MA137 – Calculus 1 with Life Science Applications
Discrete-Time Models
Sequences and Difference Equations
(Sections 2.1 and 2.2)

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September 11, 2017

What are sequences?

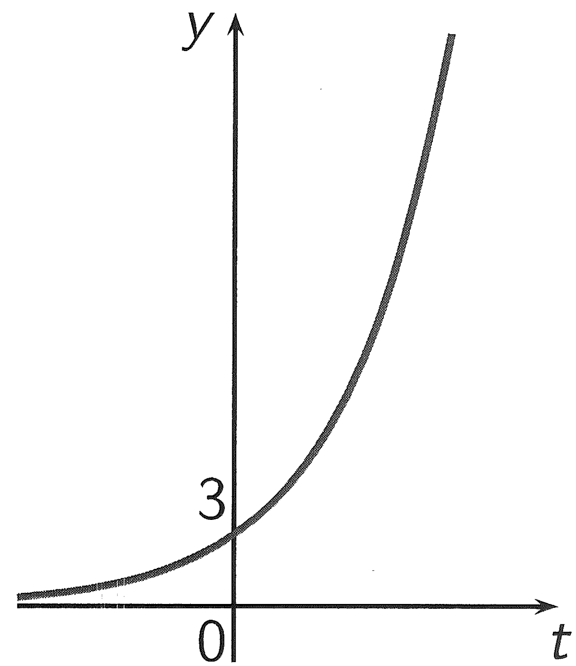
So far we have studied real valued functions whose domain consists of the real numbers, say:

$$f : \mathbb{R} \longrightarrow \mathbb{R}.$$

For example, consider the function

$$f(t) = 3 \cdot 2^t.$$

The graph of f looks like:



More generally, we have considered functions of the form

$$P(t) = P_0(1+r)^t,$$

where r is a positive real number ($r \equiv$ growth rate).

Sometimes it makes sense to change the domain of the function to the nonnegative integers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$$f : \mathbb{N} \longrightarrow \mathbb{R}, \qquad n \mapsto f(n).$$

For example, $f(n) = 3 \cdot 2^n$ with $n \in \mathbb{N}$.

A table is a useful tool to illustrate this function

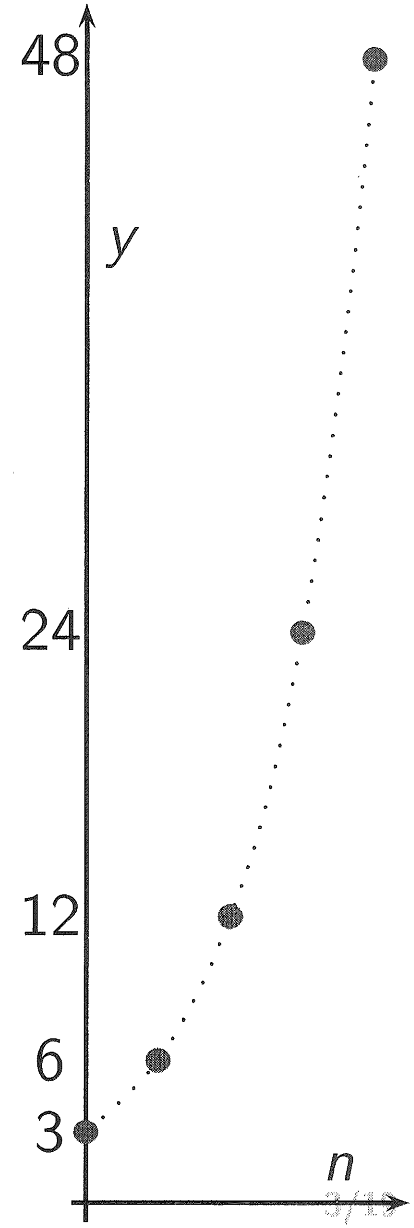
n	0	1	2	3	4	$\cdots$
$3 \cdot 2^n$	3	6	12	24	48	$\cdots$

The graph is useful too!

Because the domain consists of nonnegative integers, the graph consists of isolated points with coordinates

$$(0, f(0)) \quad (1, f(1)) \quad (2, f(2)) \quad (3, f(3)) \quad (4, f(4)) \quad \cdots$$

Note: we should not have connected the isolated points with the dotted curve. Please disregard it!!



Definition and Notation

Definition (Sequence/Notation)

We can write the function

$$f : \mathbb{N} \longrightarrow \mathbb{R}, \qquad n \mapsto f(n)$$

as a list of numbers $f_0,\ f_1,\ f_2,\ f_3,\ \ldots,$ where $f_n = f(n)$.

We refer to this list as a **sequence**.

We write $\{f_n \,|\, n \in \mathbb{N}\}$ (or $\{f_n\}$ for short) to denote the entire sequence.

We list the values of the sequence $\{f_n\}$ in order of increasing n

$$f_0,\ f_1,\ f_2,\ f_3,\ \ldots$$

Remark: Instead of 'f' we often use the letters 'a' or 'b' or 'c' ... to denote sequences.

For example: $$a_n = \frac{n}{n+1} \qquad b_n = \frac{(-1)^n}{(n+1)^2} \qquad c_n = 3 \cdot 2^n$$

Example 1:

Find a general formula for the general term a_n for each of the following sequences starting with a_0:

(a) $0, 1, 4, 9, 16, 25, 36, 49, \ldots$

(b) $1, -1, 1, -1, 1, -1, \ldots$

(c) $1, \; -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \ldots.$

Repeat this problem starting this time with a_1.

(a) Consider $0, 1, 4, 9, 16, 25, 36, 49, \ldots$

these are all squares of numbers.

We want them to be labeled as

$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 9, \quad a_4 = 16, \ldots$

thus $\boxed{a_n = n^2}$ is the nth term of the sequence

(b) We want: $a_0 = 1, \; a_1 = -1, \; a_2 = 1, \; a_3 = -1, \ldots$

So we have $\boxed{a_n = (-1)^n}$ for all $n \in \mathbb{N}$

(c) We want: $a_0 = 1, \quad a_1 = -\frac{1}{2}, \quad a_2 = \frac{1}{4}, \quad a_3 = -\frac{1}{8}$

$a_4 = \frac{1}{16}$, etc ... Notice that all denominators

are powers of 2; there is an alternating sign: $\boxed{a_n = \left(-\frac{1}{2}\right)^n}$

Repeat :

(a) this time we want : $a_1 = 0$, $a_2 = 1$, $a_3 = 4$, $a_4 = 9$, $a_5 = 16$, ... Thus we need to shift the integers :

$$a_n = (n-1)^2 \quad \text{for } n = 1, 2, 3, 4, \ldots$$

(b) We want : $a_1 = 1$, $a_2 = -1$, $a_3 = 1$, $a_4 = -1$, ...

again we shift the integers :

$$a_n = (-1)^{n-1} \quad \text{or} \quad a_n = (-1)^{n+1} \qquad n = 1, 2, 3, 4, \ldots$$

(c) We want : $a_1 = 1$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{4}$, $a_4 = -\frac{1}{8}$, ...

$$a_n = \left(-\frac{1}{2}\right)^{n-1} \qquad n = 1, 2, 3, 4, \ldots$$

Example 2:

Consider the sequence given by

$$a_n = 2 + \frac{(-1)^n}{n} \qquad n > 1.$$

List the first six terms of the sequence and plot them on the Cartesian plane.

$$a_n = 2 + \frac{(-1)^n}{n} \qquad n > 1$$

notice that the expression does <u>not</u> make sense for $n = 0$.

$$a_1 = 2 + \frac{(-1)^1}{1} = 2 - 1 = \underline{1}$$

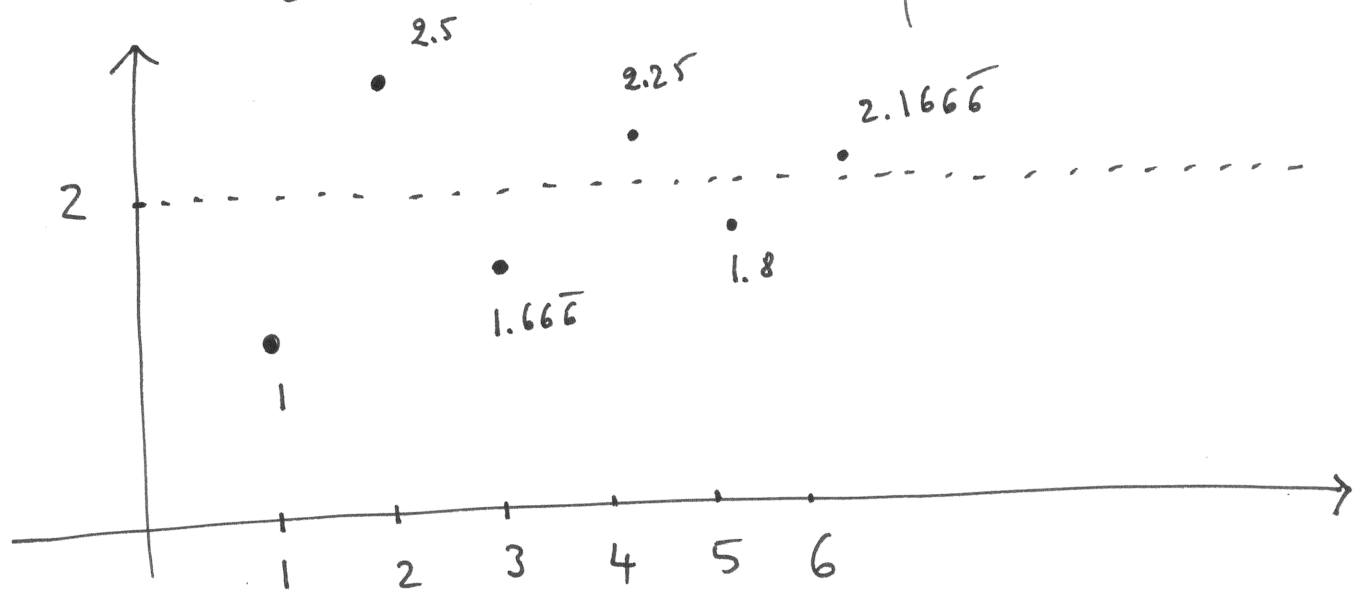
$$a_2 = 2 + \frac{(-1)^2}{2} = \underline{2.5}$$

$$a_3 = 2 + \frac{(-1)^3}{3} = 2 - \tfrac{1}{3} = \underline{1.66\bar{6}}$$

$$a_4 = 2 + \frac{(-1)^4}{4} = \underline{2.25}$$

$$a_5 = 2 + \frac{(-1)^5}{5} = \underline{1.8}$$

$$a_6 = 2 + \frac{(-1)^6}{6} = 2.166\bar{6}$$



Recursions (or Recursive Sequences)

The exponential growth model we considered earlier

$$P_n = 3 \cdot 2^n$$

is an example of a sequence. Explicitly, we have

$$P_0 = 3, \quad P_1 = 6, \quad P_2 = 12, \quad P_3 = 24, \quad P_4 = 48, \quad \cdots$$

It is not difficult to observe that this sequence of numbers describes quantities that double after each unit of time.
More explicitly, we can write

$$P_1 = 2P_0, \qquad P_2 = 2P_1, \quad P_3 = 2P_2, \quad P_4 = 2P_3, \quad \cdots$$

We can summarize the above facts into a single expression. I.e.,

$$P_{n+1} = 2P_n$$

this expression gives a rule that is applied repeatedly to go from one time step (the nth) to the next one (the $(n+1)$st).
Such an expression is called a **recursion**.

Example 3:

(a) List the first five terms of the recursively define sequence

$$a_0 = 1 \qquad a_{n+1} = (n+1)a_n.$$

Do you see something familiar?

(b) List the first five terms of the recursively define sequence

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = 1 + \frac{1}{a_n}.$$

Do you see something familiar?

Caution: While it is easy to compute terms in a recursive relation, there are 2 issues:

- In order to find a_{100}, we have to compute the previous 99 terms.
- We may not get a feeling for what will eventually happen.

$a_0 = 1 \qquad a_{n+1} = (n+1)\, a_n \qquad n = 0, 1, 2, 3, \ldots$

when $n = 0$ $\quad a_1 = 1 \cdot a_0 = 1$

when $n = 1$ $\quad a_2 = (1+1)\, a_1 = 2 \cdot 1 = 2!$

when $n = 2$ $\quad a_3 = (2+1)\, a_2 = 3 \cdot a_2 = 3 \cdot 2 \cdot 1 = 3!$

when $n = 3$ $\quad a_4 = (3+1)\, a_3 = 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$

when $n = 4$ $\quad a_5 = (4+1)\, a_4 = 5 \cdot a_4 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

In general the explicit form for the sequence is: $\boxed{a_n = n!}$ for $n = 0, 1, 2, \ldots$

$a_1 = 1 \qquad a_{n+1} = 1 + \frac{1}{a_n} \quad \text{for } n = 1, 2, 3, 4, 5, \ldots$

when $n = 1$ $\quad a_2 = 1 + \frac{1}{a_1} = 1 + \frac{1}{1} = \underline{\underline{2}}$

when $n = 2$ $\quad a_3 = 1 + \frac{1}{a_2} = 1 + \frac{1}{2} = \underline{\underline{3/2}} \cong 1.5$

when $n = 3$ $\quad a_4 = 1 + \frac{1}{a_3} = 1 + \frac{1}{3/2} = 1 + \frac{2}{3} = \underline{\underline{\frac{5}{3}}} \cong 1.\overline{666}$

when $n = 4$ $\quad a_5 = 1 + \frac{1}{a_4} = 1 + \frac{1}{5/3} = 1 + 3/5 = \underline{\underline{\frac{8}{5}}} \cong 1.6$

when $n = 5$ $\quad a_6 = 1 + \frac{1}{a_5} = 1 + \frac{1}{8/5} = 1 + \frac{5}{8} = \underline{\underline{\frac{13}{8}}} \cong 1.625$

this sequence is given by the quotient of 2 consecutive Fibonacci's numbers

when $n \to \infty$ this ratio tends to $1.618 \approx \frac{1+\sqrt{5}}{2}$

GOLDEN RATIO

Spreedsheets to Calculate Recursive Sequences

Using a spreadsheet it is possible to quickly calculate many terms in any sequence that is defined by a recurrence equation. We will explain how to do this calculation, using the specific recursive sequence of Example 3(b), that is:

$$a_1 = 1, \quad a_{n+1} = 1 + \frac{1}{a_n} \qquad (*)$$

We will use the column **A** of the spreadsheet to store the values of the index n for each term in the sequence and column **B** to store the values of the sequence a_n. Use the cells **A1** and **B1** to label the columns **n** and **a_n** respectively, and cells **A2** and **B2** to enter the index (1) and value (1) for the first term (a_1) in the sequence. To generate the next row we need to use the recursion equation (*).

In cell **A3** enter 2 (the index) and in cell **B3** enter **=1+1/B2**, as shown in the picture below

	A	B
1	n	a_n
2	1	1
3	2	=1+1/B2
4		

The value of **B3** (a_2) will then be computed from the value of **B2** (a_1) as the recurrence equation requires. We can then use the spreadsheet's **Autofill** command to generate the further terms in the sequence. Select the last row of your table (i.e., the cells **A3** and **B3**). When you select them, these two cells will be highlighted and surrounded by a colored outline. In the bottom right corner of the outline is a small colored square.

	A	B	C
1	n	a_n	
2	1	1	
3	2	2	
4			

Click and hold on the square, and then drag down several rows, as shown below

	A	B	C
1	n	a_n	
2	1	1	
3	2	2	
4	3	1.5	
5	4	1.666667	
6	5	1.6	
7	6	1.625	
8	7	1.615385	
9	8	1.619048	
10	9	1.617647	
11	10	1.618182	
12	11	1.617978	
13			

The spreadsheet will automatically fill the new rows using the recursion formula. Specifically it fills **A4** with the index 3, **A5** with 4, and so on. More importantly, it will put the formula **=1+1/B3** in **B4**. Since **B3** holds the value a_2, **B4** will hold the value $1 + 1/a_2$, which is our formula for a_3; **B5** gets filled with the formula **=1+1/B4**, which gives $1 + 1/a_3$ the formula for a_4 . The number of terms that are calculated in the sequence is the number of rows that we pull down the fill-box.

Example 4: (Online Homework HW05, # 8)

(a) Find a recursive definition for the sequence $9, 11, 13, 15, 17, \ldots$ Assume the first term in the sequence is indexed by $n = 1$.

(b) Find a closed formula for the sequence $9, 11, 13, 15, 17, \ldots$ Assume the first term in the sequence is indexed by $n = 1$.

$9, 11, 13, 15, 17, \ldots$

every number is obtained from the previous one by adding <u>two</u>:

$a_1 = 9$, $a_2 = 11 = 9 + 2 = 9 + 2(1)$, $a_3 = 13 = 9 + 4 = 9 + 2(2)$, $a_4 = 15 = 9 + 6 = 9 + 2(3)$, $a_5 = 17 = 9 + 8 = 9 + 2(4)$

<u>Recursive</u>: $\boxed{a_1 = 9 \qquad a_{n+1} = a_n + 2}$ $\quad n = 1, 2, 3, \ldots$

<u>Explicit</u>: $\boxed{a_n = 9 + 2(n-1)}$ $\quad n = 1, 2, 3, 4, \ldots$

Recap

We gave two descriptions of sequences: explicit and recursive.

- An **explicit description** is of the form $a_n = f(n)$, $n = 0, 1, 2, \ldots$ where $f(n)$ is a function of n.
- A **recursive description** is of the form $a_{n+1} = g(a_n)$, $n = 0, 1, 2, \ldots$ where $g(a_n)$ is a function of a_n.

Remark 1:

In the above situation the value of a_{n+1} depends only on the value one time step back, namely, a_n. In this case the recursion is called a **first-order recursion** .

Remark 2:

The sequence defined by

$$a_0 = 1, \quad a_1 = 1, \quad a_{n+2} = a_n + a_{n+1} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

is an example of a **second-order recursion**.

Recursive Sequences in the Life Sciences

Recursive sequences (or **difference equations**) are often used in biology to model, for example, cell division and insect populations.

In this biological context we usually replace n by t, to denote time.

If we think of t as the current time, then $t+1$ is one unit of time into the future. We also use N_t to denote the population size.

Thus a first-order difference equation modeling population size has the form

$$N_{t+1} = f(N_t) \qquad t = 0, 1, 2, 3, \ldots$$

In this context we call f an **updating function** because f 'updates' the population from N_t to N_{t+1}.

Malthusian (or Exponential) Growth Model

One of our earlier examples can be rewritten as

$$N_{t+1} = 2N_t \quad N_0 = 3 \qquad \text{or} \qquad N_t = 3 \cdot 2^t.$$

This example is a special case of the so called **Malthusian Growth Model**, named after Thomas Malthus (1766-1834):

$$N_{t+1} = (1 + r)N_t$$

which says that the next generation is proportional to the population of the current generation.

It is typical to set $R = 1 + r$ so that the recursion becomes

$$N_{t+1} = RN_t.$$

This recursion has the following explicit form

$$N_t = N_0 R^t.$$

Hence the name of Exponential Growth Model.

Example 5: (Online Homework HW05, # 11)

(a) A population of herbivores satisfies the growth equation $y_{n+1} = 1.05y_n$, where n is in years. If the initial population is $y_0 = 6,000$, then determine the explicit expression of the population.

(b) A competing group of herbivores satisfies the growth equation $z_{n+1} = 1.06y_n$ If the initial population is $z_0 = 3,200$, then determine how long it takes for this population to double.

(c) Find when the two populations are equal.

(a) $y_n = 6{,}000\,(1.05)^n$

(b) $z_n = 3{,}200\,(1.06)^n$

we want to know n such that

$$\underline{3{,}200\,(1.06)^n} = z_n = \underline{2 \cdot 3{,}200}$$

i.e. we want $(1.06)^n = 2$

take log (or ln) of both sides

$$\log (1.06)^n = \log (2) \longrightarrow n = \frac{\log 2}{\log (1.06)}$$

$$\cong \underline{\underline{11.895}}$$

(c) We want to find n such that the two populations are equal:

$$6{,}000\,(1.05)^n = 3{,}200\,(1.06)^n$$

Rewrite as:

$$\frac{6{,}000}{3{,}200} = \frac{(1.06)^n}{(1.05)^n} \quad \text{OR} \quad \frac{15}{8} = \left(\frac{1.06}{1.05}\right)^n$$

Take log (or ln) of both sides

$$\log\left(\frac{15}{8}\right) = \log\left[\left(\frac{1.06}{1.05}\right)^n\right]$$

$$\Longrightarrow \quad n \log\left(\frac{1.06}{1.05}\right) = \log(15/8)$$

$$\therefore \quad n = \frac{\log(15/8)}{\log\left(\frac{1.06}{1.05}\right)} \cong \underline{\underline{66.3177}}$$

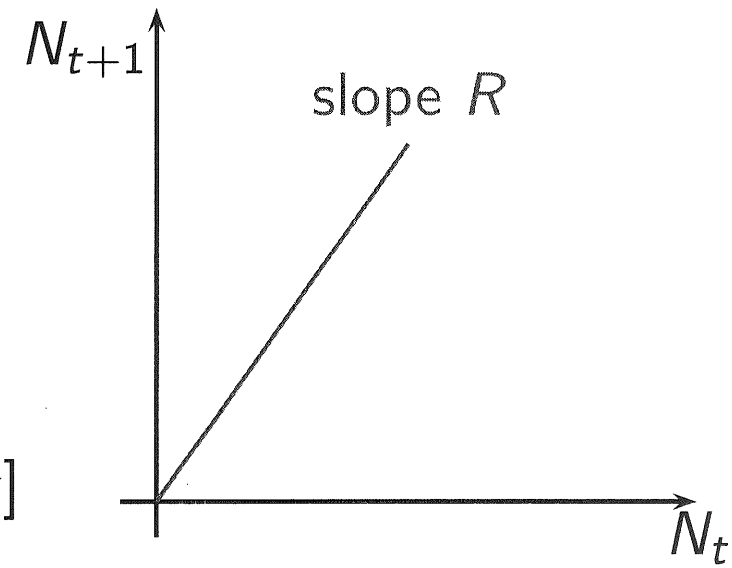
Visualizing Recursions

We can visualize recursions by plotting N_t on the horizontal axis and N_{t+1} on the vertical axis. Since $N_t \geq 0$ for biological reasons, we restrict the graph to the first quadrant.

The exponential growth recursion

$$N_{t+1} = RN_t$$

is then a straight line through the origin with slope R.
[i.e., $N_{t+1} = f(N_t)$, where $f(x) = Rx$]



For any current population size N_t, the graph allows us to find the population size in the next time step, namely, N_{t+1}.

Unless we label the points according to the corresponding t-value, we would not be able to tell at what time a point (N_t, N_{t+1}) was realized. We say that **time is implicit in this graph**.

The hallmark of exponential growth is that the ratio of successive population sizes, N_t/N_{t+1}, is constant. More precisely, it follows from $N_{t+1} = RN_t$ that

$$\frac{N_t}{N_{t+1}} = \frac{1}{R}$$

If the population consists of annual plants, we can interpret the ratio N_t/N_{t+1} as the **parent-offspring ratio**.

If this ratio is constant, parents produce the same number of offspring, regardless of the current population density. Such growth is called **density independent**.

When $R > 1$, the parent-offspring ratio, is less than 1, implying that the number of offspring exceeds the number of parents. This model yields then an ever-increasing population size. It eventually becomes **biologically unrealistic**, since any population will sooner or later experience food or habitat limitations that will limit its growth.

Below is the graph of the parent-offspring ratio N_t/N_{t+1} as a function of N_t when $N_t > 0$.