MA 137 – Calculus 1 with Life Science Applications
The Power Rule, the Basic Rules of Differentiation, and the Derivatives of Polynomials
(Section 4.2)

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Basic Rules

Since polynomials and rational functions are built up by the basic operations of addition, subtraction, multiplication, and division operating on power functions of the form $y = x^n, n = 0, 1, 2, \ldots,$ we need differentiation rules for such operations.

Theorem

Suppose c is a constant, n is a positive integer, and f(x) and g(x) are differentiable functions. Then the following relationships hold:

0. $$\frac{d}{dx}[c] = 0$$

1. $$\frac{d}{dx}[cf(x)] = c\,\frac{d}{dx}f(x)$$

2. $$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

3. $$\frac{d}{dx}[x^n] = nx^{n-1}$$

Example 1: (Neuhauser, Problem # 6, p. 149)

Differentiate $f(x) = -1 + 3x^2 - 2x^4$ with respect to x.

* $f(x) = -1 + 3x^2 - 2x^4$

* $\frac{d}{dx} f(x) = \frac{d}{dx}(-1) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(-2x^4)$

by property 2. —

$= 0 + 3 \cdot \frac{d}{dx}(x^2) - 2 \cdot \frac{d}{dx}(x^4)$

by property 1. —

$= 3(2x) - 2 \cdot (4x^3)$

by property 3. (power rule)

$= \boxed{6x - 8x^3}$

Example 2: (Neuhauser, Problem # 32, p. 150)

Differentiate

$$f(N) = \frac{bN^2 + N}{K + b}$$

with respect to N. Assume that b and K are positive constants.

$$f(N) = \frac{bN^2 + N}{K + b}$$

$$= \left(\frac{b}{K+b}\right) N^2 + \left(\frac{1}{K+b}\right) \cdot N$$

$$f'(N) = \frac{d}{dN} f(N) = \left(\frac{b}{K+b}\right) \cdot 2N + \left(\frac{1}{K+b}\right) \cdot 1$$

$$= \boxed{\frac{2bN + 1}{K+b}}$$

Example 3: (Neuhauser, Problem # 38, p. 150)

Differentiate

$$g(N) = rN\left(1 - \frac{N}{K}\right)$$

with respect to N. Assume that K and r are positive constants.

$$g(N) = rN\left(1 - \frac{N}{K}\right)$$ can be rewritten as

$$= rN - \frac{r}{K}N^2$$

Hence

$$g'(N) = \frac{dg}{dN} = r \cdot 1 - \frac{r}{K} \cdot 2N$$

$$= r - \frac{2r}{K}N$$

Example 4: (Neuhauser, Problem # 56, p. 150)

Find the tangent line to

$$f(x) = cx^3 - 2cx$$

at $x = -1$. Assume that c is a positive constant.

$f(x) = c x^3 - 2cx$

We need to find the tangent line at the point where $x = -1$.

* Hence : $P(-1, f(-1)) = \boxed{(-1, c)}$

as $f(-1) = c(-1)^3 - 2c(-1) = -c + 2c = c$

* Now, for the derivative at $x = -1$:

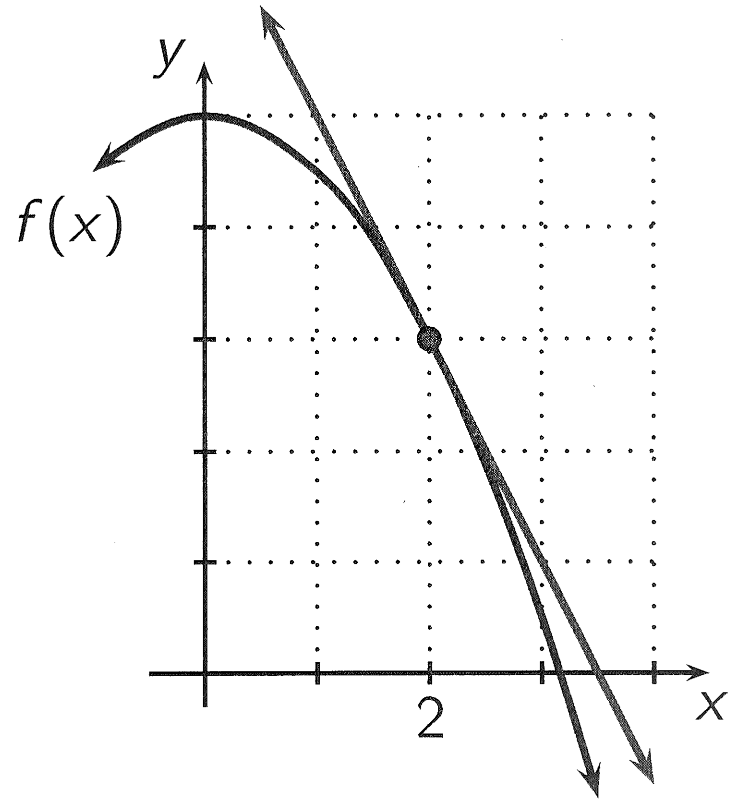
$f'(x) = 3cx^2 - 2c$

$f'(-1) = 3c(-1)^2 - 2c = 3c - 2c = c$

* Eq of tg. line : $\boxed{y - c = c(x+1)}$ or $\boxed{y = cx + 2c}$

Example 5:

A segment of the tangent line to the graph of $f(x)$ at x is shown in the picture. Using information from the graph we can estimate that



$$f(2) = \_\_\_\_\_\_ \qquad f'(2) = \_\_\_\_\_\_$$

hence the equation to the tangent line to the graph of

$$g(x) = 5x + f(x)$$

at $x = 2$ can be written in the form $y = mx + b$ where

$$m = \_\_\_\_\_\_ \qquad b = \_\_\_\_\_\_.$$

* From the graph: $f(2) = 3$ $f'(2) = -\frac{4}{2} = -2$

* $g(x) = 5x + f(x)$

At $x=2$; $g(2) = 5 \cdot 2 + f(2) = 10 + 3 = 13$

About the derivative of g: $g'(x) = 5 + f'(x)$

so that $g'(2) = 5 + f'(2) = 5 - 2 = 3$

Therefore the equation of the tg. line to the graph of g at $(2,13)$ is:

$$\boxed{y - 13 = 3(x-2)} \quad \text{OR} \quad \boxed{y = 3x + 7}$$

Example 6: (Online Homework HW12, # 11)

Lizards are cold-blooded animals whose temperatures roughly match the surrounding environment. Suppose the body temperature, $T(t)$, of a lizard is measured for a period of 18 hours from midnight until 6 PM. The body temperature (in $^\circ C$) of the lizard over this period of time (in hours) is found to be well approximated by the polynomial

$$T(t) = -0.009t^3 + 0.29t^2 - 1.7t + 15.5.$$

(a) Find the general expression for the rate of change of body temperature per hour, $T'(t)$.

(b) Use this information to find what the rate of change of body temperature is at: midnight; 4 AM; 8 AM; noon; 4 PM.

(c) Which of these times gives the fastest increase in the body temperature and which shows the most rapid cooling of the lizard?

$T(t) = -0.009t^3 + 0.29t^2 - 1.7t + 15.5$

temperature of the lizard

(a) $\boxed{T'(t) = -0.027t^2 + 0.58t - 1.7}$

(b) $T'(0) = -1.7$

$T'(4) = 0.188$

$T'(8) = 1.212$

$T'(12) = 1.372$

$T'(16) = 0.668$

(c) Fastest increase at <u>noon</u> : <u>$T'(12) = 1.372$</u>

Most rapid cooling at <u>midnight</u> <u>$T'(0) = -1.7$</u>

Proofs:

0. Define $f(x) = c$ and use the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

1. We use the definition of the derivative and one of the Limit Laws:

$$[cf(x)]' = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} = c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = cf'(x).$$

2. We use the definition of the derivative, rewrite the numerator and then use one of the Limit Laws:

$$\begin{aligned}
[f+g]'(x) &\overset{\text{def}}{=} \lim_{h\to 0} \frac{[f+g](x+h) - [f+g](x)}{h} \\
&\overset{\text{def}}{=} \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h} \\
&\overset{\text{rule}}{=} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).
\end{aligned}$$

Special product formulas: The powers of certain binomials occur so frequently that we should memorize the following formulas. We can verify them by performing the multiplications.

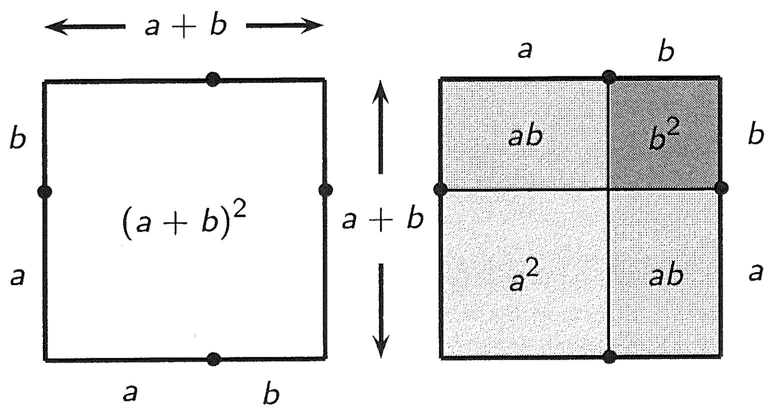
If A and B are any real numbers or algebraic expressions, then:

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
4. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Visualizing a formula:

Many of the special product formulas can be seen as geometrical facts about length, area, and volume. The ancient Greeks always interpreted algebraic formulas in terms of geometric figures.

For example, the figure below

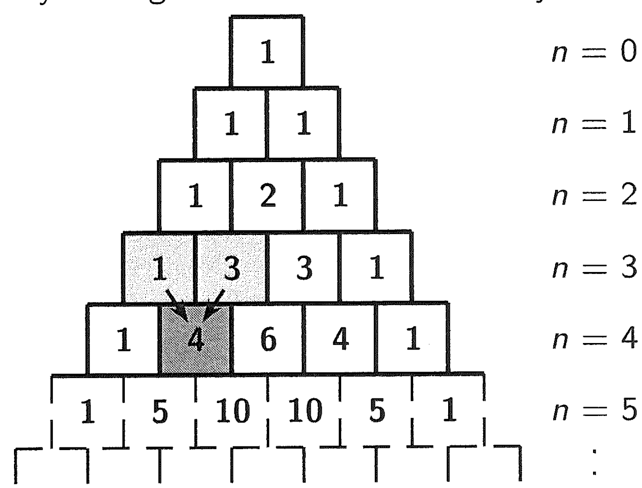


shows how the formula for the square of a binomial (formula **1**) can be interpreted as a fact about areas of squares and rectangles.

Pascal's triangle:

The coefficients (without sign) of the expansion of a binomial of the form $(a \pm b)^n$ can be read off the n-th row of the following 'triangle' named **Pascal's triangle** (after Blaise Pascal, a 17th century French mathematician and philosopher).

To build the triangle, start with '1' at the top, then continue placing numbers below it in a triangular way. Each number is simply obtained by adding the two numbers directly above it.



3. We use the definition of the derivative and the Binomial Theorem. The Binomial Theorem tells us

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

Let's now use the definition of the derivative with $f(x) = x^n$:

$$\begin{aligned}
f'(x) &\stackrel{\text{def}}{=} \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0}\frac{\{x^n + nx^{n-1}h + [n(n-1)]/2\,x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\} - x^n}{h} \\
&= \lim_{h\to 0}\frac{nx^{n-1}h + [n(n-1)]/2\,x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0}\frac{\{nx^{n-1} + [n(n-1)]/2\,x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\}h}{h} \\
&= \lim_{h\to 0}\{nx^{n-1} + [n(n-1)]/2\,x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\} \\
&= nx^{n-1}
\end{aligned}$$

3'. Define $f(x) = x^n$. We know from the alternate limit form of the definition of the derivative that the derivative $f'(x)$ is given by,

$$f'(x) = \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{x_1 \to x} \frac{x_1^n - x^n}{x_1 - x}.$$

Now we have the following formula,

$$x_1^n - x^n = (x_1 - x)(x_1^{n-1} + x x_1^{n-2} + x^2 x_1^{n-3} + \cdots + x^{n-3} x_1^2 + x^{n-2} x_1 + x^{n-1})$$

which we can verify by simply multiplying the two factors together.

Let's now use the alternative definition of the derivative with $f(x) = x^n$:

$$\begin{aligned}
f'(x) &\overset{\text{def}}{=} \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{x_1 \to x} \frac{x_1^n - x^n}{x_1 - x} \\
&= \lim_{x_1 \to x} \frac{(x_1 - x)(x_1^{n-1} + x x_1^{n-2} + x^2 x_1^{n-3} + \cdots + x^{n-3} x_1^2 + x^{n-2} x_1 + x^{n-1})}{x_1 - x} \\
&= \lim_{x_1 \to x} (x_1^{n-1} + x x_1^{n-2} + x^2 x_1^{n-3} + \cdots + x^{n-3} x_1^2 + x^{n-2} x_1 + x^{n-1}) \\
&= n x^{n-1} \qquad \text{[as there are } n \text{ equal terms in the expression]}
\end{aligned}$$