MA 137 – Calculus 1 with Life Science Applications
Related Rates
(Section 4.4)

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Related Rates

An important application of implicit differentiation is related-rates problems.

In a related-rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured).

For instance, suppose that y is a function of x and both y and x depend on time. If we know how x changes with time (i.e., if we know dx/dt), then we might want to know how y changes with time (i.e, dy/dt).

It is almost always better to use Leibniz's notation $\dfrac{dy}{dt}$, if we are differentiating, for instance, the function y with respect to time t. The y' notation is more ambiguous when working with rates and should therefore be avoided.

Neuhauser, p. 167 — ($\approx$ Online Homework HW14, # 9)

Consider a parcel of air rising quickly in the atmosphere. The parcel expands without exchanging heat with the surrounding air.

Laws of physics tell us that the volume (V) and the temperature[1] (T) of the parcel of air are related via the formula

$$TV^{\gamma-1} = C$$

where γ is approximately 1.4 for sufficiently dry air and C is a constant.

To determine how the temperature of the air parcel changes as it rises, we implicitly differentiate $TV^{\gamma-1} = C$ with respect to time t:

$$\frac{dT}{dt}V^{\gamma-1} + T(\gamma-1)V^{\gamma-2}\frac{dV}{dt} = 0 \qquad \text{or} \qquad \frac{dT}{dt} = -(\gamma-1)\frac{T}{V}\frac{dV}{dt}.$$

Since rising air expands with time, we express this relationship as $dV/dt > 0$. We conclude then that the temperature decreases (i.e., $dT/dt < 0$), since both T and V are positive and $\gamma \approx 1.4$.

[1] The temperature is measured in kelvin, a scale chosen so that the temperature is always positive. The Kelvin scale is the absolute temperature scale.

Related-rates Problems Guideline

1. Read the problem and identify the variables.

   Time is often an understood variable. If the problem involves geometry, draw a picture and label it. Label anything that does not change with a constant. Label anything that does change with a variable.

2. Write down which derivatives you are given.

   Use the units to help you determine which derivatives are given. The word "per" often indicates that you have a derivative.

3. Write down the derivative you are asked to find.

   "How fast..." or "How slowly..." indicates that the derivative is with respect to time.

4. Look at the quantities whose derivatives are given and the quantity whose derivative you are asked to find. Find a relationship between all of these quantities.

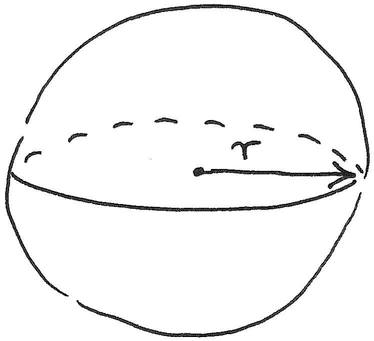
5. Use the chain rule to differentiate the relationship.

6. Substitute any particular information the problem gives you about values of quantities at a particular instant and solve the problem.

   To find all of the values to substitute, you may have to use the relationship you found in step **4**. That is, take a snapshot of the picture at that particular instant.

Example 1: (Online Homework HW14, # 7)

A spherical balloon is inflated so that its volume is increasing at the rate of 2.1 ft^3/min. How rapidly is the diameter of the balloon increasing when the diameter is 1.5 feet?



V = volume of the spherical balloon

r = radius of the balloon

$$\boxed{V = \frac{4}{3}\pi r^3}$$

$$\boxed{\frac{dV}{dt} = 2.1 \ ft^3/min}$$

We have and seek information in terms of the diameter D of the spherical balloon: $\underline{\underline{D = 2r}}$

So $\underline{\underline{V}} = \frac{4}{3}\pi\left[\frac{D}{2}\right]^3 = \frac{4}{3}\pi\frac{D^3}{8} = \underline{\boxed{\frac{\pi D^3}{6}}}$

Take derivative with respect to time:

$$\frac{dV}{dt} = \frac{\pi}{6} 3D^2 \cdot \frac{dD}{dt} \quad \therefore \quad \frac{dV}{dt} = \frac{\pi}{2}D^2 \cdot \frac{dD}{dt}$$

or $$\boxed{\frac{dD}{dt} = \frac{2}{\pi D^2} \cdot \frac{dV}{dt}}$$

Substitute our data in

$$\frac{dD}{dt} = \frac{2}{\pi D^2} \cdot \frac{dV}{dt}$$

$\frac{dV}{dt} = 2.1 \ ft^3/min$ and $D = 1.5$ feet

$$\therefore \quad \frac{dD}{dt} \text{ at that time} = \frac{2}{\pi(1.5)^2} \cdot 2.1 \ ft/min$$

$$= \underline{3.008 \ ft/min}$$

Example 2: (Online Homework HW14, # 11)

Brain weight B as a function of body weight W in fish has been modeled by the power function $B = .007W^{2/3}$, where B and W are measured in grams.

A model for body weight as a function of body length L (measured in cm) is $W = .12L^{2.53}$.

If, over 10 million years, the average length of a certain species of fish evolved from 15cm to 20cm at a constant rate, how fast was the species' brain growing when the average length was 18cm?

[**Note:** 1 nanogram (ng) $= 10^{-9}$ g.]

$B = 0.007\, W^{2/3}$ B = brain weight, W = body weight in g.

Moreover $W = .12\, L^{2.53}$
when L is the body length

We can substitute and relate directly the brain weight and the body length:

$$B = 0.007\left(0.12\, L^{2.53}\right)^{2/3} = 0.001703 \cdot L^{1.6867}$$

Hence
$$\frac{dB}{dt} = 0.001703 \cdot 1.6867\, L^{0.6867} \cdot \frac{dL}{dt}$$

$$\therefore \boxed{\frac{dB}{dt} = 0.00287245 \cdot L^{0.6867}\, \frac{dL}{dt}}$$

We need to plug in our data.

The value for L is 18 cm. What about $\frac{dL}{dt}$?

Since the average length of this species of fish has evolved at a constant rate from 15 cm to 20 cm over 10 million years:

$$\frac{dL}{dt} = \frac{20-15}{10 \text{ million}} = \frac{5}{10^7} = \underline{5 \cdot 10^{-7}}$$

Hence

$$\frac{dB}{dt} = 0.00287245\,(18)^{0.6867} \cdot 5 \cdot 10^{-7}$$

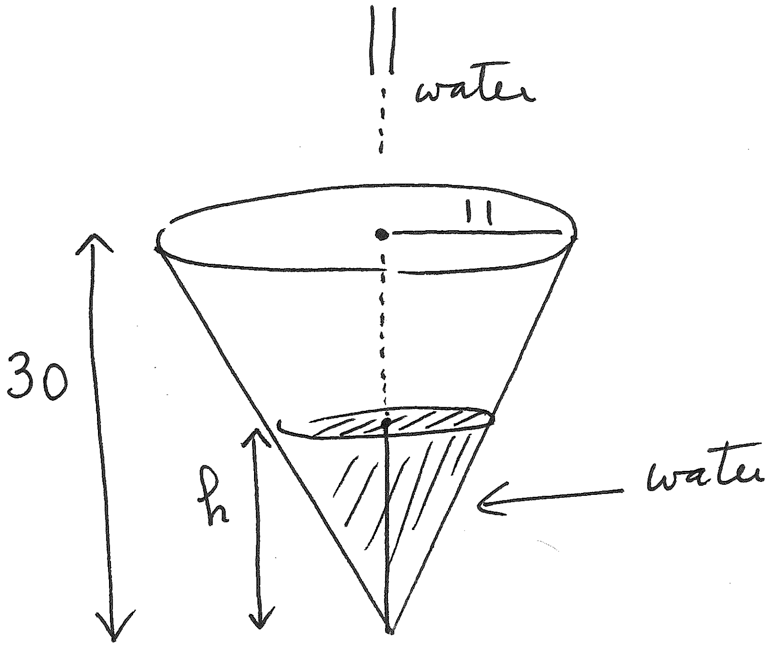
$$= 0.1045245 \cdot 10^{-7} \text{ g/year}$$

$$= 10.45245\ 10^{-9} \text{ g/year} = 10.45245 \frac{\text{ng}}{\text{year}}$$

$\boxed{\text{ng} = \text{nanograms}}$

Example 3: (Online Homework HW14, # 13)

A conical water tank with vertex down has a radius of 11 feet at the top and is 30 feet high.
If water flows into the tank at a rate of 10 $\mathrm{ft}^3/\mathrm{min}$, how fast is the depth of the water increasing when the water is 18 feet deep?



water flow in at a rate of $10\ ft^3/min$: $\boxed{\dfrac{dV}{dt} = 10}$

$\boxed{h = \text{height of water}}$

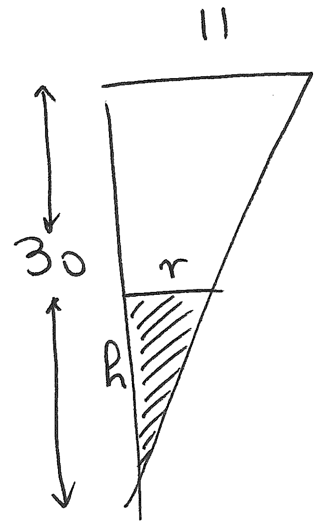
$\boxed{\dfrac{dh}{dt} = ?}$

Need to find a relation between V = volume of water and the height h.

$$V = \text{cone} = \frac{1}{3}\pi \cdot r^2 \cdot h$$

when r is the radius of the cone.

There is a relation between the h and r in this cone.



there are two similar triangles

$$\frac{30}{11} = \frac{h}{r}$$ ratio of corresponding sides is the same!

Thus $\boxed{r = \frac{11}{30} h}$

Hence $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left[\frac{11}{30} h\right]^2 \cdot h = \underline{\frac{121\pi}{2700} \cdot h^3}$

From $V = \frac{121}{2700} \cdot \pi h^3$ we get $\underline{\frac{dV}{dt} = \frac{121}{2700}\pi \frac{d}{dt}\left[h^3\right]}$

and by the chain rule

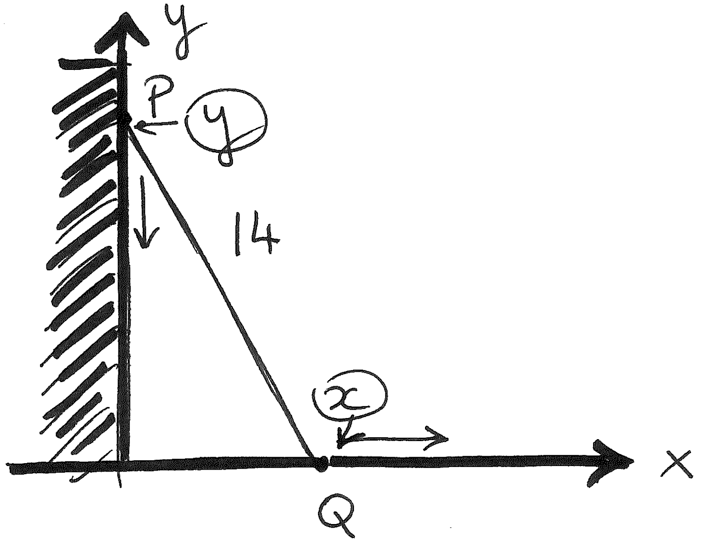
$$\frac{dV}{dt} = \frac{121}{\cancel{2700}_{900}} \pi \cancel{3} h^2 \frac{dh}{dt}$$

$\therefore$ $\boxed{\frac{dh}{dt} = \frac{900}{121\pi} \cdot \frac{1}{h^2} \frac{dV}{dt}}$

and with our data: $\frac{dh}{dt} = \frac{900}{121\pi} \cdot \frac{1}{18^2} \cdot 10 = \underline{\underline{0.07307 \frac{ft}{min}}}$

Example 4: (Online Homework HW14, # 17)

A 14 foot ladder is leaning against a wall. If the top slips down the wall at a rate of 3 ft/s, how fast will the foot be moving away from the wall when the top is 11 feet above the ground?



$\boxed{P(0,y)}$ denotes the top of the ladder

$\boxed{Q(x,0)}$ denotes the foot of the ladder

$$\frac{dy}{dt} = -3 \text{ ft/sec} \qquad \frac{dx}{dt} = ? \text{ ft/sec}$$

Relationship between the quantities:

$$\boxed{x^2 + y^2 = 14^2}$$ By Pythagoras' theorem

Use implicit differentiation

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}(14^2)$$

$$\frac{d}{dx}(x^2)\,\frac{dx}{dt} + \frac{d}{dy}(y^2)\cdot\frac{dy}{dt} = 0$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\therefore \quad \boxed{\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}}$$

With our data : top is 11 feet above the ground so $y = 11$. At that time the foot will be :

$$x^2 + 11^2 = 14^2 \Longrightarrow x = \sqrt{14^2 - 11^2} = \sqrt{75}$$

Hence $$\frac{dx}{dt} = -\frac{11}{\sqrt{75}}\cdot(-3) = \frac{33}{\sqrt{75}} \cong \boxed{3.81 \text{ ft/sec}}$$