# MA 137 – Calculus 1 with Life Science Applications
# **Implicit Differentiation**
# (Section 4.4)

**Alberto Corso**
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

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# Implicit Differentiation

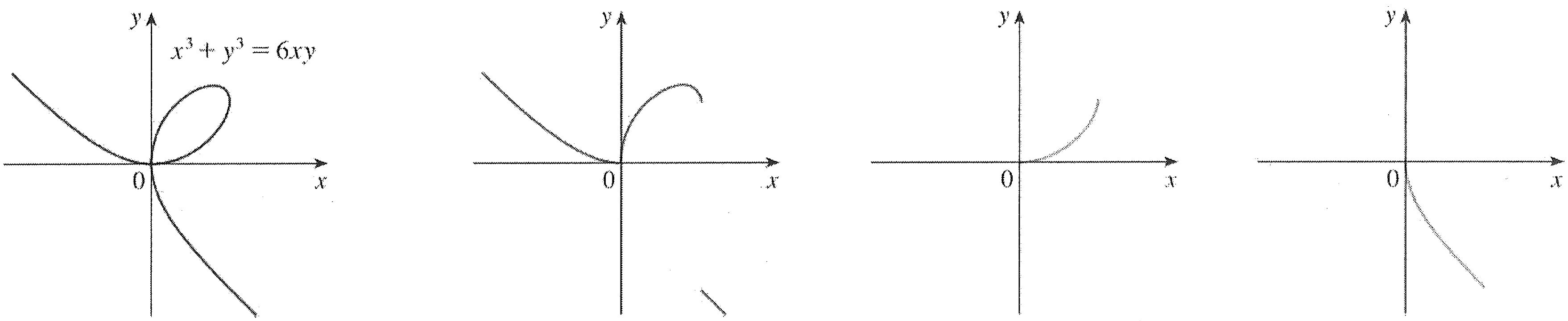
So far, we have considered only functions of the form $y = f(x)$, which define $y$ explicitly as a function of $x$.

It is also possible to define $y$ implicitly as a function of $x$, as in the following equation:

$$x^3 + y^3 = 6xy \qquad (1)$$

Here, $y$ is still given as a function of $x$ (i.e., $y$ is the dependent variable), but there is no obvious way to solve for $y$.

Below are the graphs of three such functions related to equation (1), dubbed the folium of Descartes.



When we say that $f$ is implicitly defined by the equation given in (1), we mean that the equation

$$x^3 + [f(x)]^3 = 6\,x\,f(x)$$

is true for all values of $x$ in the domain of $f$.

Fortunately, there is a very useful technique, based on the chain rule, that will allow us to find $dy/dx$ for implicitly defined functions.

This technique is called **implicit differentiation**.

We summarize the steps we take to find $dy/dx$ when an equation defines $y$ implicitly as a differentiable function of $x$:

1. Differentiate both sides of the equation with respect to $x$, keeping in mind that $y$ is a function of $x$.

   [**Note:** differentiating terms involving $y$ typically requires the chain rule.]

2. Solve the resulting equation for $dy/dx$.

## Example 1:

**(a)** Find $y'$ if $y$ is implicitly defined by $x^3 + y^3 = 6xy$.

**(b)** Find an equation for the tangent line to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

(a) Consider $x^3 + y^3 = 6xy$. Take $\frac{d}{dx}$ of both sides:

$$\frac{d}{dx}\left[x^3 + y^3\right] = \frac{d}{dx}(6xy)$$

$$\Leftrightarrow \quad \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 6\frac{d}{dx}(xy)$$

$$\Leftrightarrow \quad 3x^2 + \underbrace{\frac{d}{dy}(y^3)\cdot\frac{dy}{dx}}_{\text{chain rule}} = \underbrace{6\cdot y + 6x\frac{dy}{dx}}_{\text{product rule}}$$

$$\Leftrightarrow \quad 3x^2 + 3y^2\cdot\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$\Leftrightarrow \quad 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

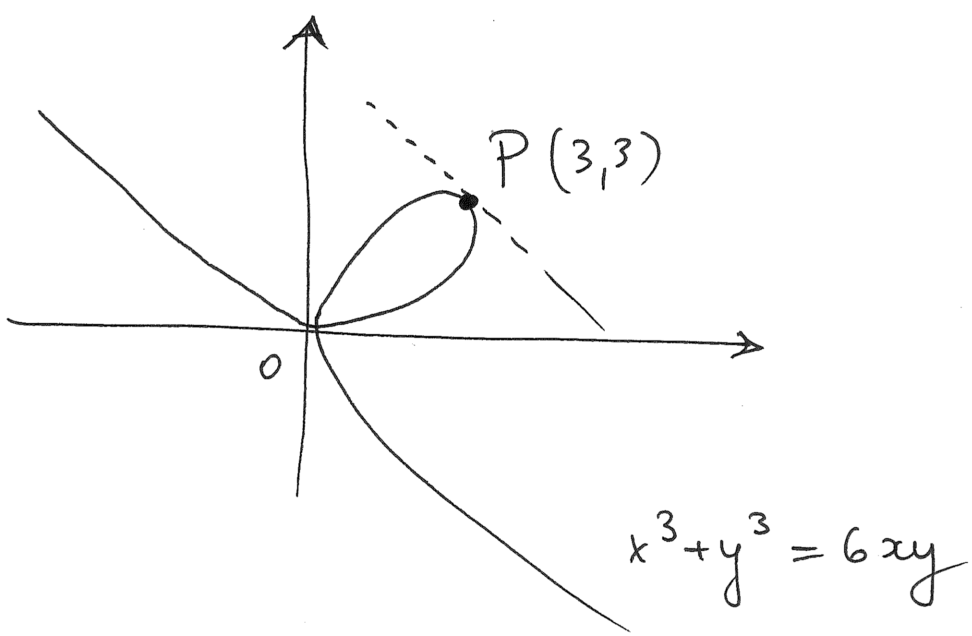
$$\Leftrightarrow \quad \boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

(b)

$$\left.\frac{dy}{dx}\right|_{P(3,3)} = \frac{2(3) - 3^2}{3^2 - 2\cdot 3} = \frac{6-9}{9-6} = \frac{-3}{3} = \boxed{-1}$$

Hence the equation of the tangent line at $P(3,3)$ is

$$y - 3 = (-1)(x-3) \qquad \boxed{y = -x + 6}$$



## Example 2: (Online Homework HW14, # 2)

Given $xy + 2x + 3x^2 = -4$:

**(a)** Find $y'$ by implicit differentiation.

**(b)** Solve the equation for $y$ and differentiate to get $y'$ in terms of $x$.
(The answers should be consistent!)

(a) Given $xy + 2x + 3x^2 = -4$. Take $\frac{d}{dx}$ of both sides

$$\frac{d}{dx}\left[xy + 2x + 3x^2\right] = \frac{d}{dx}(-4)$$

$$\frac{d}{dx}(xy) + 2\frac{d}{dx}(x) + 3\frac{d}{dx}(x^2) = 0$$

$$1\cdot y + x\frac{dy}{dx} + 2 + 3(2x) = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-2 - 6x - y}{x}}$$

(b) $xy = -2x - 3x^2 - 4 \Rightarrow y = \dfrac{-2x - 3x^2 - 4}{x}$

So explicitely $\boxed{y = -2 - 3x - \frac{4}{x}}$

$$y' = -3 - 4(-1)x^{-2} = \boxed{-3 + \frac{4}{x^2}}$$

## Example 3: (Neuhauser, Problem # 54, p. 172)

Find $dy/dx$ by implicit differentiation if

$$\frac{x}{xy+1} = 2xy.$$

$$\frac{x}{(xy+1)} = 2xy \iff x = 2xy(xy+1)$$

$$\iff x = 2x^2y^2 + 2xy.$$

Take the derivative of both sides w.r.t. $x$:

$$\frac{d}{dx}[x] = \frac{d}{dx}\left[2x^2y^2 + 2xy\right]$$

$$1 = 4xy^2 + 2x^2\left(2y\frac{dy}{dx}\right) + 2\cdot1\cdot y + 2x\frac{dy}{dx}$$

$$1 - 4xy^2 - 2y = 4x^2y\frac{dy}{dx} + 2x\frac{dy}{dx}$$

hence

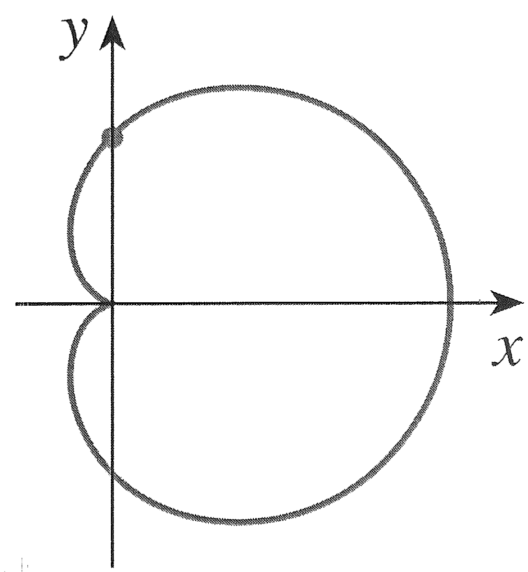
$$\boxed{\frac{dy}{dx} = \frac{1-4xy^2-2y}{4x^2y+2x}}$$

## Example 4: (Online Homework HW14, # 6)

Use implicit differentiation to find an equation of the tangent line to the curve (called **cardioid**)

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

at the point $(0, 1/2)$.



$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \qquad \text{cardioid}$$

We need the tangent line at $P(0, 1/2)$.

We need the slope: so $\left.\frac{dy}{dt}\right|_P = ?$

We use implicit differentiation:

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}\left[(2x^2 + 2y^2 - x)^2\right]$$

$$\Longleftrightarrow$$

$$2x + \frac{d}{dx}(y^2) = 2(2x^2 + 2y^2 - x)^{2-1} \cdot \frac{d}{dx}(2x^2 + 2y^2 - x)$$

chain rule

$$\Longleftrightarrow$$

$$2x + \underbrace{2y \cdot \frac{dy}{dx}}_{\text{chain rule}} = 2(2x^2+2y^2-x) \cdot \left[4x + 4y\frac{dy}{dx} - 1\right]$$

factor

$$2y\frac{dy}{dx} - 8y(2x^2+2y^2-x)\frac{dy}{dx} = 2(2x^2+2y^2-x)(4x-1) - 2x$$

$$\Longleftrightarrow$$

$$\frac{dy}{dx}\left(2y - 8y(2x^2+2y^2-x)\right) = 2(2x^2+2y^2-x)(4x-1) - 2x$$

simplify 2 on both sides

$$\therefore \boxed{\frac{dy}{dx} = \frac{(2x^2+2y^2-x)(4x-1) - x}{y - 4y(2x^2+2y^2-x)}}$$

Evaluate the derivative when $x = 0$ and $y = 1/2$

$$\left.\frac{dy}{dx}\right|_P = \frac{\left(2\left(\frac{1}{2}\right)^2\right)(-1) - 0}{\frac{1}{2} - 4\left(\frac{1}{2}\right)\left(2\left(\frac{1}{2}\right)^2\right)} = \frac{-\frac{1}{2}}{\frac{1}{2} - 1}$$

$$= \frac{-1/2}{-1/2} = \boxed{1}$$

Thus the equation of the tg. line is

$$\boxed{y - \frac{1}{2} = 1 \cdot (x - 0)} \quad \text{or} \quad \boxed{y = x + \frac{1}{2}}$$

# Power Rule for Rational Exponents

We now provide a proof of the generalized form of the power rule when the exponent $r$ is a rational number: $\frac{d}{dx}(x^r) = r\,x^{r-1}$.
We write $r = p/q$, where $p$ and $q$ are integers and are in lowest terms. (If $q$ is even, we require $x$ and $y$ to be positive.) Then

$$y = x^r \quad \iff \quad y = x^{p/q} \quad \iff \quad y^q = x^p.$$

Differentiating both sides of $y^q = x^p$ with respect to $x$, we find that

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

Hence

$$\begin{aligned}\frac{dy}{dx} &= \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q}x^{p-1-(p-p/q)} \\ &= \frac{p}{q}x^{p/q-1} = r\,x^{r-1}\end{aligned}$$