MA137 – Calculus 1 with Life Science Applications
Semilog and Double Log Plots
(Section 1.3)

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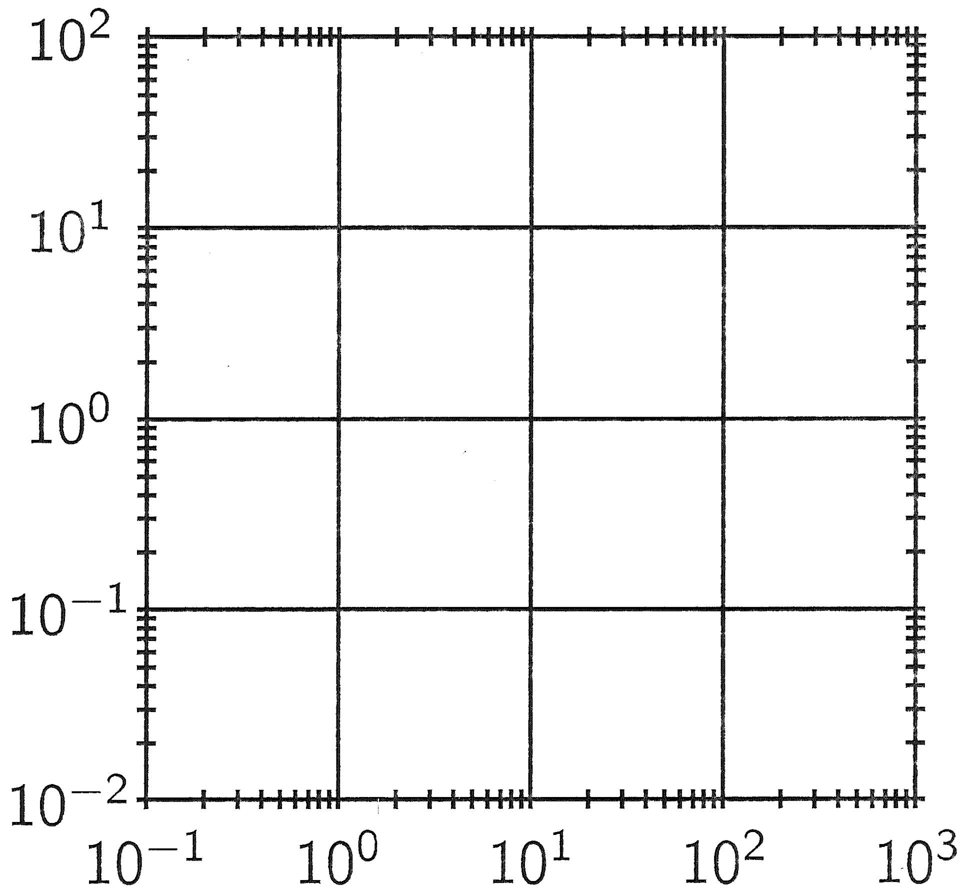
Double-log (or Log-Log) Plots

- If we use logarithmic scales on both the horizontal and vertical axes, the resulting graph is called a log-log plot.



Lines in Double-Log Plots

- A log-log plot is used when we suspect that a power function might be a good model for our data.
- Recall that power functions are frequently found in "scaling relations" between biological variables (e.g., organ sizes). Finding such relationships is the objective of **allometry**.
- If we start with a **power function** $y = Cx^p$ and take logarithms of both sides, we get

$$\log y = \log(Cx^p) = \log C + \log x^p$$

$$\log y = \log C + p \log x$$

Let $Y = \log y$, $A = \log C$, and $X = \log x$. Then the latter equation becomes

$$Y = A + pX$$

We recognize that Y is a linear function of X, so the points $(\log x, \log y)$ lie on a straight line.

Conversely, suppose we have a straight line in a log-log plot:

$$Y = pX + B$$

when

$Y = \log y$

$X = \log x$

Hence we have

$$\log y = p \log x + B$$

trick!

$$\Longleftrightarrow$$

$$\log y = \log(x^p) + \log(10^B)$$

set $C = 10^B$.

$$\Longleftrightarrow$$

$$\log y = \log(10^B \cdot x^p)$$

$$\Longleftrightarrow \quad y = 10^B \cdot x^p$$

to get $y = C x^p$

Example 1: (Problem # 58, Section 1.3, p. 53)

When $\log y$ is graphed as a function of $\log x$, a straight line results. Graph the straight line given by the following two points

$$(x_1, y_1) = (2, 5) \qquad (x_2, y_2) = (5, 2)$$

on a log-log plot. determine the functional relationship between x and y. (**Note:** The original x-y coordinates are given.)

1st method:

A line in a log-log plot corresponds to a power relation of the form: $y = Cx^p$.

Since $(2,5)$ and $(5,2)$ satisfy this relation we obtain:

$$5 = C\,2^p \quad \text{and} \quad 2 = C\,5^p$$

Thus $\dfrac{5}{2^p} = C = \dfrac{2}{5^p}$. This implies

$$\frac{5^p}{2^p} = 2/5 \quad \text{or} \quad \left(\frac{5}{2}\right)^p = 2/5$$

Take log of both sides and we get

$$\log\left[\left(\tfrac{5}{2}\right)^p\right] = \log(2/5) \rightsquigarrow p\log(2.5) = \log(0.4)$$

$$\Rightarrow p = \frac{\log(0.4)}{\log(2.5)} = \boxed{-1} \qquad \Rightarrow C = \frac{5}{2^{(-1)}} = \boxed{10}$$

Thus the functional relationship is $\boxed{y = \frac{10}{x}}$

2nd method :

$$m = \text{slope} = \frac{\log 5 - \log 2}{\log 2 - \log 5} = \frac{\log(5/2)}{\log(2/5)} = \boxed{-1}$$

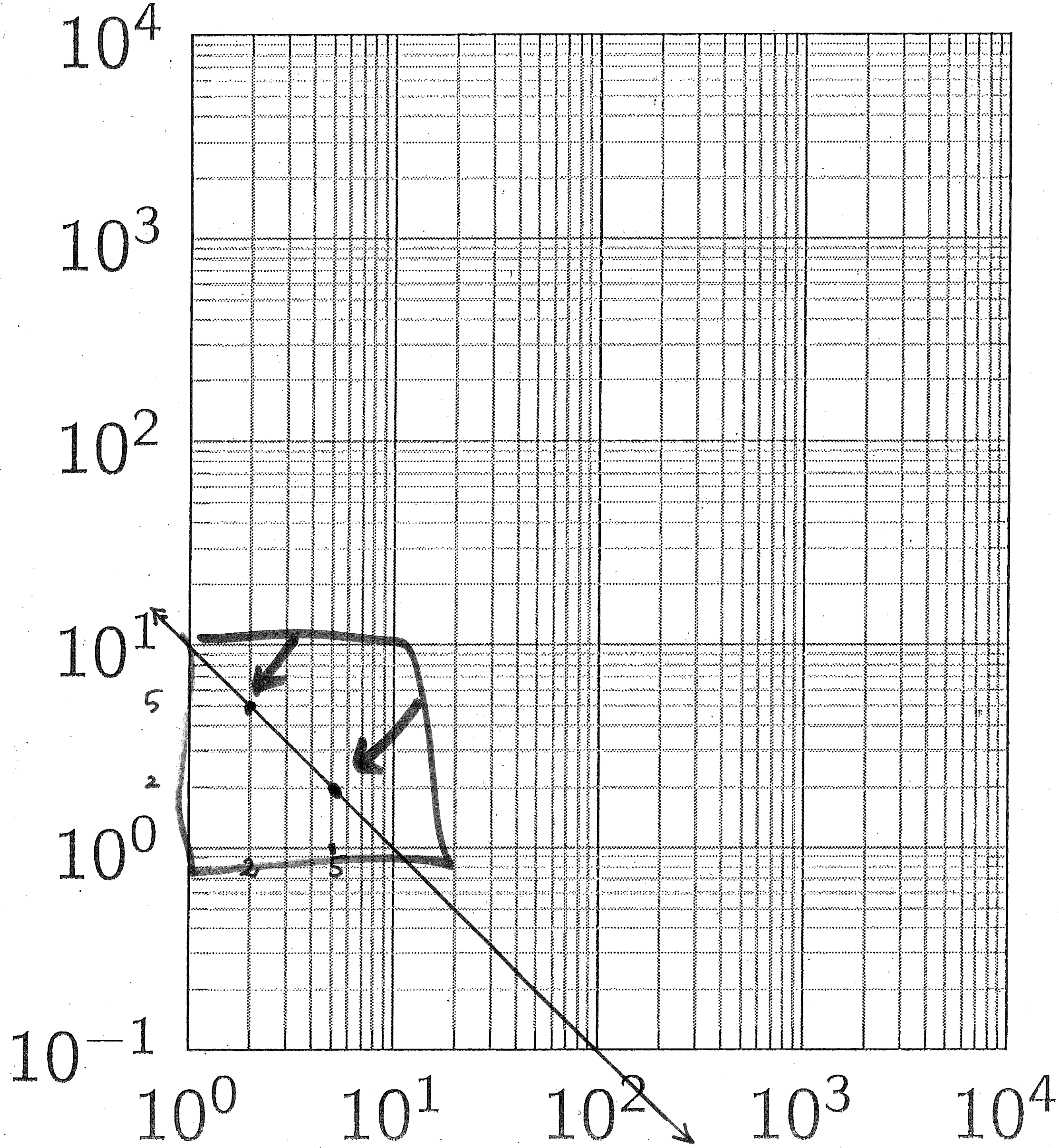
Hence the equation in point slope form is :

$$(\underbrace{\log y}_{Y} - \log 2) = -1\,(\underbrace{\log x}_{X} - \log 5)$$

$$\Rightarrow \log\left(\frac{y}{2}\right) = -\left(\log\left(\frac{x}{5}\right)\right)$$

$$\Rightarrow \underline{\log\left(\frac{y}{2}\right)} = \log\left[\left(\frac{x}{5}\right)^{-1}\right] = \underline{\log\left(\frac{5}{x}\right)}$$

$$\Rightarrow \frac{y}{2} = \frac{5}{x} \Longrightarrow y = \frac{10}{x}$$



$$y = \frac{10}{x}$$

$$= 10x^{-1}$$

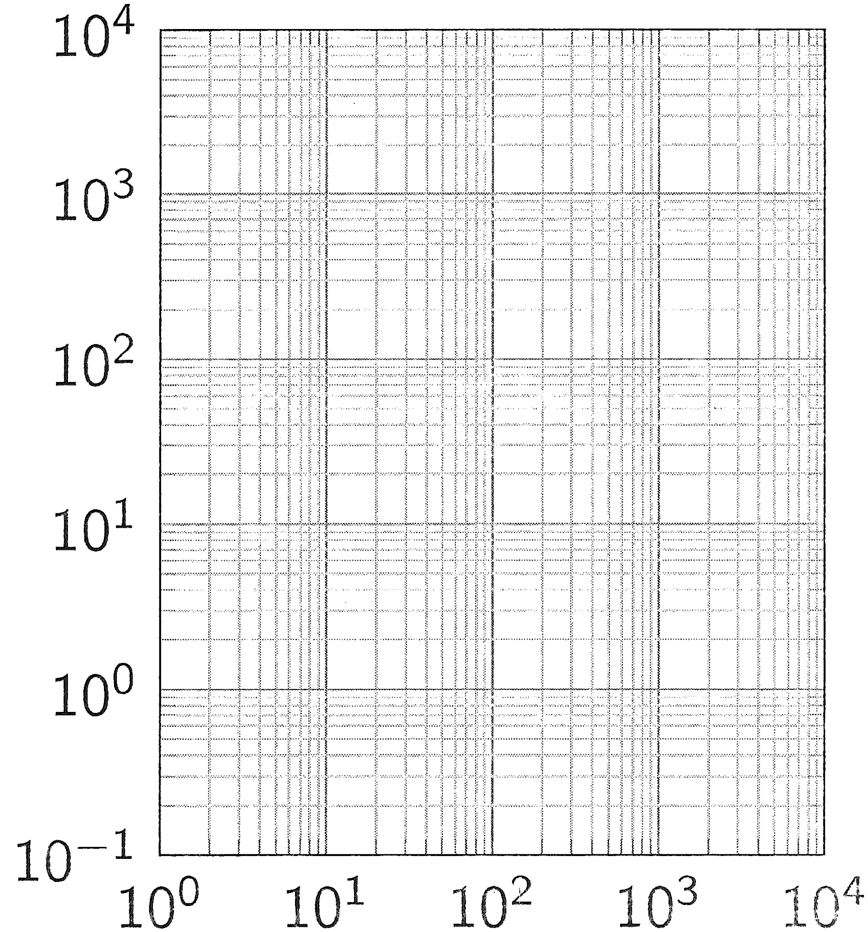
hyperbola

Example 2: (Exam 1, Fall 13, # 4)

There are several possible functional relationships between height and diameter of a tree. One particularly simple model is given by

$$H = AD^{3/4}$$

where A is a constant that depends on the species of tree, H is the height, and D is the diameter. If $A = 50$ plot this relationship in the double log plot below.



Is your graph a straight line? If so, what is its slope?

Consider the function $\underline{H = 50\,D^{3/4}}$

We can construct the following table of values.

D	$H = 50\,D^{3/4}$
1	50
10	281.17
10^2	1,581.14
10^3	8,891.4
10^4	$50{,}000 = 5 \cdot 10^4$

In a log–log plot this power relationship becomes a straight line:

$$\log H = \log(50\,D^{3/4})$$

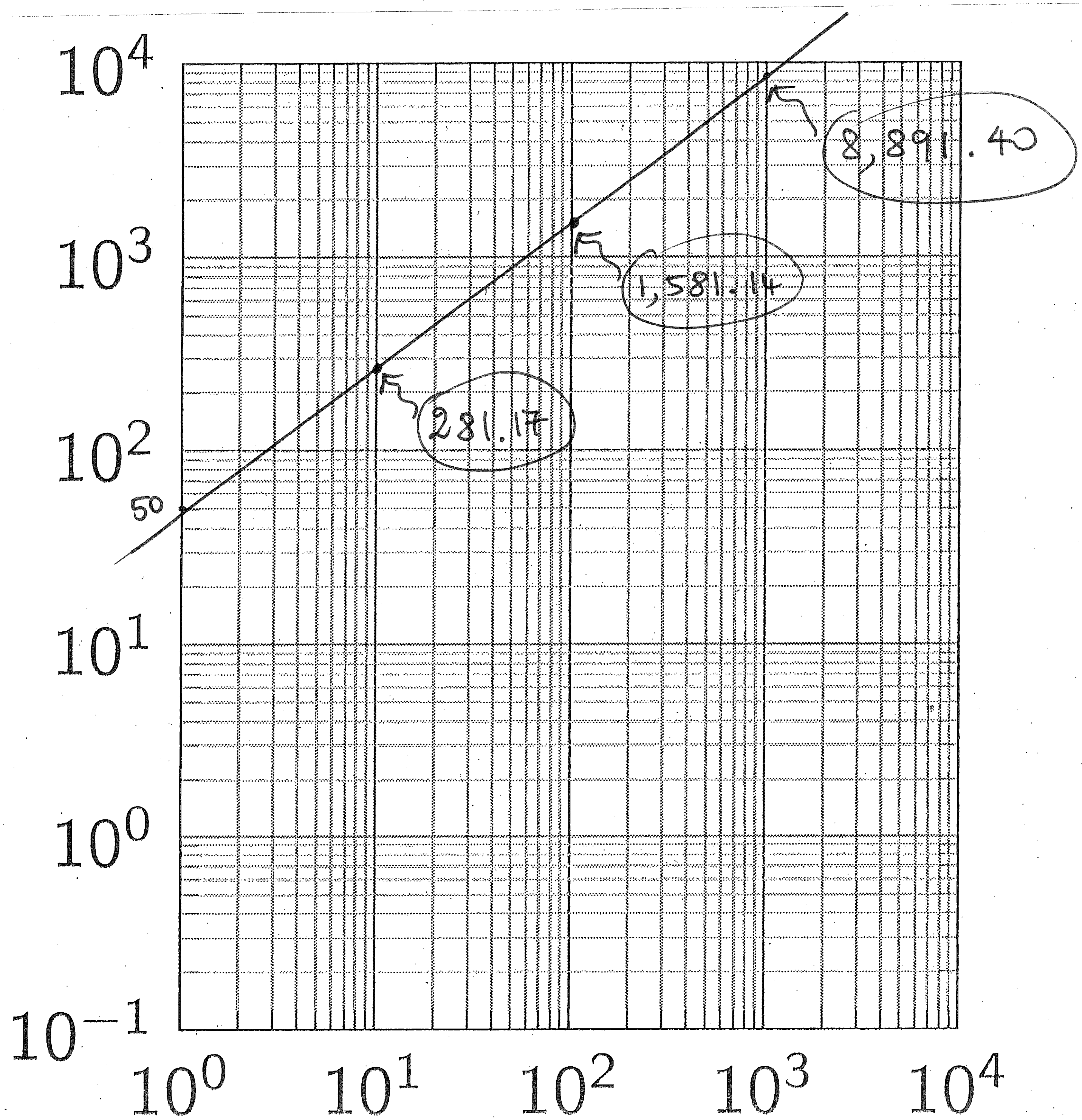
$$\Longleftrightarrow$$

$$\log H = \log 50 + \log(D^{3/4})$$

$$\Longleftrightarrow$$

$$\log(H) = 3/4 \log(D) + \log(50)$$

<u>slope is</u> $\boxed{3/4}$



10^4
10^3
10^2
10^1
10^0
10^{-1}
50
281.17
1,581.14
8,891.40
10^0
10^1
10^2
10^3
10^4

Example 3: (Problem # 74, Section 1.3, p. 54)

The following table is based on a functional relationship between x and y that is either an exponential or a power function:

x	y
0.5	7.81
1	3.4
1.5	2.09
2	1.48
2.5	1.13

Use an appropriate logarithmic transformation and a graph to decide whether the table comes from a power function or an exponential function, and find the functional relationship between x and y.

First, let's see if there is an exponential relationship among our data points. This means that in the semi log plot we have a straight line.

x	y	$\log y$
0.5	~~7.81~~	0.893
1	~~3.4~~	0.531
1.5	~~2.09~~	0.32
2	~~1.48~~	0.17
2.5	~~1.13~~	0.053

Let's compute the slope of the line between two pairs of points of the form $(x, \log y)$

$(0.5, 0.893)$ & $(1.5, 0.32)$ $\longrightarrow$ slope $m = \dfrac{0.32 - 0.893}{1} \cong -0.573$

$(1.5, 0.32)$ & $(2.5, 0.053)$ $\longrightarrow$ slope $m = \dfrac{0.053 - 0.32}{1} \cong -0.267$

Since we do not get similar values, these points do not lie on a straight line.

Let's see if the points of the form $(\log x, \log y)$ lie on a straight line in a log-log plot:

$\log x$	$\log y$
-0.301	0.893
0	0.531
0.176	0.32
0.301	0.17
0.398	0.053

Pick: $(-0.301, 0.893)$ & $(0.176, 0.32)$

$$\text{slope} = \frac{0.32 - 0.893}{0.176 - (-0.301)} \cong -1.201$$

Pick $(0, 0.531)$ & $(0.398, 0.053)$

$$\text{slope} = \frac{0.053 - 0.531}{0.398 - 0} \cong -1.201$$

it seems that we can choose as a slope $\boxed{-1.20}$

The equation of the line in point slope form is (we choose the simplest point $(0, 0.531)$)

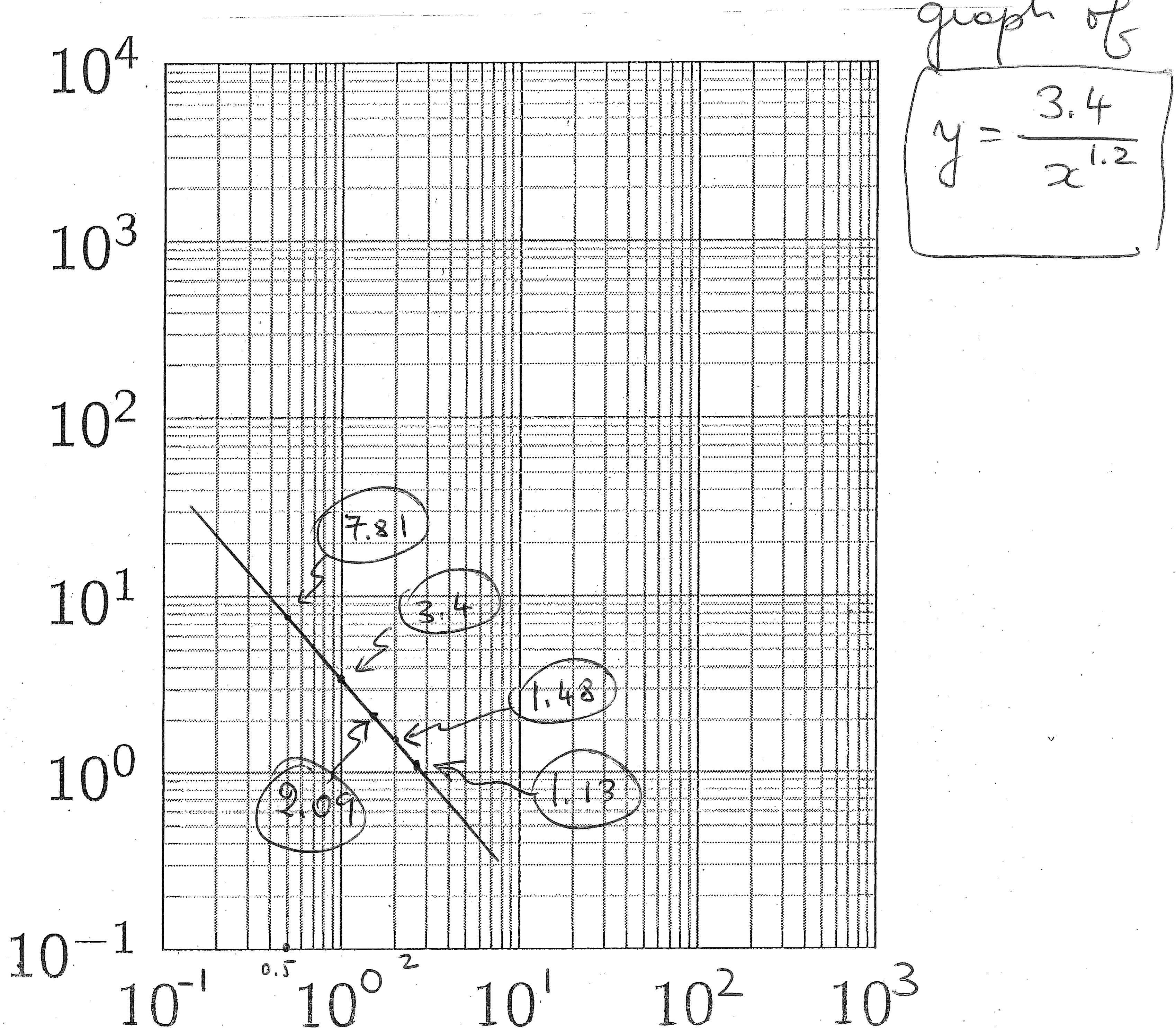
$$(\log y - 0.531) = -1.2(\log x - 0)$$

$$\log y - \log 10^{0.531} = -1.2 \log x$$

$$\Longleftrightarrow$$

$$\log\left(\frac{y}{10^{0.531}}\right) = \log x^{-1.2} \Longleftrightarrow \frac{y}{10^{0.531}} = x^{-1.2}$$

$$\therefore \quad y = 10^{0.531} x^{-1.2} \quad \text{OR} \quad \boxed{y = \frac{3.4}{x^{1.2}}}$$



graph of
$y = \frac{3.4}{x^{1.2}}$
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-1}
0.5
10^0
2
10^1
10^2
10^3
7.81
3.4
1.48
2.09
1.13

Example 4 (Forgetting):

Ebbinghaus's Law of Forgetting states that if a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies

$$\log P = \log P_0 - c\log(t+1),$$

where c is a constant that depends on the type of task and t is measured in <u>months</u>.

(a) Solve the equation for P.

(b) Use Ebbinghaus's Law of Forgetting to estimate a student's score on a biology test two <u>years</u> after he got a score of 80 on a test covering the same material. Assume $c = 0.3$.

(a) $\log P = \log P_0 - c \log(t+1)$

We want to solve for P:

$$\log P = \log P_0 - \log\left[(t+1)^c\right]$$

$$\Longleftrightarrow$$

$$\log P = \log\left[\frac{P_0}{(t+1)^c}\right]$$

Hence $10^{\log P} = 10^{\log\left[\frac{P_0}{(t+1)^c}\right]}$

$$\Longrightarrow \boxed{P(t) = \frac{P_0}{(t+1)^c}}$$

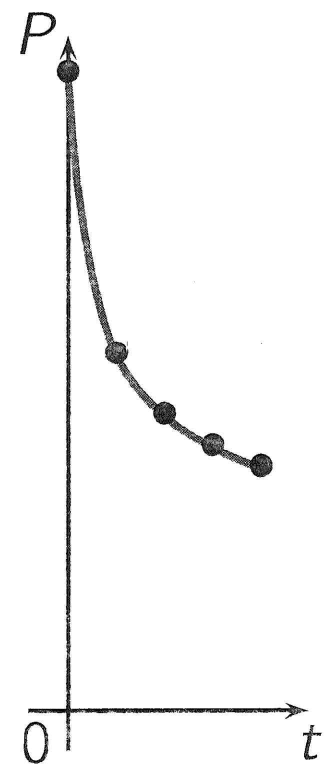
(b) With our data $P_0 = 80$ $c = 0.3$

$P(t) = \dfrac{80}{(t+1)^{0.3}}$ hence $P(\underbrace{24}_{\text{2 years in months}}) = \dfrac{80}{(24+1)^{0.3}} \cong \underline{\underline{30.46}}$

Comment (about Example 4)

Below is the graph of the function $P = 80/(t+1)^{0.3}$ in standard coordinates:

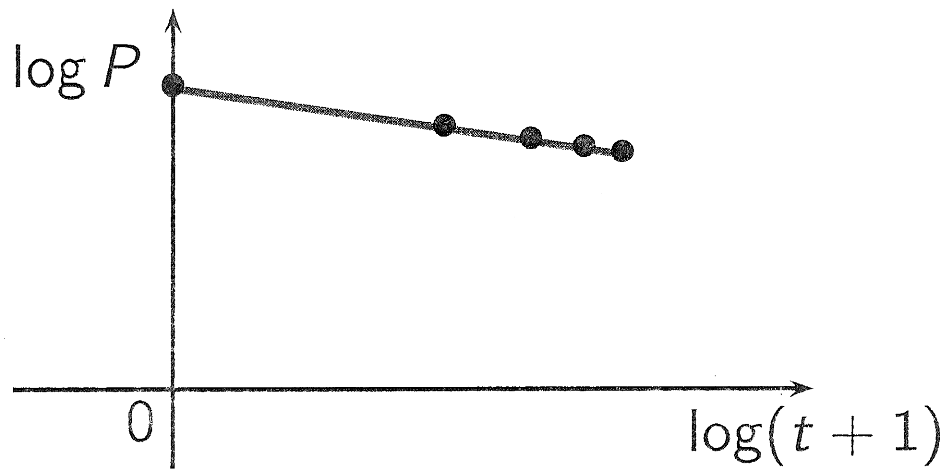
t	$P = 80/(t+1)^{0.3}$
0	80
6	44.62
12	37.06
18	33.072
24	30.458



Comment (cont.d)

Below is the graph of $\log P = \log 80 - 0.3\log(t+1)$ in a log-log plot:

t	$\log(t+1)$	$\log P = \log 80 - 0.3\log(t+1)$
0	0	1.903
6	0.845	1.650
12	1.114	1.569
18	1.279	1.519
24	1.398	1.484



Example 5 (Biodiversity):

Some biologists model the number of species S in a fixed area A (such as an island) by the **Species-Area relationship**

$$\log S = \log c + k \log A,$$

where c and k are positive constants that depend on the type of species and habitat.

(a) Solve the equation for S.

(b) Use part **(a)** to show that if $k = 3$ then doubling the area increases the number of species eightfold.

(a) $\log S = \log c + k \log A$

$\iff$

$\log S = \log c + \log [A^k]$

$\iff$

$\log S = \log [c A^k]$

$\iff 10^{\log S} = 10^{\log [c A^k]} \iff \boxed{S = c A^k}$

(b) Suppose $k = 3$, i.e. $\underline{S = c A^3}$

For $A = a_o$ we get that $S(a_o) = c a_o^3$.

However if we double the area, i.e. $A = 2a_o$,

we get $S(2a_o) = c(2a_o)^3 = 8 \underbrace{c a_o^3} = 8 S(a_o)$

<u>i.e.</u> doubling the area increases the number of species eightfold.