MA 137 – Calculus 1 with Life Science Applications
Limits
(Section 3.1)

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Computing a limit means computing what happens to the value of a function as the variable in the expression gets closer and closer to (but does not equal) a particular value.

Intuitive Definition

Let f be a function of x. The expression $\lim_{x \to c} f(x) = L$ means that as x gets closer and closer to c, through values both smaller and larger than c, but not equal to c, then the values of $f(x)$ get closer and closer to the value L.

Note 1: If $\lim_{x \to c} f(x) = L$ and L is a finite number, we say that the limit exists and that $f(x)$ **converges** to L as x tends to c.
If the limit does not exist, we say that $f(x)$ **diverges** as x tends to c.

Note 2: when finding the limit of $f(x)$ as x approaches c, we do not simply plug c into $f(x)$. (OK...often we do!)
In fact, we will see examples in which $f(x)$ is not even defined at $x = c$.
The value of $f(c)$ is irrelevant when we compute the value of $\lim_{x \to c} f(x)$.

Example 1:

Compute $\lim_{x \to 2} \frac{x^2+8}{x+2}$.

x gets close to 2 from the left				
x	1.8	1.9	1.99	1.999
$f(x) = \frac{x^2+8}{x+2}$				

x gets close to 2 from the right				
2.001	2.01	2.1	2.2	x
				$f(x) = \frac{x^2+8}{x+2}$

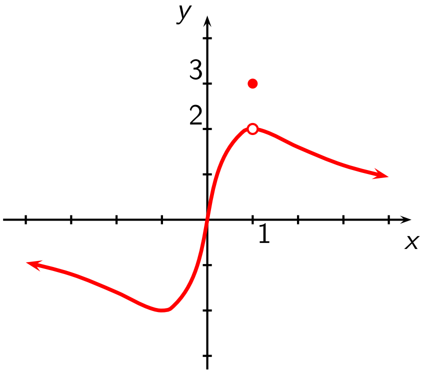
Example 2:

The graph of the function

$$g(x) = \begin{cases} \dfrac{4x}{x^2+1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$

is shown to the right.

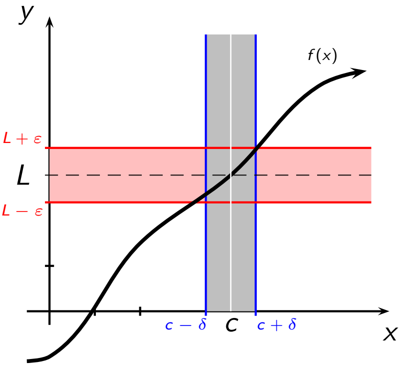
Compute $\displaystyle\lim_{x\to 1} g(x)$.



x	0.8	0.9	0.99	1.001	1.1	1.2
$g(x)$	1.95121	1.98895	1.9999	1.9999	1.99095	1.96721

Formal Definition

The statement $\lim_{x \to c} f(x) = L$ means that, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.



Limit Laws

The operations of arithmetic, namely, addition, subtraction, multiplication, and division, all behave reasonably with respect to the idea of getting closer to as long as nothing illegal happens.

This is summarized by the following laws:

If $\lim_{x \longrightarrow c} f(x)$ and $\lim_{x \longrightarrow c} g(x)$ exist and a is a constant, then

1. $\lim_{x \longrightarrow c} [f(x) + g(x)] = [\lim_{x \longrightarrow c} f(x)] + [\lim_{x \longrightarrow c} g(x)]$
2. $\lim_{x \longrightarrow c} [a\, f(x)] = a [\lim_{x \longrightarrow c} f(x)]$
3. $\lim_{x \longrightarrow c} [f(x) \cdot g(x)] = [\lim_{x \longrightarrow c} f(x)] \cdot [\lim_{x \longrightarrow c} g(x)]$
4. $\lim_{x \longrightarrow c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \longrightarrow c} f(x)}{\lim_{x \longrightarrow c} g(x)}$, provided $\lim_{x \longrightarrow c} g(x) \neq 0$

Theorem (**Substitution Theorem 1)**

If $p(x)$ is a polynomial, then $\lim_{x \longrightarrow c} p(x) = p(c)$.

Proof: A polynomial is a sum of terms, say $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$.

The result now follows from the Limit Laws:

$$\lim_{x \to c} p(x) = \lim_{x \to c} \left[a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \right]$$

(the limit of the sum is the sum of the limits)

$$= \lim_{x \to c} [a_n x^n] + \lim_{x \to c} [a_{n-1} x^{n-1}] + \cdots + \lim_{x \to c} [a_2 x^2] + \lim_{x \to c} [a_1 x] + \lim_{x \to c} [a_0]$$

(each of the terms is a product and the limit of the product is the product of the limits)

$$= \lim_{x \to c} [a_n] \lim_{x \to c} [x^n] + \lim_{x \to c} [a_{n-1}] \lim_{x \to c} [x^{n-1}] + \cdots + \lim_{x \to c} [a_2] \lim_{x \to c} [x^2] + \lim_{x \to c} [a_1] \lim_{x \to c} [x] + \lim_{x \to c} [a_0]$$

(each of these terms is either a constant or a power of x)

$$= a_n c^n + a_{n-1} c^{n-1} + \cdots + a_2 c^2 + a_1 c + a_0 = p(c)$$

Theorem (Substitution Theorem 2)

If $f(x)$ is a rational function, that is $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \longrightarrow c} f(x) = \lim_{x \to c} \frac{p(x)}{q(x)} = \frac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)} = \frac{p(c)}{q(c)} = f(c),$$

provided $q(c) \neq 0$.

The usual issue is that we often have to compute limits when these conditions are not met.

Example 3:

(a) Compute $\displaystyle\lim_{x\to 1} \frac{x^2 - 2x + 1}{x+1}$.

(b) Suppose $\displaystyle\lim_{x\to 3} f(x) = -2$ and $\displaystyle\lim_{x\to 3} g(x) = 4$. Determine

$$\lim_{x\to 3}\left[(x+1)\cdot f(x)^2 + \frac{x+2}{g(x)}\right]$$

When Limits Fail to Exist

There are two basic ways that a limit can fail to exist.

(a) The function attempts to approach multiple values as $x \to c$.

Geometrically, this behavior can be seen as a jump in the graph of a function.

Algebraically, this behavior typically arises with piecewise defined functions.

(b) The function grows without bound as $x \to c$.

Geometrically, this behavior can be seen as a vertical asymptote in the graph of a function.

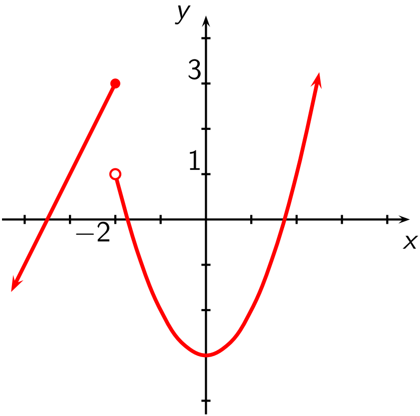
Algebraically, this behavior typically arises when the denominator of a function approaches zero.

Example 4:

The graph of the function

$$h(x) = \begin{cases} x^2 - 3 & \text{if } x > -2 \\ 2x + 7 & \text{if } x \leq -2 \end{cases}$$

is shown to the right.



Analyze $\lim_{x \to -2} h(x)$.

One-sided Limits

The previous example brings us to the following notions:

One-sided limits

A one-sided limit expresses what happens to the values of an expression as the variable in the expression gets closer and closer to some particular value c from either the left on the number line (that is, through values less than c) or from the right on the number line (that is, through values greater than c).
The notation is:

$$\underbrace{\lim_{x\to c^-} f(x)}_{\text{limit from the left of } c} \qquad \underbrace{\lim_{x\to c^+} f(x)}_{\text{limit from the right of } c}$$

Fact: $\lim_{x\to c} f(x)$ exists if and only if

both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and have the same value.

Example 5:

(a) Analyze $\displaystyle \lim_{x\to 1} \frac{5}{(x-1)^2}$.

(b) Analyze $\displaystyle \lim_{x\to 1} \frac{2}{x-1}$.

(c) Analyze the limit $\displaystyle \lim_{x\to 0} \frac{2}{\sqrt{x}}$.

The most interesting and important situation with limits is when a substitution yields $0/0$.

The result $0/0$ yields absolutely no information about the limit. It does not even tell us that the limit does not exist. The only thing it tells us is that we have to do more work to determine the limit.

Example 6:

Find the limit $$\lim_{x\to 3} \frac{x^2 - 2x - 3}{x - 3}.$$

Example 7: (Online Homework HW07, #5)

Guess the value of the limit (if it exists) by evaluating the function at values close to where the limit is to be done.

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

Example 8:

Find the limits

$$\lim_{x \to 0^+} \frac{|x|}{x} \qquad \lim_{x \to 0^-} \frac{|x|}{x} \qquad \lim_{x \to 0} \frac{|x|}{x}.$$

Example 9: (Neuhauser, Example 9, p. 97)

Find the limit $$\lim_{x \to 0} \frac{\sqrt{x^2+16}-4}{x^2}$$