MA137 – Calculus 1 with Life Science Applications
Semilog and Double Log Plots
(Section 1.3)

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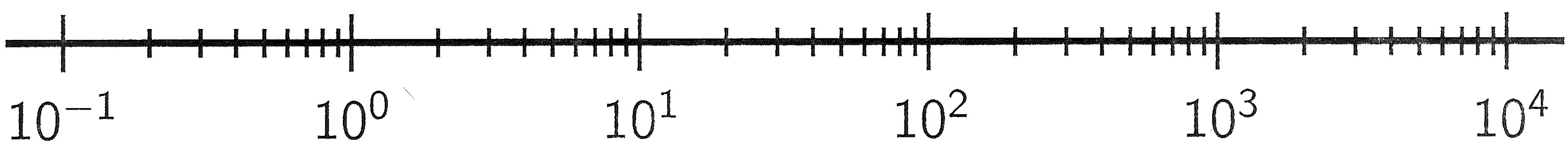
Logarithmic Scales

- When a physical quantity varies over a very large range, it is often convenient to take its logarithm in order to have a more manageable set of numbers.
- Quantities that are measured on logarithmic scales include
  - acidity of a solution (the **pH scale**),
  - earthquake intensity (Richter scale),
  - loudness of sounds (decibel scale),
  - light intensity,
  - information capacity,
  - radiation.
- In such cases, the equidistant marks on a logarithmic scale represent consecutive powers of 10.



The pH Scale

Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Sorensen, in 1909, defined a more convenient measure:

$$\boxed{\mathrm{pH} = -\log[H^+]}$$

where $[H^+]$ is the concentration of hydrogen ions measured in moles per liter (M).

Solutions are defined in terms of the pH as follows:

those with $\mathrm{pH} = 7$ (or $[H^+] = 10^{-7}\mathrm{M}$) are *neutral*,

those with $\mathrm{pH} < 7$ (or $[H^+] > 10^{-7}\mathrm{M}$) are *acidic*,

those with $\mathrm{pH} > 7$ (or $[H^+] < 10^{-7}\mathrm{M}$) are *basic*.

Example 1 (Finding pH):

The hydrogen ion concentration of a sample of each substance is given. Calculate the pH of the substance.

(a) Lemon juice: $[H^+] = 5.0 \times 10^{-3}$M

(b) Tomato juice: $[H^+] = 3.2 \times 10^{-4}$M

(c) Seawater: $[H^+] = 5.0 \times 10^{-9}$M

(a) Lemon juice $[H^+] = 5.0 \times 10^{-3}$ M

$$pH = -\log(5.0 \times 10^{-3}) = -\log(5) - \log(10^{-3})$$

$$= 3 - \log(5) = \underline{\underline{2.301}}$$

(b) Tomato juice $[H^+] = 3.2 \times 10^{-4}$ M

$$pH = -\log(3.2 \times 10^{-4}) = -\log(3.2) - \log(10^{-4})$$

$$= 4 - \log(3.2) = \underline{\underline{3.49485}}$$

(c) Seawater $[H^+] = 5.0 \times 10^{-9}$ M

$$pH = -\log(5.0 \times 10^{-9}) = -\log(5) - \log(10^{-9})$$

$$= 9 - \log(5) = \underline{\underline{8.301}}$$

Example 2 (Ion Concentration):

Calculate the hydrogen ion concentration of each substance from its pH reading.

(a) Vinegar: pH $= 3.0$

(b) Milk: pH $= 6.5$

(a) Vinegar : $pH = 3.0$

$\Rightarrow \quad 3.0 = -\log[H^+] \Rightarrow \log[H^+] = -3$

$\Rightarrow \quad \boxed{[H^+] = 10^{-3}}$

(b) Milk : $pH = 6.5$

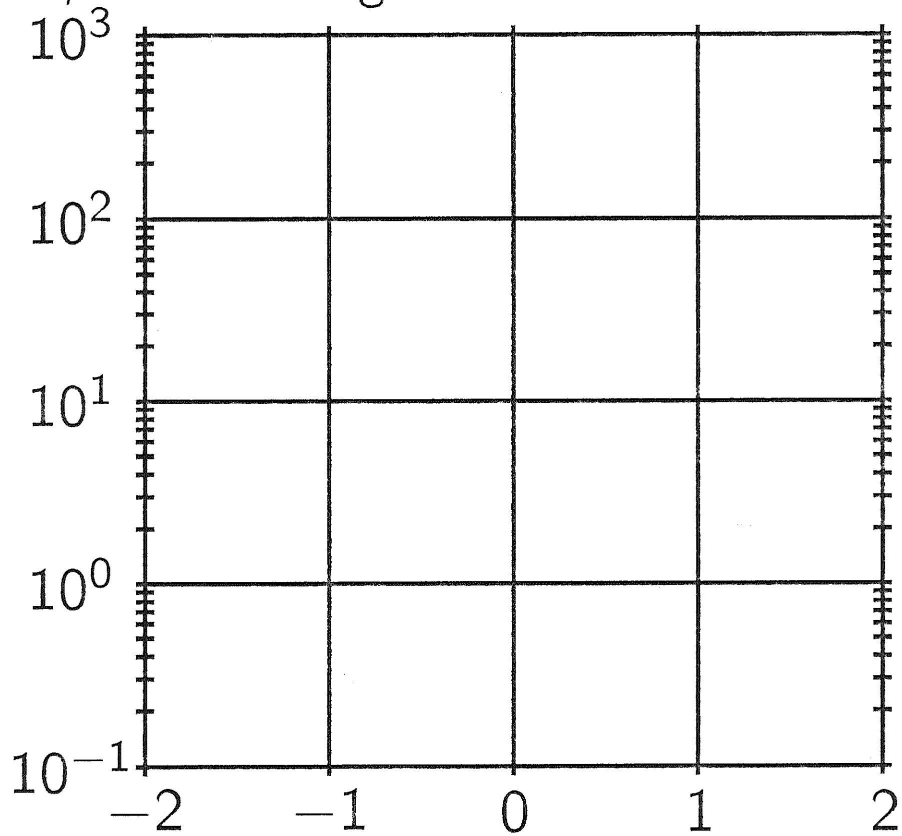
$\Rightarrow \quad 6.5 = -\log[H^+] \longrightarrow \log[H^+] = -6.5$

$\Rightarrow \quad [H^+] = 10^{-6.5} = 10^{0.5} \cdot 10^{-0.5} \cdot 10^{-6.5}$

$= 3.2 \times 10^{-7}$

Semilog Plots

- In biology its common to use a semilog plot to see whether data points are appropriately modeled by an exponential function.
- This means that instead of plotting the points (x, y), we plot the points $(x, \log y)$.
- In other words, we use a logarithmic scale on the vertical axis.



Graphs for a Science article

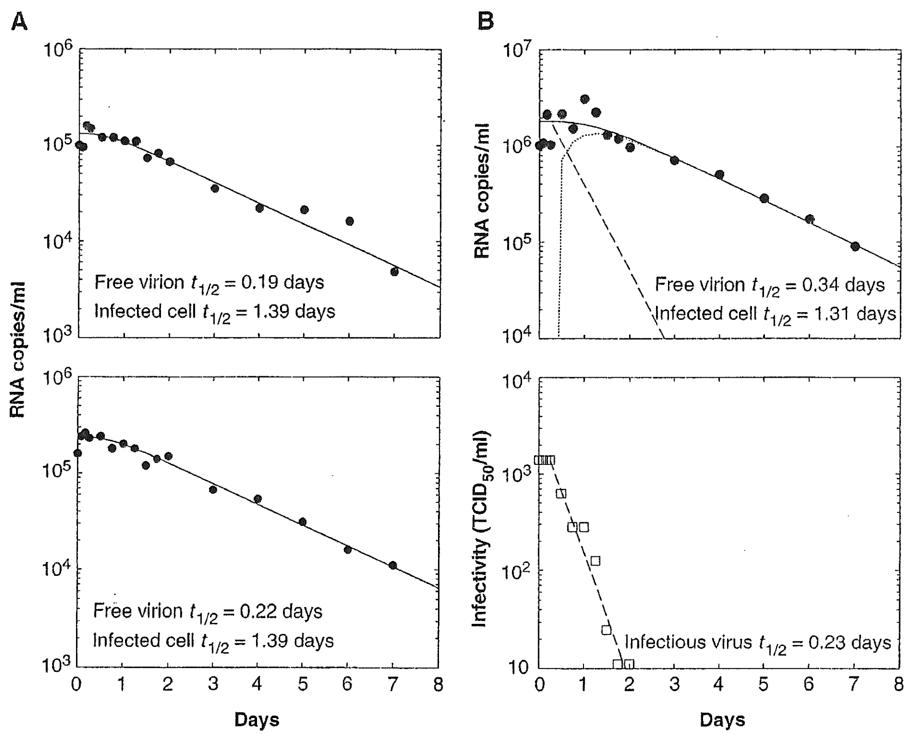


Fig. 1. (**A**) Plasma concentrations (copies per milliliter) of HIV-1 RNA (circles) for two representative patients (upper panel, patient 104; lower panel, patient 107) after ritonavir treatment was begun on day 0. The theoretical curve (solid line) was obtained by nonlinear least squares fitting of Eq. 6 to the data. The parameters c (virion clearance rate), δ (rate of loss of infected cells), and V_0 (initial viral load) were simultaneously estimated. To account for the pharmacokinetic delay, we assumed $t = 0$ in Eq. 6 to correspond to the time of the pharmacokinetic delay (if measured) or selected 2, 4, or 6 hours as the best-fit value (see Table 1). The logarithm of the experimental data was fitted to the logarithm of Eq. 6 by a nonlinear least squares method with the use of the subroutine DNLS1 from the Common Los Alamos Software Library, which is based on a finite difference Levenberg-Marquardt algorithm. The best fit, with the smallest sum of squares per data point, was chosen after eliminating the worst outlying data point for each patient with the use of the jackknife method. (**B**) Plasma concentrations of HIV-1 RNA (upper panel; circles) and the plasma infectivity titer (lower panel; squares) for patient 105. (Top panel) The solid curve is the best fit of Eq. 6 to the RNA data; the dotted line is the curve of the noninfectious pool of virions, $V_{NI}(t)$; and the dashed line is the curve of the infectious pool of virions, $V_I(t)$. (Bottom panel) The dashed line is the best fit of the equation for $V_I(t)$ to the plasma infectivity data. $TCID_{50}$, 50% tissue culture infectious dose.

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The graphs are taken from the article

HIV-1 Dynamics in Vivo: Virion Clearance Rate, Infected Cell Life-Span, and Viral Generation Time,

by Alan S. Perelson, Avidan U. Neumann, Martin Markowitz, John M. Leonard and David D. Ho,

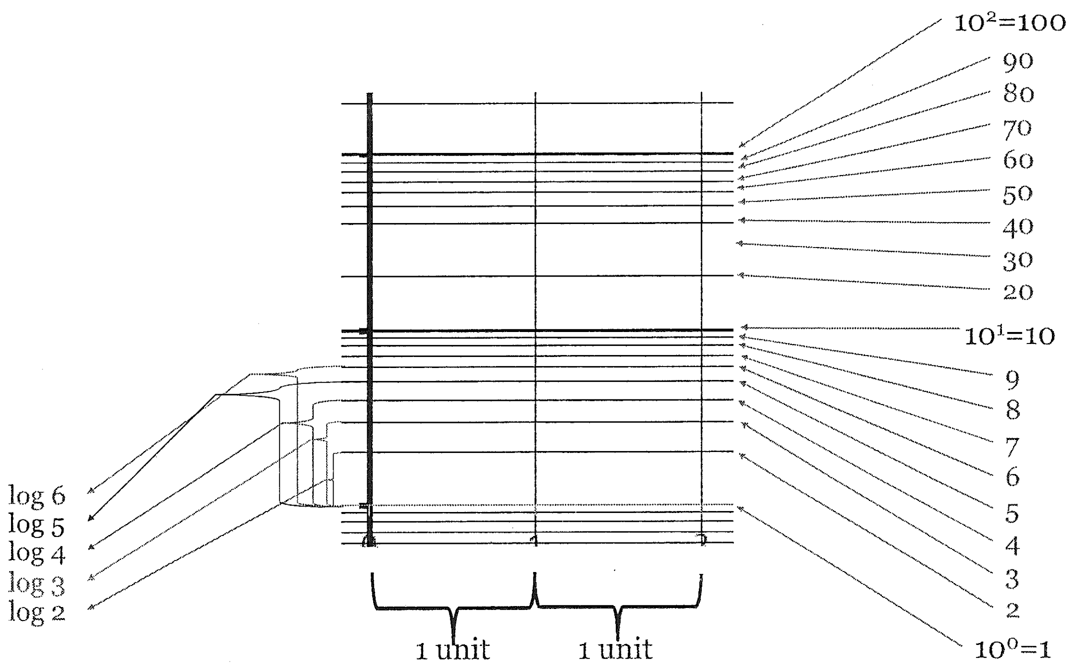
Science, New Series, Vol. 271, No. 5255 (Mar. 15, 1996), pp. 1582-1586.

David Ho was Time magazine's 1996 Man of the Year.

How to Read a Semilog Plot

You need remember is that the log axis runs in exponential cycles.
Each cycle runs linearly in 10's but the increase from one cycle to another is an increase by a factor of 10.
So within a cycle you would have a series of: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (this could also be 0.1-1, etc.).
The next cycle begins with 10 and progresses as 20, 30, 40, 50, 60, 70, 80, 90, 100.
The cycle after that would be 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000.

Below is a picture of semilog graph paper.



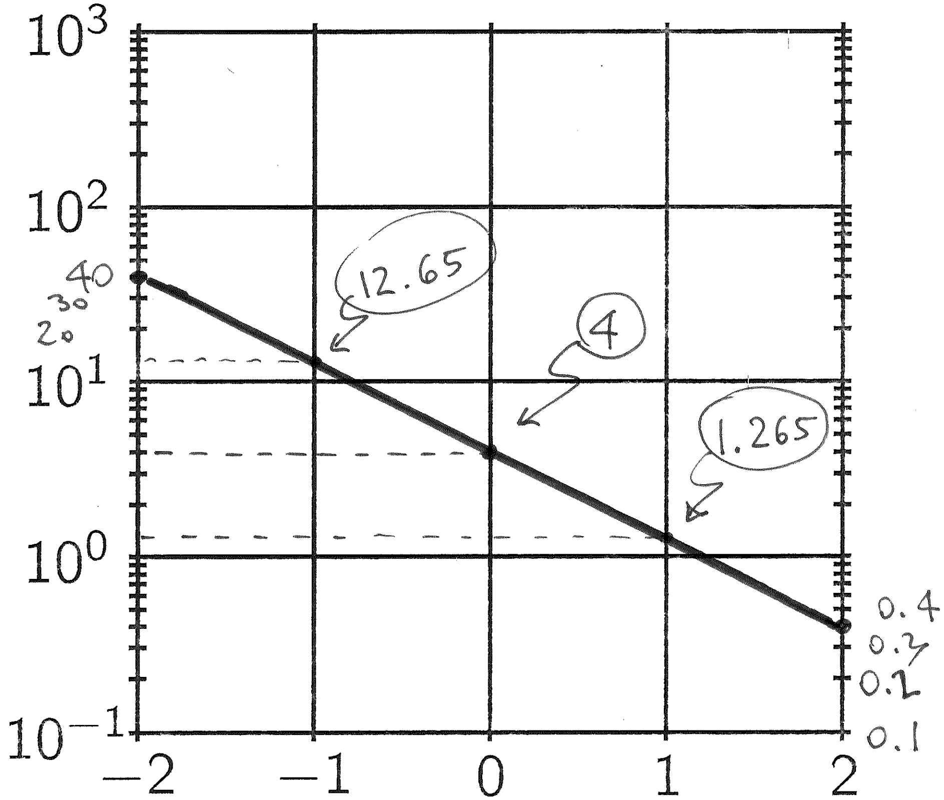
Number	log
100	2
90	1.9542
80	1.9031
70	1.8451
60	1.7782
50	1.6990
40	1.6021
30	1.4771
20	1.3010
10	1
9	0.9542
8	0.9031
7	0.8451
6	0.7782
5	0.6990
4	0.6021
3	0.4771
2	0.3010
1	0.0000

Example 3:

Suppose that x and y are related by the expression

$$y = 4 \cdot 10^{-x/2} \quad [= 4 \cdot (10^{-1/2})^x = 4 \cdot (0.316)^x].$$

Use a logarithmic transformation to find a linear relationship between the given quantities and graph the resulting linear relationship in the semilog (or log-linear) plot.



Let's plot a few values of that function.
We can build the table

x	
-2	$4 \cdot 10^{-(-2/2)} = 4 \cdot 10 = 40$
-1	$4 \cdot 10^{-(-1/2)} = 12.65$
0	4
1	1.265
2	0.4

From $y = 4 \cdot 10^{-x/2}$

take $\log = \log_{10}$ of both sides:

$$\log y = \log\left(4 \cdot 10^{-x/2}\right)$$
$$= \log(4) + \log\left(10^{-x/2}\right)$$
$$= \log(4) + \left(-\tfrac{1}{2}\right) \cdot x$$

Set $Y = \log y$, so the above equation becomes

$$\boxed{Y = \left(-\tfrac{1}{2}\right) \cdot x + \log(4)}$$

slope of line

at ($\log 4$) when the intercept is plotted

Lines in Semilog Plots

- If we start with an **exponential function** of the form $y = a \cdot b^x$ and take logarithms of both sides, we get

$$\log y = \log(a \cdot b^x) = \log a + \log b^x$$

$$\log y = \log a + x \log b$$

If we let $Y = \log y$, $M = \log b$, and $B = \log a$, then we obtain

$$Y = B + Mx,$$

i.e., the equation of a line with slope M and Y-intercept B.

- So if we obtain experimental data that we suspect might possibly be exponential, then we could graph a semilog scatter plot and see if it is approximately linear.

Conversely, suppose we have a straight line in a semilog plot:

$$Y = Mx + B \qquad \text{where } Y = \log y$$

Then from $\log y = Mx + B$ we obtain

$$10^{\log y} = 10^{Mx+B}$$

$$\Longleftrightarrow$$

$$y = 10^{Mx} \cdot 10^{B}$$

$$\Longleftrightarrow$$

$$y = \underbrace{(10^{B})}_{a} \cdot \underbrace{(10^{M})}_{b}{}^{x} = \underline{a \cdot b^{x}}$$

where $\underline{a = 10^{B}}$ $\underline{b = 10^{M}}$

Example 4:

When $\log y$ is graphed as a function of x, a straight line results. Graph the straight line given by the following two points

$$(x_1, y_1) = (0, 40) \qquad (x_2, y_2) = (2, 600)$$

on a log-linear plot. Determine the functional relationship between x and y. (**Note:** The original x-y coordinates are given.)

First method: a line in a semilog plot corresponds to an exponential function of the form $y = a \cdot b^x$

when $x=0$ then $y=40$
$x=2$ then $y=600$
$$\Longrightarrow \begin{cases} 40 = a \cdot \overbrace{b^0}^{=1} \Rightarrow \boxed{a=40} \\ 600 = a\,b^2 \end{cases}$$

$\therefore$ $a = 40$ and $600 = 40 \cdot b^2 \Longrightarrow b^2 = \frac{600}{40} = 15$

$\therefore$ $b = \sqrt{15} \cong 3.873$ $\therefore$ $\boxed{y = 40 \cdot (3.873)^x}$

Second method: in the $(x, \log y)$ plot we need to compute the equation of the line through $(0, \log 40)$ and $(2, \log 600)$

Slope of the line is $m = \dfrac{\log 600 - \log 40}{2 - 0}$

$$= \frac{\log\left(\frac{600}{40}\right)}{2} = \frac{1}{2}\log(15) = \boxed{\log(\sqrt{15})}$$

Hence the point-slope form of the line is

$$(\underbrace{Y}_{\log y} - \log 40) = \log(\sqrt{15}) \cdot (x - 0)$$

$$\therefore \quad \log(y/40) = \log(\sqrt{15}) \cdot x$$

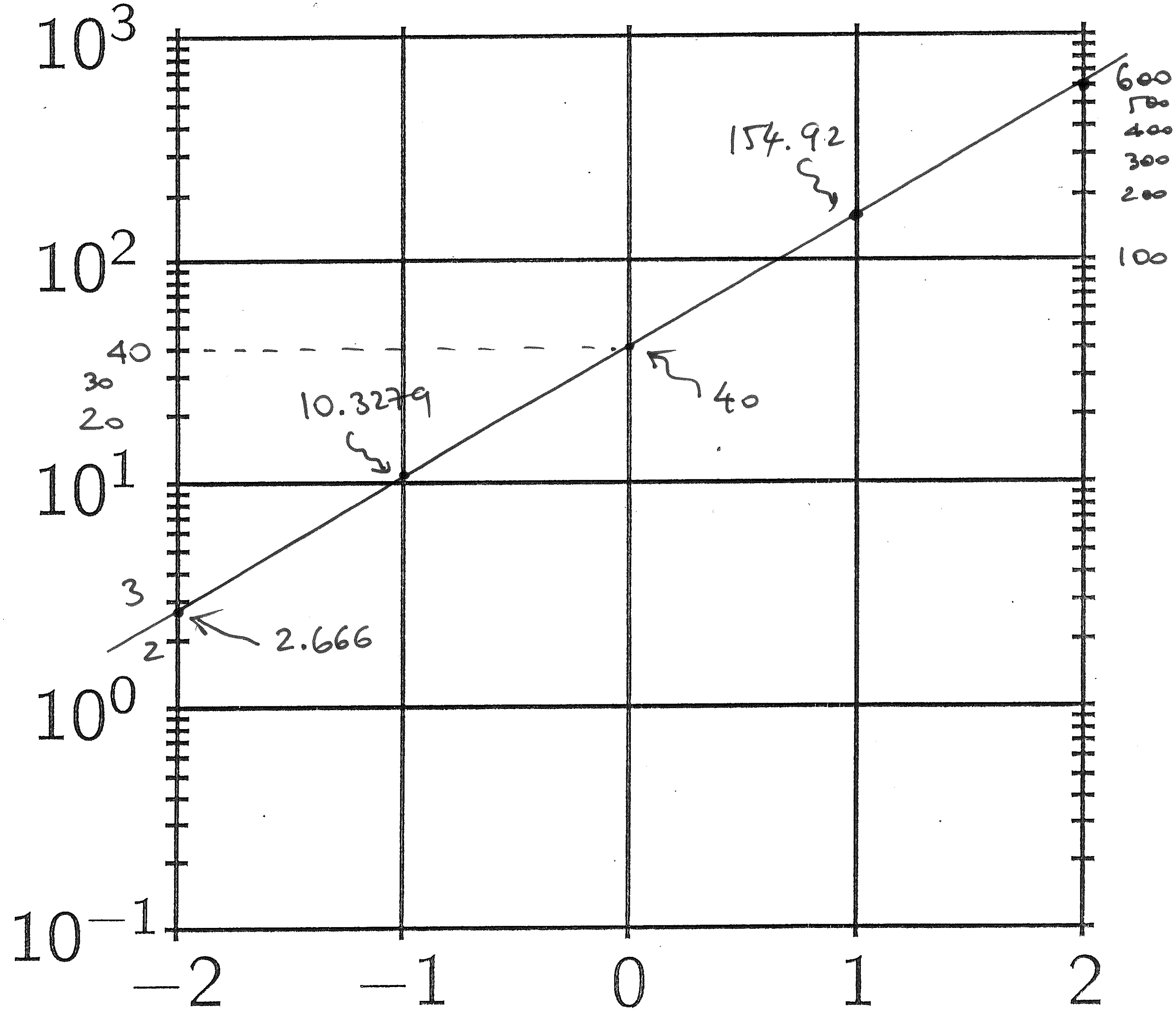
OR

$$\log(y/40) = x \log(\sqrt{15}) = \log\left((\sqrt{15})^x\right)$$

$$\frac{y}{40} = (\sqrt{15})^x$$

OR

$$\boxed{y = 40(3.873)^x}$$



x	$40(3.873)^x$
-2	2.666
-1	10.3279
0	40
1	154.92
2	600

Example 5: (Problem # 46, Section 1.3, p. 53)

When $\log y$ is graphed as a function of x, a straight line results. Graph the straight line given by the following two points

$$(x_1, y_1) = (1, 4) \qquad (x_2, y_2) = (6, 1)$$

on a log-linear plot. Determine the functional relationship between x and y. (**Note:** The original x-y coordinates are given.)

First method: since we obtain a straight line in a semi log plot, the functional relation between x and y is exponential: $y = a \cdot b^x$

Hence $\left.\begin{array}{l} x_1 = 1 \Rightarrow y_1 = 4 \\ x_2 = 6 \Rightarrow y_2 = 1 \end{array}\right\} \Longrightarrow \boxed{\begin{array}{l} 4 = a \cdot b^1 \\ 1 = a \cdot b^6 \end{array}}$

Solve in both eq. for a: $\frac{4}{b} = a = \frac{1}{b^6}$

$\therefore \quad \frac{4}{b} = \frac{1}{b^6} \Longrightarrow \frac{b^6}{b} = 1/4 \Longrightarrow b^5 = 1/4$

$\therefore \quad b = \sqrt[5]{1/4} \cong 0.7578$

Now $a = \frac{4}{b} = \frac{4}{0.7578}$
$\cong \underline{5.278}$

$\therefore \quad \boxed{y = 5.278\,(0.7578)^x}$

2nd method

Let's compute the equation of the line in the semi log plot through $(1, \log 4)$ and $(6, \log 1)$

$$m = \frac{\log 1 - \log 4}{6-1} = \frac{-\log 4}{5} = \left(-\tfrac{1}{5}\right)\log 4 = \log\left(4^{-1/5}\right)$$

$$= \log\left(\frac{1}{\sqrt[5]{4}}\right)$$

Hence

$$\underbrace{Y}_{\log y} - \log 1 = \log\left(\frac{1}{\sqrt[5]{4}}\right)(x-6)$$

point-slope form

$$\Rightarrow \log y = (x-6)\log\left(\frac{1}{\sqrt[5]{4}}\right)$$

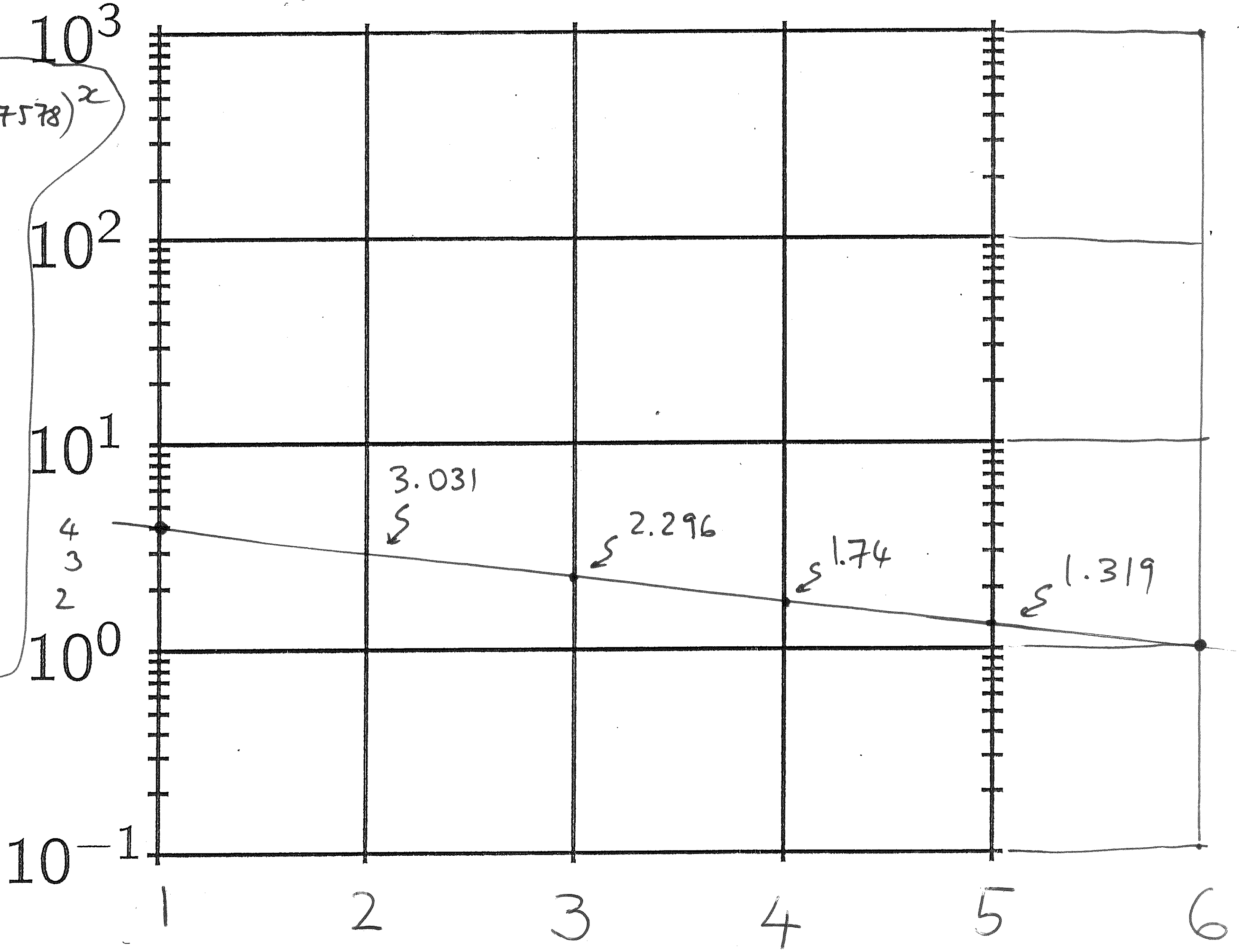
$$\log y = \log\left[\left(\frac{1}{\sqrt[5]{4}}\right)^{x-6}\right] \rightarrow y = \left(\frac{1}{\sqrt[5]{4}}\right)^{x-6}$$

$$y = \left(\frac{1}{\sqrt[5]{4}}\right)^{x} \cdot (4)^{6/5}$$

$$= \boxed{5.278\,(0.7578)^{x}}$$

Here is the plot of $y = 5.278(0.7578)^x$

x	$5.278(0.7578)^x$
1	4
2	3.031
3	2.296
4	1.74
5	1.319
6	1



Example 6: (Problem # 52, Section 1.3, p. 53)

Consider the relationship $y = 6 \times 2^{-0.9x}$ between the quantities x and y. Use a logarithmic transformation to find a linear relationship of the form

$$Y = mx + b$$

between the given quantities.

x	$y = 6 \cdot 2^{-0.9x}$
-2	20.89
-1	11.196
0	6
1	3.215
2	1.723

$$y = 6 \cdot 2^{-0.9x}$$

Take $\log = \log_{10}$ of both sides:

$$\log y = \log\left(6 \cdot 2^{-0.9x}\right)$$

$$= \log 6 + \log\left(2^{-0.9x}\right)$$

$$= \left[(-0.9)\log 2\right] x + \log 6$$

$$\boxed{\underset{\underset{Y}{\shortparallel}}{\log y} = -0.27092x + 0.7782}$$

approx. graph of $y = 6 \cdot 2^{-0.9x}$ in semilog plot