MA 137 – Calculus 1 with Life Science Applications
Limits at Infinity & Properties of Continuous Functions
(Sections 3.3 & 3.5)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

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Asymptotic Behavior

When studying sequences $\{a_n\}$, say in the context of populations over time, we were interested in their long-term behavior: $\lim_{n \longrightarrow \infty} a_n$.

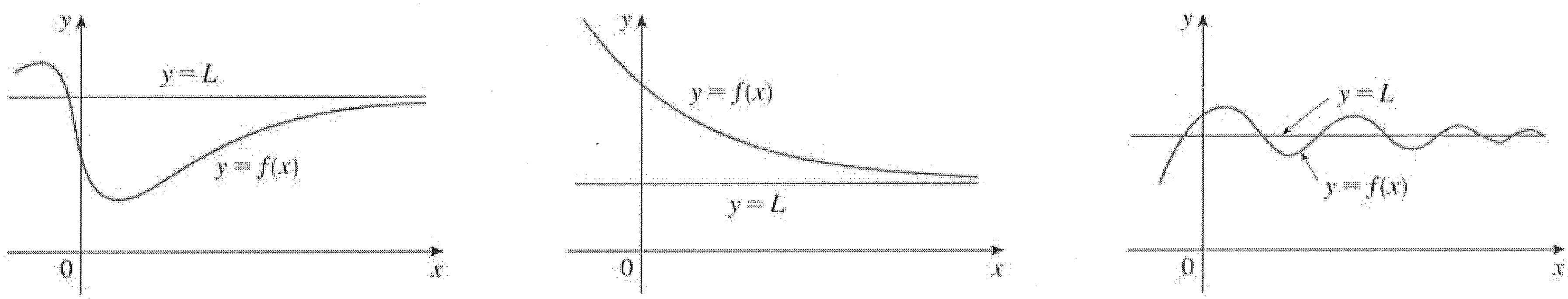
Now we do something similar: We ask what happens to function values $f(x)$ when x becomes large. Now, x is no longer restricted to be an integer.

Definition

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L whenever x is sufficiently large.



- A similar definition holds for limits where x tends to $-\infty$.
- The Limit Laws that we discussed earlier also hold as x tends to $\pm\infty$.
- It is easy to convince yourself that

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0.$$

From the Limit Laws it also follows that

$$\lim_{x \to \pm\infty} \frac{1}{x^p} = 0 \qquad \text{for any integer } p > 0.$$

Example 1:

Evaluate

$$\lim_{x\to\infty} \frac{1-x+2x^2}{3x-5x^2} \qquad \lim_{x\to\infty} \frac{1-x^3}{1+x^5}$$

$$\lim_{x\to\infty} \frac{1-x+2x^2}{3x-5x^2}$$ = if we use directly the Rules for Limits we obtain $= \frac{\infty}{-\infty}$

Thus we need to rewrite first the fraction by dividing top and bottom by x^2:

$$= \lim_{x\to\infty} \frac{\frac{1-x+2x^2}{x^2}}{\frac{3x-5x^2}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x^2} - \frac{1}{x} + 2}{\frac{3}{x} - 5}$$

= now we can use the Rules for Limits

$$= \frac{\left(\lim_{x\to\infty} \frac{1}{x^2}\right) - \left(\lim_{x\to\infty} \frac{1}{x}\right) + \lim_{x\to\infty} 2}{\left(\lim_{x\to\infty} \frac{3}{x}\right) - \lim_{x\to\infty} 5} = \frac{2}{-5}$$

$$= \boxed{-0.4}$$

As before

$$\lim_{x\to\infty} \frac{1-x^3}{1+x^5} = \frac{-\infty}{\infty} = \text{hence we rewrite}$$

$$= \lim_{x\to\infty} \frac{\frac{1-x^3}{x^3}}{\frac{1+x^5}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x^3}-1}{\frac{1}{x^3}+x^2} =$$

$$= \frac{\left(\lim_{x\to\infty}\frac{1}{x^3}\right) - \lim_{x\to\infty} 1}{\left(\lim_{x\to\infty}\frac{1}{x^3}\right) + \lim_{x\to\infty} x^2} = \frac{-1}{\infty} = \boxed{0}$$

General Fact

If $f(x)$ is a rational function of the form $f(x) = p(x)/q(x)$, where $p(x)$ is a polynomial of degree $\deg(p)$ and $q(x)$ is a polynomial of $\deg(q)$, then:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg(p) < \deg(q) \\ L \neq 0 & \text{if } \deg(p) = \deg(q) \\ \text{does not exist} & \text{if } \deg(p) > \deg(q) \end{cases}$$

Here, L is a real number that is the ratio of the coefficients of the leading terms in the numerator and denominator.

The same behavior holds as $x \to -\infty$.

Example 2: (Online Homework HW09, # 5)

A function is said to have a **horizontal asymptote** if either the limit at infinity exists or the limit at negative infinity exists. Show that each of the following functions has a horizontal asymptote by calculating the given limit.

$$\lim_{x \to \infty} \frac{\sqrt{x^2+2x}}{8-7x} \qquad \lim_{x \to -\infty} \frac{\sqrt{x^2+2x}}{8-7x}.$$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2x}}{8-7x} = \lim_{x\to\infty} \frac{\left(\sqrt{x^2+2x}\right)\frac{1}{x}}{(8-7x)\frac{1}{x}} =$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{x^2+2x}{x^2}}}{\frac{8}{x}-7} = \lim_{x\to\infty} \frac{\sqrt{1+2/x}}{\frac{8}{x}-7}$$

$$= \frac{\sqrt{1+\lim_{x\to\infty} 2/x}}{\left(\lim_{x\to\infty}\frac{8}{x}\right)-7} = \frac{1}{-7} = \boxed{-1/7}$$

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we can bring inside the sqrt only positive values

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+2x}}{8-7x} = \text{tricky point} = \lim_{x\to-\infty} \frac{\left(\sqrt{x^2+2x}\right)\left(-\frac{1}{x}\right)(-1)}{(8-7x)\frac{1}{x}}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{1+\frac{2}{x}}}{8/x-7} = \frac{-\sqrt{1+\lim_{x\to-\infty}\frac{2}{x}}}{\left(\lim_{x\to-\infty}\frac{8}{x}\right)-7} = \boxed{\frac{1}{7}}$$

Rational functions are not the only functions that involve limits as $x \to \infty$ (or $x \to -\infty$).

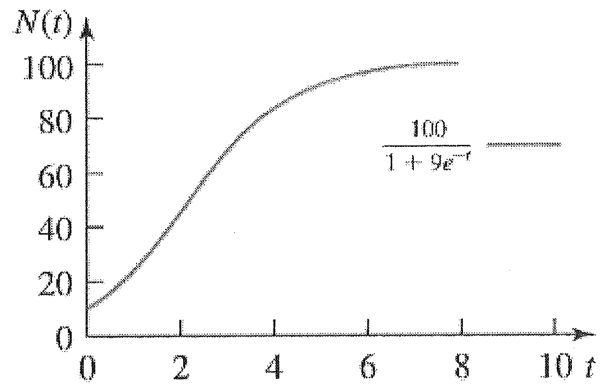
Many important applications in biology involve exponential functions. We will use the following result repeatedly — it is one of the most important limits:

$$\lim_{x \to \infty} e^{-x} = 0.$$

Another formulation of the same result is:

$$\lim_{x \to -\infty} e^{x} = 0.$$

For example: $$\lim_{t \to \infty} \frac{100}{1 + 9e^{-t}} = 100$$



Example 3: (Neuhauser, Example 3, p. 112)

The **logistic curve** describes the density of a population over time, where the rate of growth depends on the population size. It is characterized by the fact that *the per capita rate of growth decreases linearly with increasing population size.* If $N(t)$ denotes the size of the population at time t, then the logistic curve is given by

$$N(t) = \frac{K}{1 + (\frac{K}{N_0} - 1)e^{-rt}} \qquad \text{for} \quad t \geq 0.$$

The parameters K and r are positive numbers that describe the population dynamics.

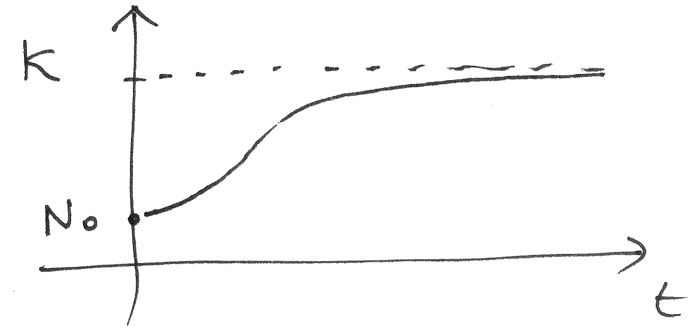
If we seek the long-term behavior of the population as it evolves in accordance with the logistic growth curve, we find that

$$\lim_{t \to \infty} N(t) = K.$$

That is, as $t \to \infty$, the population size approaches K, which is called the **carrying capacity** of the population.

logistic growth function

$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}}$$



eg: $N(t) = \dfrac{100}{1 + 9e^{-rt}}$ with $K = 100$, $N_0 = 10$, $r = 1 = 100\%$

Notice

$$N(0) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) \underbrace{e^0}_{1}} = \frac{K}{\cancel{1} + \frac{K}{N_0} - \cancel{1}} = \underline{\underline{N_0}}$$

Also

$$\lim_{t \to \infty} \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}} = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) \underbrace{\lim_{t \to \infty} e^{-rt}}_{0}}$$

$$= \frac{K}{1 + 0} = \boxed{K}$$ carrying capacity.

Example 4:

The **von Bertalanffy growth function**

$$L(t) = L_\infty - (L_\infty - L_0)e^{-kt}$$

where k is a positive constant, models the length L of a fish as a function of t, the age of fish. This model assume that the fish has a well defined length L_0 at birth ($t = 0$).

Calculate $\lim_{t \to \infty} L(t)$.

$$L(t) = L_\infty - (L_\infty - L_0)e^{-kt}$$

e.g. $\quad L(t) = 10 - 8e^{-0.2t}$

$L_\infty = 10$

$L_0 = 2$

$k = 0.2 = 20\%$

Notice that

$$L(0) = L_\infty - (L_\infty - L_0)\underbrace{e^0}_{1} = \cancel{L_\infty} - \cancel{L_\infty} + L_0$$

$$= L_0$$

$$\lim_{t\to\infty} L_\infty - (L_\infty - L_0)e^{-kt} = L_\infty - (L_\infty - L_0)\underbrace{\lim_{t\to\infty} e^{-kt}}_{0}$$

$$= \boxed{L_\infty}$$

asymptotic length of the fish

A Spiritual Journey

- One morning, at sunrise, a (tibetan) monk began to climb a tall mountain from his monastery. The narrow path, no more than a foot or two wide, spiraled around the mountain to a glittering temple at the summit. The monk ascended the path at varying rates of speed, stopping many times along the way to rest and to eat the dried fruit he carried with him or to look at the flowers. He reached the temple at sunset.

  After several days of fasting and meditation, he began his journey back along the same path, starting at sunrise and again walking at varying speeds with many pauses along the way. He reached the bottom at sunset.

- I assert that there is at least one spot along the path the monk occupied at precisely the same time of day on both trips.?

- Is my assertion true? How do you decide?

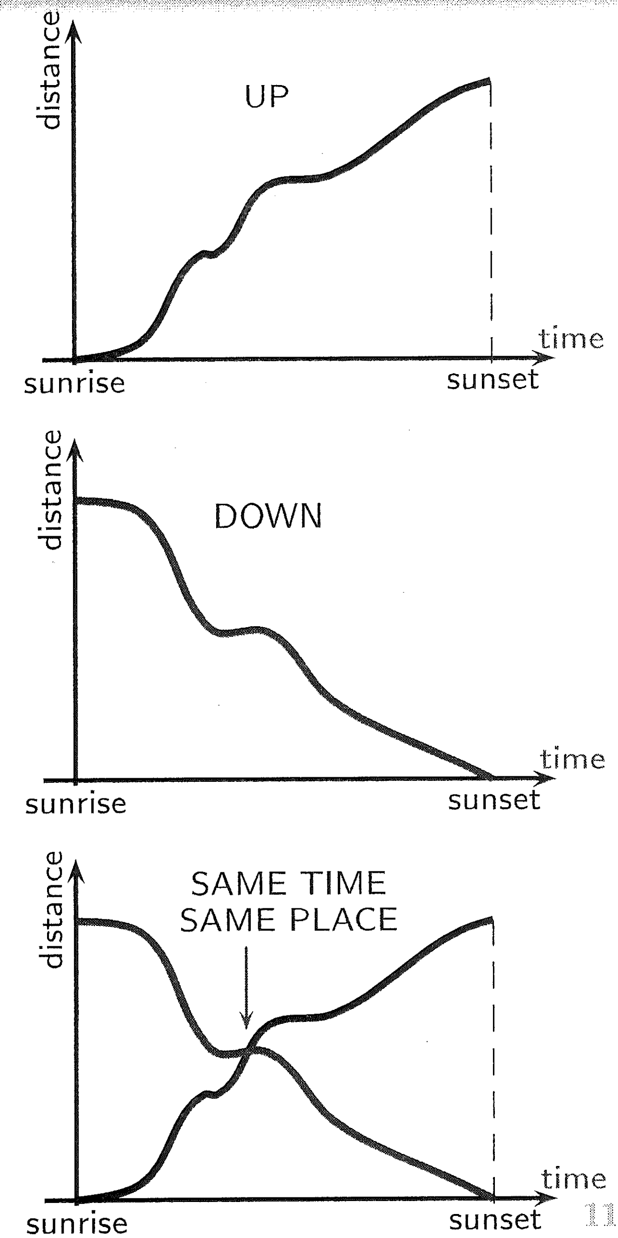
Visual Thinking: the monk and the mountain

The monk travels along the same path on both days and his position is determined by the distance from the monastery.

Position is a continuous function of time.

If we plot the path up the mountain in a time-distance coordinate system, then the curve goes from (sunrise, monastery) to (sunset, temple). Flat regions on the graph are rest times, dips arise from, say, retracing his steps to look at a flower. The path down the mountain is a curve from the point (sunrise, temple) to (sunset, monastery).

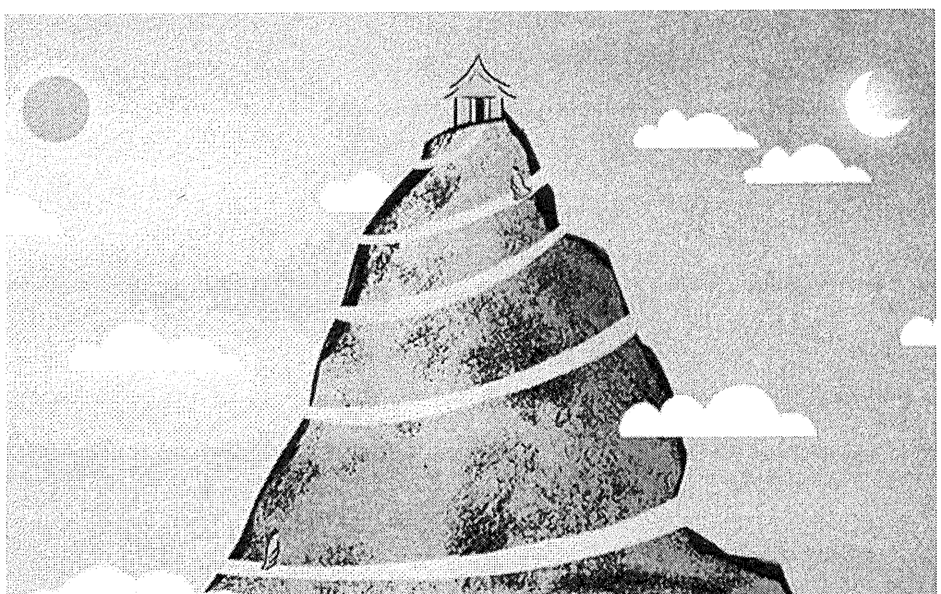
When the two paths are plotted on the same axes, it is obvious that the curves intersect — this is a point where the monk is at the same point at the same time on the two days.



Two Monks and the Mountain

There is an insightful solution to the problem that is equivalent to the previous graphical one, but there is no need to graph a plot. One monk and two days does not make the solution as transparent as it could be. So we look at a similar, equivalent problem.

Suppose there are two monks and they both start at sunrise, one at the bottom of the mountain (≡ monastery), the other at the top (≡ temple). It seems obvious now that the two monks must meet somewhere along the path — at the same time and at the same place.



Mathematical Thinking

Let $dist_{\text{up}}(t)$ represent the distance the monk is away from the monastery on day 1, where t represents any time between sunrise and sunset.
Note that $dist_{\text{up}}(\text{sunrise}) = 0$ and $dist_{\text{up}}(\text{sunset}) = d$, the distance from the monastery to the temple on the top of the mountain.

Let $dist_{\text{down}}(t)$ represent the distance the monk is away from the monastery on day 2, where t represents any time between sunrise and sunset. Note that $dist_{\text{down}}(\text{sunrise}) = d$ and $dist_{\text{down}}(\text{sunset}) = 0$.

Observe that these functions are continuous, since they correspond to the path that the monk is walking.

Now consider the function $f = dist_{\text{up}} - dist_{\text{down}}$.
Since $dist_{\text{up}}$ and $dist_{\text{down}}$ are continuous, so is the function f.

Observe that $f(\text{sunrise}) = -d$ and $f(\text{sunset}) = d$. Because f is continuous there must be a value t_0 between sunrise and sunset such that $f(t_0) = 0$.

$$f(t_0) = 0 \iff [dist_{\text{up}} - dist_{\text{down}}](t_0) = 0 \iff dist_{\text{up}}(t_0) = dist_{\text{down}}(t_0).$$

Thus t_0 is the time when the monk is at the same point at the same time on the two days.

The Intermediate Value Theorem (IVT)

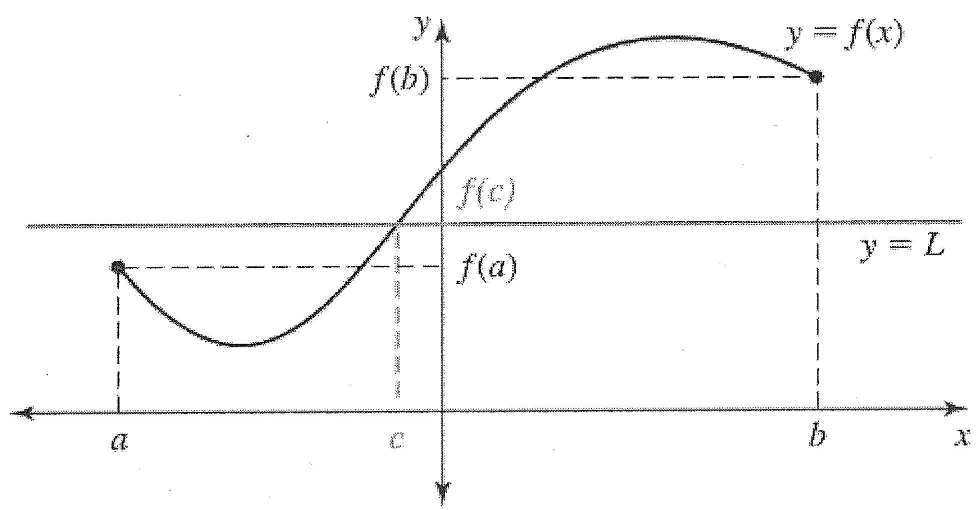
The previous story (≡ the monk and the mountain) represents an illustration of the content of the following result.

The Intermediate Value Theorem (B. Bolzano, 1817)

Suppose that f is **continuous** on the **closed** interval $[a, b]$.

If L is any real number with $f(a) < L < f(b)$ [or $f(b) < L < f(a)$],

then there exists **at least one** value $c \in (a, b)$ such that $f(c) = L$.



- In applying the Intermediate Value Theorem, it is important to check that f is continuous.
- Discontinuous functions can easily miss values; for example, the floor function misses all numbers that are not integers.
- The Intermediate Value Theorem gives us only the existence of a number c; it does not tell us how many such points there are or where they are located.
- As an application, the Theorem can be used to find approximate roots (or solutions) of equations of the form $f(x) = 0$.

Example 5:

Use the Intermediate Value Theorem to conclude that $e^{-x} = x$ has a solution in $[0, 1]$

$$e^{-x} = x \iff e^{-x} - x = 0$$

So consider the function $\boxed{f(x) = e^{-x} - x}$

It is a continuous function because it is the difference of two continuous functions.

In particular it is continuous for $x \in [0,1]$.

Notice that $f(0) = e^{-0} - 0 = e^{0} = \underline{\underline{1}}$

$$f(1) = e^{-1} - 1 = \frac{1}{e} - 1 = -0.6321$$

Thus there must be a value c in $(0,1)$ such that $f(c) = 0$ ∴ i.e. $e^{-c} - c = 0$

c is a root of our equation (by the IVT)

Example 6: (Online Homework HW09, #9)

Determine if the Intermediate Value Theorem implies that the equation $x^3 - 3x - 3.9 = 0$ has a root in the interval $[0, 1]$.

Determine if the Intermediate Value Theorem implies that the equation $x^3 - 3x + 1.2 = 0$ has a root in the interval $[0, 1]$.

(a) Consider $f(x) = x^3 - 3x - 3.9$ on $[0, 1]$

The function f is continuous on $[0, 1]$ because it is a polynomial.

$f(0) = -3.9$ $\quad$ $f(1) = 1 - 3 - 3.9 = -5.9$

Hence the IVT does not apply. We cannot conclude whether there is a root of $x^3 - 3x - 3.9$ on $[0, 1]$

(b) Consider $g(x) = x^3 - 3x + 1.2$ on $[0, 1]$

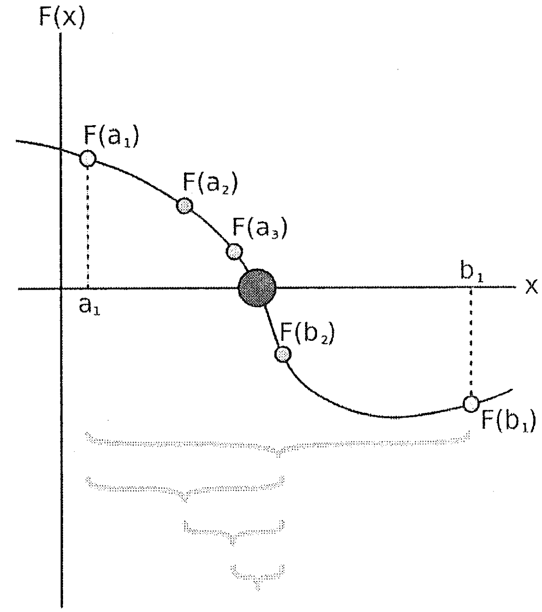
g is a continuous function: $g(0) = 1.2$

and $g(1) = 1 - 3 + 1.2 = -0.8$

Hence, by the IVT, there is a $c \in (0, 1)$ such that $g(c) = 0$ i.e. $c^3 - 3c + 1.2 = 0$

Bisection Method

The bisection method is used for numerically finding a root of the equation $f(x) = 0$, where f is a continuous function defined on an interval $[a, b]$ and where $f(a)$ and $f(b)$ have opposite signs.



In this method one repeatedly bisects an interval and then selects a subinterval in which a root must lie for further inspection.

At each step the method divides the interval in two by computing the midpoint $c = (a + b)/2$ of the interval and the value of the function $f(c)$ at that point.

Unless c is itself a root (which is very unlikely, but possible) there are only two possibilities: either $f(a)$ and $f(c)$ have opposite signs and $[a, c]$ contains a root, or $f(c)$ and $f(b)$ have opposite signs and $[c, b]$ contains a root.

In this way an interval that contains a zero of f is reduced in width by 50% at each step. The process is repeated until the interval is sufficiently small.

Example 7: (Online Homework HW09, #10)

Carry out three steps of the Bisection Method for $f(x) = 2^x - x^4$ as follows:

(a) Show that $f(x)$ has a zero in $[1, 2]$.

(b) Determine which subinterval, $[1, 1.5]$ or $[1.5, 2]$, contains a zero.

(c) Determine which interval, $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, or $[1.75, 2]$, contains a zero.

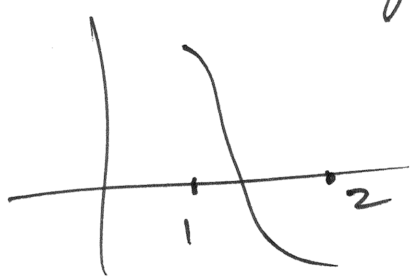
$f(x) = 2^x - x^4$

is a continuous function for all values of x as it is the difference of two continuous functions.

In particular it is continuous on $[1, 2]$.

Notice $f(1) = \underline{1}$ and $f(2) = 2^2 - 2^4 = \underline{-12}$

Thus the graph of f must cross the x-axis at some point $c \in (1, 2)$



Consider the midpoint $x = 1.5$

$f(1.5) = 2^{1.5} - (1.5)^4 = -2.23$

Hence the root c lies in the interval $(1, 1.5)$

Let's keep computing middle points.

The next is $\frac{1+1.5}{2} = 1.25$

Now, $f(1.25) = 2^{1.25} - (1.25)^4 = -0.0629$

Hence the root $\boxed{c \in (1, 1.25)}$

WE KEEP GOING

The next midpoint is $\frac{1+1.25}{2} = 1.125$

and $f(1.125) = 0.5792$

Thus $c \in (1.125, 1.25)$.

Now, the next midpoint is $\frac{1.125+1.25}{2} = 1.1875$

and $f(1.1875) = 0.2890$ So $c \in (1.1875, 1.25)$

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