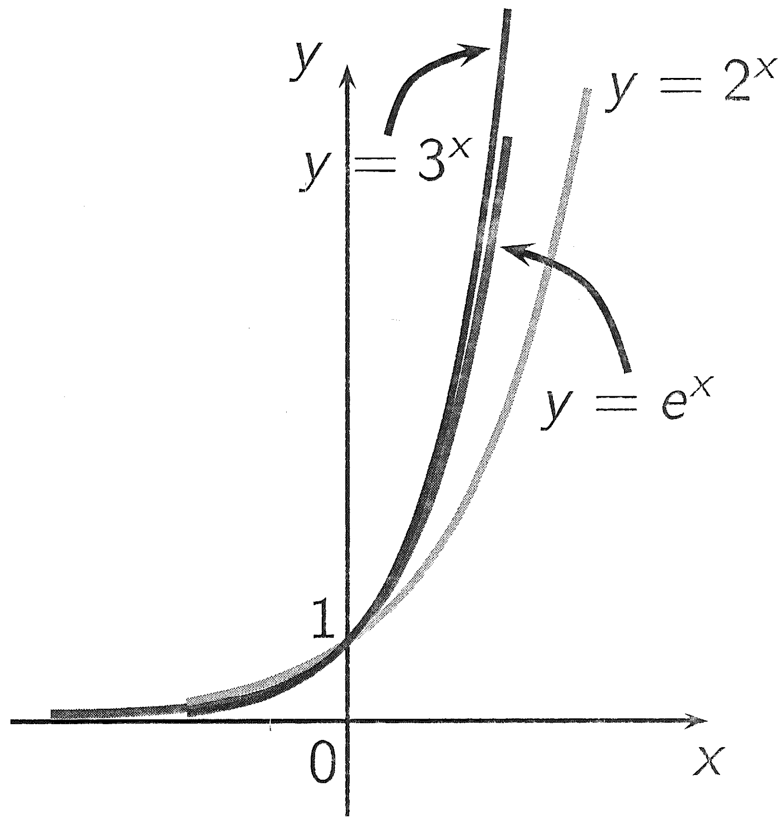
Note:

Sometimes we write

$$f(x) = \exp(x)$$

to denote the exponential function.

Since $2 < e < 3$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.



Example 2:

When a certain drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after t hours is modeled by

$$D(t) = 50\, e^{-0.2t}.$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

(*) Notice that $D(0) = 50\,e^{-0.2\cdot 0} = 50\,\underbrace{e^{0}}_{1}$

$= 50$

Thus 50 is the initial amount of drug administered to the patient.

(**) $D(3) = 50\,e^{-0.2\cdot 3} = 50\,e^{-0.6} \cong 27.44$ mg

(***) It would have been more interesting to ask: How long do we need to wait so that the blood stream of the patient only has 25 mg left of drug?

$25 = D(\bar{t}) = 50\,e^{-0.2\,\bar{t}} \iff \boxed{\frac{1}{2} = e^{-0.2\,\bar{t}}}$

How do we solve for $\bar{t}$?

Logarithmic Functions

Every exponential function $f(x) = a^x$, with $0 < a \neq 1$, is a one-to-one function (Horizontal Line Test). Thus, it has an inverse function, called the *logarithmic function with base a* and denoted by $\log_a x$.

Definition

Let a be a positive number with $a \neq 1$. The **logarithmic function** with base a, denoted by $\log_a$, is defined by

$$y = \log_a x \iff a^y = x.$$

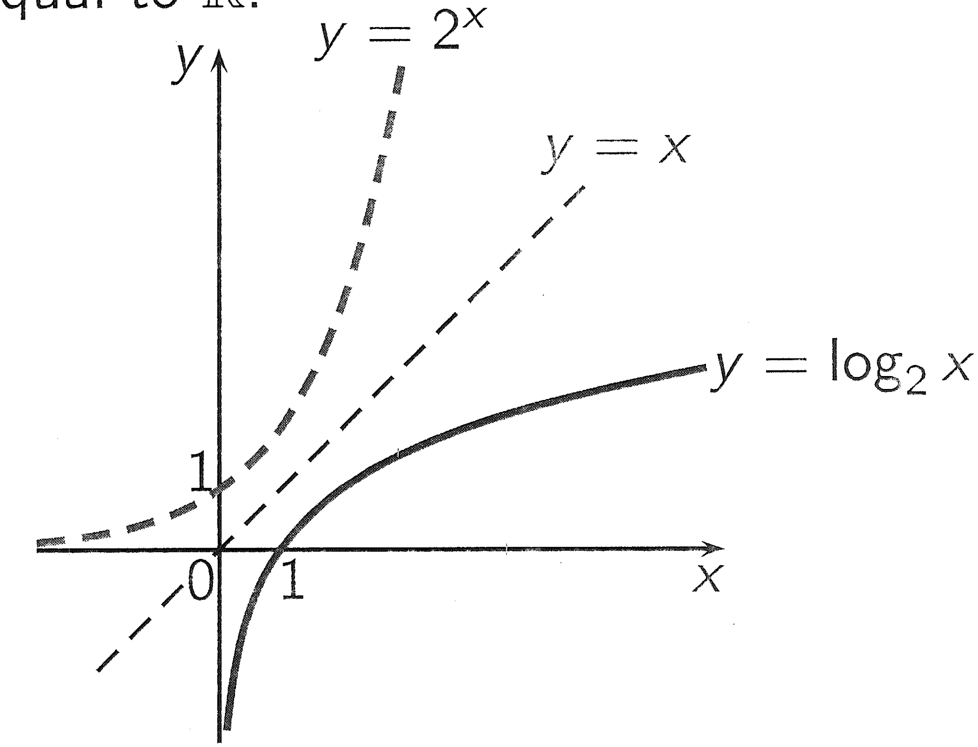
That is, $\log_a x$ is the exponent to which a must be raised to give x.

Properties of Logarithms

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a a^x = x$
4. $a^{\log_a x} = x$

Graphs of Logarithmic Functions

The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$. Thus, the function $y = \log_a x$ is defined for $x > 0$ and has range equal to $\mathbb{R}$.



The point $(1, 0)$ is on the graph of $y = \log_a x$ (as $\log_a 1 = 0$) and the y-axis is a vertical asymptote.

Natural Logarithms

Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of Calculus is the number e.

Definition

The logarithm with base e is called the **natural logarithm** and denoted:

$$\boxed{\ln x := \log_e x.}$$

We recall again that, by the definition of inverse functions, we have

$$y = \ln x \qquad \Longleftrightarrow \qquad e^y = x.$$

Properties of Natural Logarithms

1. $\ln 1 = 0$
2. $\ln e = 1$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

Common Logarithms

Another convenient choice of base for the purposes of the Life Sciences is the number 10.

Definition

The logarithm with base 10 is called the **common logarithm** and denoted:

$$\boxed{\log x := \log_{10} x.}$$

We recall again that, by the definition of inverse functions, we have

$$y = \log x \qquad \Longleftrightarrow \qquad 10^y = x.$$

Properties of Natural Logarithms

1. $\log 1 = 0$
2. $\log 10 = 1$
3. $\log 10^x = x$
4. $10^{\log x} = x$

Laws of Logarithms

Since logarithms are 'exponents', the Laws of Exponents give rise to the Laws of Logarithms:

Laws of Logarithms

Let a be a positive number, with $a \neq 1$. Let A, B and C be any real numbers with $A > 0$ and $B > 0$.

1. $\log_a(AB) = \log_a A + \log_a B$;
2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$;
3. $\log_a(A^C) = C \log_a A$.

Proof of **Law 1.**: $\log_a(AB) = \log_a A + \log_a B$

Let us set

$$\log_a A = u \quad \text{and} \quad \log_a B = v.$$

When written in exponential form, they become

$$a^u = A \quad \text{and} \quad a^v = B.$$

Thus:

$$\begin{aligned} \underline{\log_a(AB)} &= \log_a(a^u a^v) \\ &= \log_a(a^{u+v}) \\ &\overset{\text{why?}}{=} u + v \\ &= \underline{\log_a A + \log_a B}. \end{aligned}$$

In a similar fashion, one can prove **2.** and **3.**

Expanding and Combining Logarithmic Expressions

Example 3:

Use the Laws of Logarithms to combine the expression

$$\log_a b + c \log_a d - r \log_a s - \log_a t$$

into a single logarithm.

$$\log_a b + c \log_a d - r \log_a s - \log_a t$$

$$= \left[\log_a b + \log_a (d^c)\right] - \left[\log_a (s^r) + \log_a t\right]$$

$$= \log_a (b d^c) - \log_a (s^r t)$$

$$= \log_a \left(\frac{b d^c}{s^r t} \right)$$

We used properties 1.–3. of Logarithms.

Change of Base

For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. One can prove that:

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}.}$$

Proof: Set $y = \log_b x$. By definition, this means that $b^y = x$. Apply now $\log_a(\cdot)$ to $b^y = x$. We obtain

$$\log_a(b^y) = \log_a x \quad \rightsquigarrow \quad y \log_a b = \log_a x.$$

Thus

$$\log_b x = y = \frac{\log_a x}{\log_a b}.$$

Example: $\log_5 2 = \dfrac{\log 2}{\log 5} = \dfrac{\ln 2}{\ln 5} \approx 0.43068.$

Exponential Equations

An exponential equation is one in which the variable occurs in the exponent. For example,

$$3^{x+2} = 7.$$

We take the (either common or natural) logarithm of each side and then use the Laws of Logarithms to 'bring down the variable' from the exponent:

$$\log(3^{x+2}) = \log 7$$

$$\rightsquigarrow \quad (x+2)\log 3 = \log 7$$

$$\rightsquigarrow \quad x + 2 = \frac{\log 7}{\log 3}$$

$$\rightsquigarrow \quad x = \frac{\log 7}{\log 3} - 2 \approx -0.228756$$

Example 4: (Online Homework HW03, # 6)

Solve the given equation for x:

$$2^{5x-4} = 3^{10x-10}$$

$$2^{5x-4} = 3^{10x-10}$$

Take log of both sides (OR ln)

$$\log\left(2^{5x-4}\right) = \log\left(3^{10x-10}\right)$$

$$\Longleftrightarrow (5x-4)\log 2 = (10x-10)\log 3$$

$$\Longleftrightarrow (5\log 2)x - 4\log 2 = (10\log 3)x - 10\log 3$$

$$\Longleftrightarrow (10\log 3)x - (5\log 2)x = 10\log 3 - 4\log 2$$

$$\Longleftrightarrow \left[10\log 3 - 5\log 2\right]x = 10\log 3 - 4\log 2$$

$$\Longleftrightarrow x = \frac{(10\log 3 - 4\log 2)}{(10\log 3 - 5\log 2)} = \frac{\log\left(\frac{3^{10}}{2^4}\right)}{\log\left(3^{10}/2^5\right)} \cong \underline{\underline{1.09216}}$$

Logarithmic Equations

A logarithmic equation is one in which a logarithm of the variable occurs. For example,

$$\log_2(25 - x) = 3.$$

To solve for x, we write the equation in exponential form, and then solve for the variable:

$$25 - x = 2^3 \quad \leadsto \quad 25 - x = 8 \quad \leadsto \quad x = 17.$$

Alternatively, we raise the base, 2, to each side of the equation; we then use the Laws of Logarithms:

$$2^{\log_2(25-x)} = 2^3 \quad \leadsto \quad 25 - x = 2^3 \quad \leadsto \quad x = 17.$$

Example 5: (Online Homework HW03, # 5)

Solve the given equation for x:

$$\log_{10} x + \log_{10}(x + 21) = 2$$

$$\log_{10} x + \log_{10}(x+21) = 2$$

$$\iff \log_{10}\left[x(x+21)\right] = 2$$

$$\iff 10^{\log_{10}\left[x(x+21)\right]} = 10^2$$

$$\iff \boxed{x(x+21) = 100}$$

$$\iff x^2 + 21x - 100 = 0$$

$$\iff (x+25)(x-4) = 0$$

$$\iff \underline{x = -25,\ 4}$$

HOWEVER, $\log_{10}(-25) + \log_{10}(-25+21) = 2$ does not make any sense! So $\boxed{x = 4}$ is the only solution.