

MA 137 — Calculus 1 with Life Science Applications
**Derivatives of Logarithmic Functions and
Logarithmic Differentiation**
(Section 4.7)

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The Derivative of the Natural Logarithmic Function

Theorem

The function $\ln x$ is differentiable for all $x > 0$, and $\frac{d}{dx} \ln x = \frac{1}{x}$.

In particular, if $g(x)$ is a differentiable function, it follows from the chain rule that

$$\frac{d}{dx} \ln g(x) = \frac{1}{g(x)} g'(x).$$

We can use the derivative of e^x and the relationship between the exponential and the natural logarithmic functions to find the derivative of the function $\ln x$. Namely, we start by taking the derivative with respect to x of both sides of $e^{\ln x} = x$. We obtain

$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x \iff e^{\ln x} \frac{d}{dx} \ln x = 1 \iff \frac{d}{dx} \ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Alternative Proof

We use the formal definition of the derivative and $e^x = \lim_{u \rightarrow \infty} \left(1 + \frac{x}{u}\right)^u$

$$\begin{aligned} \frac{d}{dx} \ln x &\stackrel{\text{def}}{=} \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h} \\ &\stackrel{\text{ln prop.}}{=} \lim_{h \rightarrow 0} \frac{1}{h} \ln \left(\frac{x+h}{x} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{x} \frac{x}{h} \ln \left(1 + \frac{1}{x/h} \right) && u = x/h \\ &\stackrel{\text{laws}}{=} \frac{1}{x} \lim_{u \rightarrow \infty} \ln \left(1 + \frac{1}{u} \right)^u \\ &\stackrel{\text{cont.}}{=} \frac{1}{x} \ln \left[\lim_{u \rightarrow \infty} \left(1 + \frac{1}{u} \right)^u \right] \\ &= \frac{1}{x} \ln e = \frac{1}{x} \end{aligned}$$

The Derivative of ANY Logarithmic Function

Theorem

The function $\log_a x$ is differentiable for $x > 0$, and $\frac{d}{dx} \log_a x = \frac{1}{(\ln a)x}$.

In particular, if $g(x)$ is a differentiable function, it follows from the chain rule that

$$\frac{d}{dx} \log_a g(x) = \frac{1}{(\ln a)g(x)} g'(x).$$

From the base change formula for logarithms we have that

$$\log_a x = \frac{\ln x}{\ln a}$$

Thus it is enough to find the derivative of $\ln x$. Hence the formula.

Example 1: (Nuehauser, Problems # 28/34/52, p. 192)

Find $\frac{dy}{dx}$ when $y = \ln(1 - x^2)$.

Find $\frac{dy}{dx}$ when $y = [\ln(1 - x^2)]^3$.

Find $\frac{dy}{ds}$ when $y = \ln(\ln s)$.

Example 2: (Nuehauser, Problem # 56, p. 193)

Find $\frac{dy}{dx}$ when $y = \log(3x^2 - x + 2)$.

[Note: $\log = \log_{10}$]

Example 3: (Nuehauser, Problem # 62, p. 193)

Assume that $f(x)$ is differentiable with respect to x . Show that

$$\frac{d}{dx} \ln \left[\frac{f(x)}{x} \right] = \frac{f'(x)}{f(x)} - \frac{1}{x}$$

Logarithmic Differentiation

In 1695, Leibniz introduced logarithmic differentiation, following Johann Bernoulli's suggestion to find derivatives of functions of the form

$$y = [f(x)]^x.$$

Bernoulli generalized this method and published his results two years later.

The **basic idea** is to take logarithms on both sides and then to use implicit differentiation.

Example 4: (Neuhauser, Example # 10, p. 190)

Find $\frac{dy}{dx}$ when $y = x^x$.

What about $\frac{d}{dx}[(2x)^{2x}]$?

Example 5: (Neuhauser, Problems # 66/73/74, p. 193)

Use logarithmic differentiation to find the first derivative of the functions

$$y = (\ln x)^{3x}$$

$$y = x^{\cos x}$$

$$y = (\cos x)^x$$

Example 6: (Neuhauser, Problem # 75, p. 193)

Use logarithmic differentiation to find the first derivative of the function

$$y = \frac{e^{2x}(9x - 2)^3}{\sqrt[4]{(x^2 + 1)(3x^3 - 7)}}$$

Power Rule (General Form)

Theorem

Let $f(x) = x^r$, where r is any real number. Then

$$\frac{d}{dx} x^r = r x^{r-1}$$

Proof: We set $y = x^r$ and use logarithmic differentiation to obtain

$$\begin{aligned}\frac{d}{dx} [\ln y] &= \frac{d}{dx} [\ln x^r] \\ \frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx} [r \ln x] \\ \frac{1}{y} \frac{dy}{dx} &= r \frac{1}{x}\end{aligned}$$

Solving for dy/dx yields

$$\frac{dy}{dx} = r \frac{1}{x} y = r \frac{1}{x} x^r = r x^{r-1}$$