

MA 137 — Calculus 1 with Life Science Applications
Derivatives of Trigonometric Functions
(Section 4.5)

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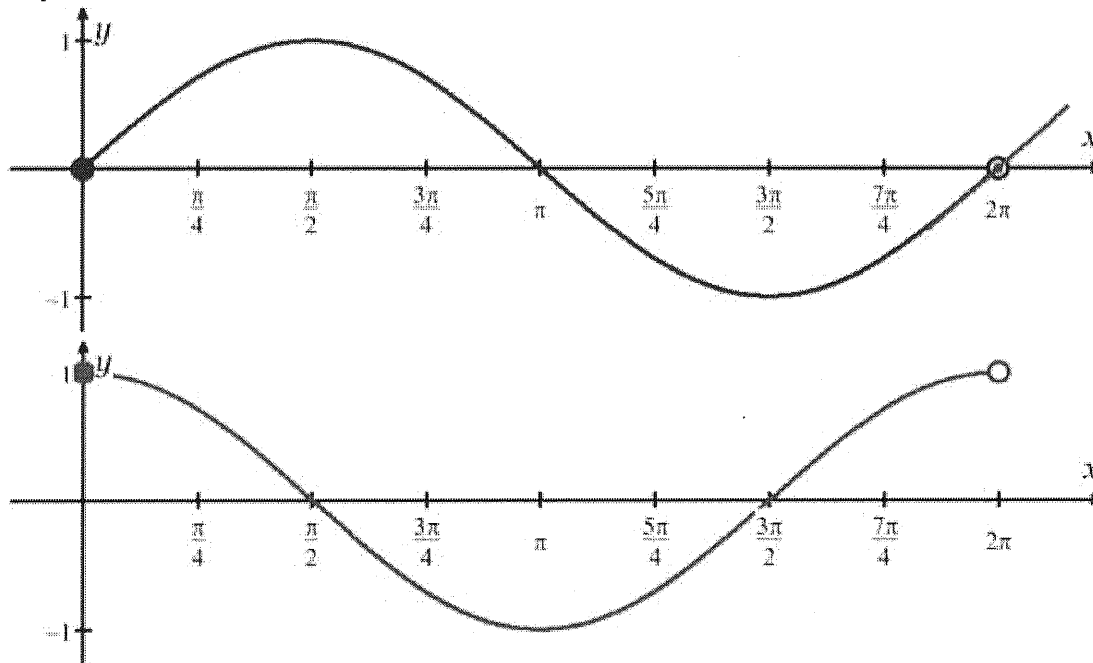
October 23, 2017

Cyclic phenomena are most easily modeled by sines and cosines:

- length of day;
- length of season;
- some population models (e.g. ideal predator-prey models).

We need to know how fast they change.

Let's compare $\sin x$ and $\cos x$:



The Derivative of Sine and Cosine

Theorem

The functions $\sin x$ and $\cos x$ are differentiable for all x , and

$$\frac{d}{dx} \sin x = \cos x \quad \text{and} \quad \frac{d}{dx} \cos x = -\sin x$$

We need the trigonometric limits from Section 3.4 to compute the derivatives of the sine and cosine functions. Namely,

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0.$$

We also need the addition formulas for sine and cosine

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

Note that all angles are measured in radians.

Proof for Cosine

We use the formal definition of derivatives:

$$\begin{aligned}
 \frac{d}{dx} \cos x &\stackrel{\text{def}}{=} \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\
 &\stackrel{\text{add. form.}}{=} \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h} \\
 &= \lim_{h \rightarrow 0} \left[\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right] \\
 &\stackrel{\text{laws}}{=} \cos x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &\stackrel{\text{fund. lim.}}{=} \cos x \cdot 0 - \sin x \cdot 1 \\
 &= -\sin x
 \end{aligned}$$

Proof for Sine

We use the formal definition of derivatives:

$$\begin{aligned}
 \frac{d}{dx} \sin x &\stackrel{\text{def}}{=} \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\
 &\stackrel{\text{add. form.}}{=} \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \\
 &= \lim_{h \rightarrow 0} \left[\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right] \\
 &\stackrel{\text{laws}}{=} \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &\stackrel{\text{fund. lim.}}{=} \sin x \cdot 0 + \cos x \cdot 1 \\
 &= \cos x
 \end{aligned}$$

Derivatives of Remaining Trigonometric Functions

The derivatives of the other trigonometric functions can be found using the following identities and the quotient rule:

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

For example: $\frac{d}{dx}(\tan x) = \dots = \sec^2 x = 1 + \tan^2 x.$

Example 1: (Online Homework HW15, # 3)

Find the equation of the tangent line to the curve $y = 6x \cos x$ at the point $(\pi, -6\pi)$.

$$y = 6x \cos x$$

We want the equation of the tangent line at

$P(\pi, -6\pi)$. Notice $y(\pi) = 6 \cdot \pi \cdot \underbrace{\cos(\pi)}_{-1}$
 $= -6\pi$ ✓

We need the derivative evaluated at $x = \pi$:

$$y' = 6 \cdot 1 \cdot \cos x + 6x \cdot (-\sin x)$$
$$= 6 \cos x - 6x \sin x$$

$$y' \Big|_{x=\pi} = 6 \cdot \underbrace{\cos \pi}_{-1} - 6\pi \underbrace{\sin(\pi)}_{=0} = \boxed{-6}$$

Hence $\boxed{y - (-6\pi) = -6(x - \pi)}$ |||

OR $y = -6x + \cancel{6\pi} - (\cancel{6\pi}) \quad \therefore \boxed{y = -6x}$

Example 2: (Online Homework HW15, # 4)

(a) Let $f(x) = \sin^3(x)$. Find $f'(x)$.

(b) Let $g(x) = \sin(x^3)$. Find $g'(x)$.

$$(a) \quad f(x) = \sin^3(x) = [\sin(x)]^3$$

that's the meaning

So, using the power chain rule, we get

$$\begin{aligned} f'(x) &= 3 [\sin(x)]^{3-1} \cdot \cos(x) \\ &= \boxed{3 \sin^2(x) \cdot \cos(x)} \quad (L) \end{aligned}$$

$$(b) \quad g(x) = \sin(x^3)$$

Using the chain rule we get

$$g'(x) = \cos(x^3) \cdot \underbrace{3x^2}$$

$$= \boxed{3x^2 \cos(x^3)}$$

Example 3: (Online Homework HW15, # 7)

Find the derivative of the following function:

$$f(x) = \frac{\cos(2x)}{6 - \sin(2x)}$$

$$f(x) = \frac{\cos(2x)}{6 - \sin(2x)}$$

$$f'(x) = \frac{(\cos(2x))' \cdot (6 - \sin(2x)) - \cos(2x) (6 - \sin(2x))'}{[6 - \sin(2x)]^2}$$

$$= \frac{-\sin(2x) \cdot 2(6 - \sin(2x)) - \cos(2x) \cdot [-\cos(2x) \cdot 2]}{[6 - \sin(2x)]^2}$$

$$= \frac{-12 \sin(2x) + \underbrace{2 \sin^2(2x) + 2 \cos^2(2x)}_{=2}}{[6 - \sin(2x)]^2}$$

$$= \boxed{\frac{2 - 12 \sin(2x)}{[6 - \sin(2x)]^2}} \quad \checkmark \checkmark$$

Example 4: (Online Homework HW15, # 8)

Find the derivative of the following function:

$$f(x) = (x^3 - \cos(6x^2))^5$$

$$f(x) = [x^3 - \cos(6x^2)]^5$$

We need to use the chain rule to find $f'(x)$:

$$\begin{aligned} f'(x) &= 5 [x^3 - \cos(6x^2)]^{5-1} \cdot (x^3 - \cos(6x^2))' \\ &= 5 [x^3 - \cos(6x^2)]^4 \cdot (3x^2 - (-\sin(6x^2) \cdot 12x)) \\ &= \boxed{5 [x^3 - \cos(6x^2)]^4 \cdot (3x^2 + 12x \sin(6x^2))} \end{aligned}$$

Example 5:

Human heart goes through cycles of contraction and relaxation (called systoles). During cycles, blood pressure goes up and down repeatedly; as heart contracts, pressure rises, and as heart relaxes (for a split second), pressure drops.

Consider approximate function for blood pressure of a patient

$$P(t) = 100 + 20 \cos\left(\frac{\pi t}{35}\right) \text{ mmHg}$$

where t is measured in minutes . Find and interpret $P'(t)$.

$$P(t) = 100 + 20 \cos\left(\frac{\pi t}{35}\right)$$

$$P'(t) = 0 + 20 \cdot \left[-\sin\left(\frac{\pi t}{35}\right) \cdot \frac{\pi}{35} \right]$$

$$= -\frac{20 \cdot \pi}{35} \sin\left(\frac{\pi t}{35}\right)$$

$$= \boxed{-\frac{4\pi}{7} \sin\left(\frac{\pi t}{35}\right)} \quad \text{mmHg/min}$$

$P'(t)$ is measured in mmHg/min; it is the change in blood pressure due to normal cycle of the heart.

Example 6: (Online Homework HW15, # 9)

During the human female menstrual cycle, the gonadotropin, FSH or follicle stimulating hormone, is released from the pituitary in a sinusoidal manner with a period of approximately 28 days.

Guyton's text on Medical Physiology shows that if we define day 0 ($t = 0$) as the beginning of menstruation, then FSH, $F(t)$, cycles with a high concentration of about 4.4 (relative units) around day 9 and a low concentration of about 1.2 around day 23.

- a. Consider a model of the concentration FSH (in relative units) given by

$$F(t) = A + B \cos(\omega(t - \varphi)),$$

where A , B , ω , and φ (with $0 \leq \varphi \leq 28$) are constants and t is in days. Use the data above to find the four parameters.

If ovulation occurs around day 14, then what is the approximate concentration of FSH at that time?

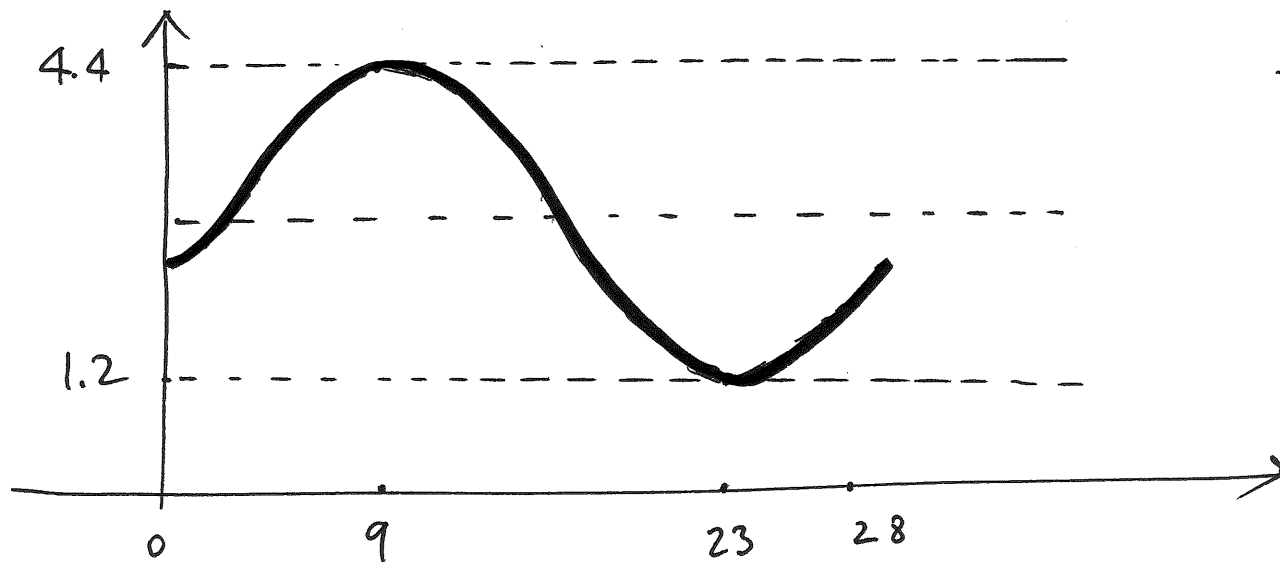
You should sketch a graph of the concentration of FSH over one period.

- b. Find the derivative of $F(t)$. Give its value at the time of ovulation ($t = 14$).

We need to find the correct values in

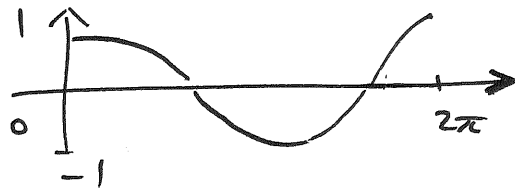
$$F(t) = A + B \cos(\omega(t - \varphi))$$

So that we have a complete cycle in 28 days;
the high concentration is around day nine
and has value 4.4; the low concentration
is around day 23 and has value 1.2.



Here is how
the graph
should
look like!

We need to shift, stretch and translate the graph of $\cos(t)$:



$A =$ gives the vertical shift of the x -axis

$B =$ gives half of the amplitude of the cycle.

A is the middle point between 4.4 and 1.2

$$= \textcircled{2.8} = \frac{4.4 + 1.2}{2}$$

$$B = \frac{4.4 - 1.2}{2} = \text{half of the amplitude} = \textcircled{\underline{1.6}}$$

the peak that occurs when $t=0$ for $\cos(t)$

now occurs when $t=9$. Thus $\textcircled{\underline{\underline{\varphi = 9}}}$

gives the horizontal shift to the right

Finally we need to find the frequency " ω ":

Notice that $F(t) = F(t+28) = F(t+56) = \dots$

i.e., the values repeat every 28 days:

Thus: $F(t) = F(t+28)$ gives

$$A + B \cos(\omega(t-\varphi)) = A + B \cos(\omega(t+28-\varphi))$$

$$\cos(\omega(t-\varphi)) = \cos(\omega(t-\varphi) + \underline{\underline{\omega \cdot 28}})$$

Thus $\omega \cdot 28$ must be 2π

$$\therefore \omega = \frac{2\pi}{28} \approx \underline{\underline{0.2244}}$$

Hence

$$F(t) = 2.8 + 1.6 \cos(0.2244(t-9))$$

$$(b) \quad F'(t) = 1.6 \left[-\sin(0.2244(t-9)) \cdot (0.2244) \right]$$

$$= -0.3590 \sin(0.2244(t-9)) \quad | \quad \cup$$

$$F'(14) = -0.3590 \cdot \sin(1.122)$$

$$\cong \underline{\underline{-0.32345}}$$

$$F(14) = 2.8 + 1.6 \cos(1.122) \cong \underline{\underline{3.49421}}$$