MA 137 – Calculus 1 with Life Science Applications
The Product and Quotient Rule
and the Derivatives of Rational and Power Functions
(Section 4.3)

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October 9, 2017

Basic Rules (cont'd)

Theorem

Suppose $f(x)$ and $g(x)$ are differentiable functions. Then the following relationships hold:

4. $$\frac{d}{dx}[f(x)\cdot g(x)] = \frac{d}{dx}[f(x)]\cdot g(x) + f(x)\cdot\frac{d}{dx}[g(x)]$$

(*in prime notation*) $(fg)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

5. $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]\cdot g(x) - f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

(*in prime notation*) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}$

The Power Rule for Negative Exponents

The quotient rule allows us to extend the power rule to the case where the exponent is a negative integer:

Theorem

If $f(x) = x^{-n}$, where n is a positive integer, then $f'(x) = -nx^{-n-1}$.

Proof: We write $f(x) = \dfrac{1}{x^n}$ and use the quotient rule

$$f'(x) = \frac{0 \cdot x^n - 1 \cdot nx^{n-1}}{[x^n]^2} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{(n-1)-2n} = -nx^{-n-1}.$$

There is a general form of the power rule in which the exponent can be any real number. In Section 4.4, we give the proof for the case when the exponent is rational; we prove the general case in Section 4.7.

Theorem (General Form)

If $f(x) = x^r$, where r is any real number, then $f'(x) = r\,x^{r-1}$.

Example 1: (Neuhauser, Example # 1, p. 153)

Differentiate $f(x) = (3x+1)(2x^2-5)$.

* $f(x) = (3x+1)(2x^2-5)$

Let's use the product rule

$$f'(x) = 3\cdot(2x^2-5) + (3x+1)(4x)$$

If we want to expand we get

$$f'(x) = 6x^2 - 15 + 12x^2 + 4x$$

$$= 18x^2 + 4x - 15$$

* We could also have multiply the factors in $f(x)$:

$$f(x) = 6x^3 - 15x + 2x^2 - 5$$

$$= 6x^3 + 2x^2 - 15x - 5$$

so that

$$f'(x) = 18x^2 + 4x - 15$$

Example 2: (Online Homework HW12, # 17)

Differentiate $Y(u) = (u^{-2} + u^{-3})(u^5 + u^2)$.

* We use the product and power rule for integer exponents.

$$Y(u) = (u^{-2} + u^{-3})(u^5 + u^2)$$

$$Y'(u) = \left[-2u^{-3} - 3u^{-4}\right](u^5 + u^2) + (u^{-2} + u^{-3})(5u^4 + 2u)$$

We could simplify it now....

* I personally would have simplified first $Y(u)$ as follows:

$$Y(u) = u^{-2}u^5 + u^{-2}u^2 + u^{-3}u^5 + u^{-3}u^2$$

$$= u^3 + 1 + u^2 + u^{-1}$$

Hence $Y(u) = u^3 + u^2 + 1 + u^{-1}$

and $Y'(u) = 3u^2 + 2u - u^{-2}$

$= 3u^2 + 2u - \frac{1}{u^2}$

$= \boxed{\frac{3u^4 + 2u^2 - 1}{u^2}}$

Example 3: (Neuhauser, Problem # 39, p. 158)

Assume that $f(x)$ is differentiable.

Find an expression for the derivative of

$$y = -5x^3 f(x) - 2x$$

at $x = 1$, assuming that $f(1) = 2$ and $f'(1) = -1$.

$$y = -5x^3 f(x) - 2x \qquad f(1) = 2$$

$$f'(1) = -1$$

$$y' = -15x^2 f(x) - 5x^3 f'(x) - 2$$

Hence when $x = 1$

$$y'(1) = -15(1)^2 f(1) - 5(1)^3 f'(1) - 2$$

$$= -15(2) - 5(-1) - 2$$

$$= -30 + 5 - 2 = \boxed{-27}$$

Example 4: (Online Homework HW12, # 19)

Differentiate $f(x) = \dfrac{ax+b}{cx+d}$,

where a, b, c, and d are constants and $ad - bc \neq 0$.

$$f(x) = \frac{ax+b}{cx+d}$$

We use the quotient rule

$$f'(x) = \frac{a(cx+d) - (ax+b)c}{(cx+d)^2}$$

$$= \frac{\cancel{acx} + ad - \cancel{acx} - bc}{(cx+d)^2}$$

$$= \frac{ad-bc}{(cx+d)^2} \neq 0 \quad \left(\text{unless } x = -\frac{d}{c}\right)$$

Example 5: (Online Homework HW12, # 22)

Find an equation of the tangent line to the given curve at the specified point:

$$y = \frac{\sqrt{x}}{x+3} \qquad P(4, 2/7).$$

$$y = \frac{\sqrt{x}}{x+3} \qquad P\left(4, \frac{2}{7}\right)$$

notice that indeed $y(4) = \frac{\sqrt{4}}{4+3} = \frac{2}{7}$

We need $y'(4)$ to write the equation of the tangent line at P.

We use the quotient rule; also recall that

$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ by the generalized power rule.

$$y' = \frac{\frac{1}{2\sqrt{x}}(x+3) - \sqrt{x}\,(1)}{(x+3)^2} = \frac{\frac{x+3 - (2\sqrt{x})(\sqrt{x})}{2\sqrt{x}}}{(x+3)^2}$$

$$= \frac{x+3-2x}{2\sqrt{x}\,(x+3)^2} = \boxed{\frac{3-x}{2\sqrt{x}\,(x+3)^2}}$$

Since $y' = \dfrac{3-x}{2\sqrt{x}\,(x+3)^2}$ we have that

$$y'(4) = \frac{3-4}{2\sqrt{4}\,(4+3)^2} = \frac{-1}{4 \cdot 49}$$

So:

$$y - \frac{2}{7} = -\frac{1}{4 \cdot 49}(x-4)$$

$$y = -\frac{1}{196}x + \frac{1}{49} + \frac{2}{7}$$

$$\boxed{y = -\frac{1}{196}x + \frac{15}{49}}$$

Example 6: (Neuhauser, Example # 6, p. 155)

Differentiate the Monod growth function

$$f(R) = \frac{aR}{k + R}$$

where a and k are positive constants.

$$f(R) = \frac{aR}{k+R}$$ where a, k are positive constants

$$f'(R) = \frac{d}{dR} f = \frac{(a \cdot 1)(k+R) - aR(1)}{(k+R)^2}$$

$$= \frac{ak + \cancel{aR} - \cancel{aR}}{(k+R)^2} = \boxed{\frac{ak}{(k+R)^2}}$$

Since $a, k > 0$ then notice that

$\boxed{f'(R) > 0}$ <u>always</u> except for $R = -k$

Example 7: (Neuhauser, Problem # 84, p. 159)

Assume that $f(x)$ is differentiable.

Find an expression for the derivative of

$$y = \frac{f(x)}{x^2 + 1}$$

at $x = 2$, assuming that $f(2) = -1$ and $f'(2) = 1$.

$$y = \frac{f(x)}{x^2+1} \qquad f(2) = -1 \qquad f'(2) = 1$$

Want $y'(2)$.

$$y' = \frac{f'(x)(x^2+1) - f(x)\cdot 2x}{(x^2+1)^2}$$

Hence:

$$y'(2) = \frac{f'(2)(2^2+1) - f(2)\cdot 2\cdot 2}{(2^2+1)^2}$$

$$= \frac{1\cdot 5 - (-1)\cdot 4}{25} = \boxed{\frac{9}{25}}$$

Proofs:

4. We use the definition of the derivative, rewrite the numerator in a 'tricky' way and use the limit laws and the continuity of the functions.

$$
\begin{aligned}
(fg)'(x) &\stackrel{\text{def}}{=} \lim_{h\to 0} \frac{(fg)(x+h)-(fg)(x)}{h} \\
&\stackrel{\text{def}}{=} \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&\stackrel{\text{trick}}{=} \lim_{h\to 0} \frac{f(x+h)g(x+h) \boxed{-f(x)g(x+h)+f(x)g(x+h)} - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h}\, g(x+h) + f(x)\, \frac{g(x+h)-g(x)}{h} \right] \\
&\stackrel{\text{rule}}{=} \left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \right] \left[\lim_{h\to 0} g(x+h) \right] + f(x) \left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \right] \\
&\stackrel{\text{cont.}}{=} f'(x)\, g(x) + f(x)\, g'(x).
\end{aligned}
$$

5. We use the definition of the derivative, rewrite the numerator in a 'tricky' way and use the limit laws and the continuity of the functions.

$$(f/g)'(x) =$$

$$\overset{\text{def}}{=} \lim_{h\to 0} \frac{(f/g)(x+h) - (f/g)(x)}{h}$$

$$\overset{\text{def}}{=} \lim_{h\to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x)\,g(x+h)}$$

$$\overset{\text{trick}}{=} \lim_{h\to 0} \frac{f(x+h)g(x) \boxed{-f(x)g(x) + f(x)g(x)} - f(x)g(x+h)}{h\,g(x)\,g(x+h)}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h\,g(x+h)} - f(x)\,\frac{g(x+h) - g(x)}{h\,g(x)g(x+h)} \right]$$

$$\overset{\text{rule}}{=} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} \frac{1}{g(x+h)} - \lim_{h\to 0} \frac{f(x)}{g(x)g(x+h)} \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$\overset{\text{cont.}}{=} f'(x)\frac{1}{g(x)} - \frac{f(x)}{[g(x)]^2} g'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}.$$