MA 137 — Calculus 1 with Life Science Applications
Derivatives of Logarithmic Functions and Logarithmic Differentiation
(Section 4.7)

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The Derivative of the Natural Logarithmic Function

Theorem

The function $\ln x$ *is differentiable for all* $x > 0$*, and* $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$.

In particular, if $g(x)$ *is a differentiable function, it follows from the chain rule that*

$$\frac{d}{dx}\ln g(x) = \frac{1}{g(x)}\,g'(x).$$

We can use the derivative of e^x and the relationship between the exponential and the natural logarithmic functions to find the derivative of the function $\ln x$. Namely, we start by taking the derivative with respect to x of both sides of $e^{\ln x} = x$. We obtain

$$\frac{d}{dx}e^{\ln x} = \frac{d}{dx}x \iff e^{\ln x}\frac{d}{dx}\ln x = 1 \iff \frac{d}{dx}\ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Alternative Proof

We use the formal definition of the derivative and $e^x = \lim_{u\to\infty}\left(1+\frac{x}{u}\right)^u$

$$\begin{aligned}
\frac{d}{dx}\ln x &\overset{\text{def}}{=} \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} \\
&\overset{\text{ln prop.}}{=} \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) \\
&= \lim_{h\to 0}\frac{1}{x}\,\frac{x}{h}\ln\left(1+\frac{1}{x/h}\right) \qquad u = x/h \\
&\overset{\text{laws}}{=} \frac{1}{x}\lim_{u\to\infty}\ln\left(1+\frac{1}{u}\right)^u \\
&\overset{\text{cont.}}{=} \frac{1}{x}\ln\left[\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u\right] \\
&= \frac{1}{x}\ln e = \frac{1}{x}
\end{aligned}$$

The Derivative of ANY Logarithmic Function

Theorem

The function $\log_a x$ *is differentiable for* $x > 0$, *and* $\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x}$.

In particular, if $g(x)$ *is a differentiable function, it follows from the chain rule that*

$$\frac{d}{dx}\log_a g(x) = \frac{1}{(\ln a)g(x)}\,g'(x).$$

From the base change formula for logarithms we have that

$$\log_a x = \frac{\ln x}{\ln a}$$

Thus it is enough to find the derivative of $\ln x$. Hence the formula.

Example 1: (Nuehauser, Problems # 28/34/52, p. 192)

Find $\dfrac{dy}{dx}$ when $y = \ln(1 - x^2)$.

Find $\dfrac{dy}{dx}$ when $y = \left[\ln(1 - x^2)\right]^3$.

Find $\dfrac{dy}{ds}$ when $y = \ln(\ln s)$.

(a) $y = \ln(1-x^2)$

$$\frac{dy}{dx} = \underbrace{\frac{1}{1-x^2} \cdot (-2x)}_{\text{chain rule}} = \boxed{\frac{-2x}{1-x^2}}$$

(b) $y = \left[\ln(1-x^2)\right]^3$

$$\frac{dy}{dx} = 3\left[\ln(1-x^2)\right]^{3-1} \cdot \left(\ln(1-x^2)\right)'$$

$$= 3\left[\ln(1-x^2)\right]^2 \cdot \frac{1}{1-x^2} \cdot (-2x) = \boxed{\frac{-6x\left[\ln(1-x^2)\right]^2}{1-x^2}}$$

(c) $y = \ln(\ln s)$

$$\frac{dy}{ds} = \frac{1}{\ln s} \cdot (\ln s)' = \boxed{\frac{1}{\ln s} \cdot \frac{1}{s}}$$

Example 2: (Nuehauser, Problem # 56, p. 193)

Find $\dfrac{dy}{dx}$ when $y = \log(3x^2 - x + 2)$.

[**Note:** $\log = \log_{10}$]

$$y = \log(3x^2 - x + 2)$$

$$\frac{dy}{dx} = \frac{1}{\ln(10) \cdot (3x^2 - x + 2)} \cdot (6x - 1)$$

Recall the formula:

$$\frac{d}{dx}\left[\log_a g(x)\right] = \frac{1}{[\ln(a)] \cdot g(x)} \cdot g'(x)$$

Example 3: (Nuehauser, Problem # 62, p. 193)

Assume that $f(x)$ is differentiable with respect to x. Show that

$$\frac{d}{dx} \ln \left[\frac{f(x)}{x} \right] = \frac{f'(x)}{f(x)} - \frac{1}{x}$$

$$y = \ln\left[\frac{f(x)}{x}\right]$$

We want to compute $\frac{dy}{dx}$.

1st method direct computation using the chain rule and the quotient rule:

$$y' = \frac{1}{\frac{f(x)}{x}} \cdot \left[\frac{f(x)}{x}\right]' = \frac{x}{f(x)} \cdot \frac{f'(x) \cdot x - f(x)}{x^2}$$

$$= \frac{f'(x)\, x - f(x)}{x\, f(x)} = \underline{\underline{\frac{f'(x)}{f(x)} - \frac{1}{x}}}$$

2nd method $y = \ln\left[\frac{f(x)}{x}\right] = \ln\left[f(x)\right] - \ln x$

Hence $y' = \frac{1}{f(x)} \cdot f'(x) - \frac{1}{x} = \boxed{\frac{f'(x)}{f(x)} - \frac{1}{x}}$

Logarithmic Differentiation

In1695,Leibniz introduced logarithmic differentiation, following Johann Bernoulli's suggestion to find derivatives of functions of the form

$$y = [f(x)]^x.$$

Bernoulli generalized this method and published his results two years later.

> The **basic idea** is to take logarithms on both sides and then to use implicit differentiation.

Example 4: (Neuhauser, Example # 10, p. 190)

Find $\dfrac{dy}{dx}$ when $y = x^x$.

What about $\dfrac{d}{dx}\left[(2x)^{2x}\right]$?

$y = x^x$

to find y' we can take ln of both sides:

$\ln y = \ln x^x$ **OR** $\ln y = x \cdot \ln x$

Hence, now take $\frac{d}{dx}$ of both sides and use the chain rule:

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[x \cdot \ln x\right]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y\left[\ln x + 1\right] = x^x\left[\ln x + 1\right]$$

Alternative method:

$$y = x^x = e^{\ln x^x} = e^{\underline{\underline{x \ln x}}}$$

Hence $$y' = e^{x \ln x} \cdot (x \ln x)'$$

$$= e^{x \ln x} \cdot \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right)$$

$$= e^{\ln x^x} \cdot (\ln x + 1)$$

$$= x^x \cdot (\ln x + 1)$$

same answer as before !!!

About $y = (2x)^{2x}$

1st method: $\ln y = \ln\left[(2x)^{2x}\right] \iff \ln y = 2x \cdot \ln(2x)$

Take the derivative: $\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[2x \cdot \ln(2x)\right]$

$$\iff \frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \ln(2x) + 2x \cdot \left[\frac{1}{2x} \cdot 2\right]$$

$$\frac{dy}{dx} = y\left[2\ln(2x) + 2\right] = (2x)^{2x}\left[2\ln(2x) + 2\right]$$

2nd method: $y = (2x)^{2x} = e^{\ln\left[(2x)^{2x}\right]} = e^{2x \cdot \ln(2x)}$

Hence

$$y' = e^{2x \cdot \ln(2x)} \cdot \left[2x \cdot \ln(2x)\right]'$$

$$\overset{!}{=} e^{\ln(2x)^{2x}} \cdot \left[2\ln(2x) + 2x \cdot \left(\frac{1}{2x} \cdot 2\right)\right]$$

$$= (2x)^{2x} \cdot \left[2\ln(2x) + 2\right]$$

Example 5: (Neuhauser, Problems # 66/73/74, p. 193)

Use logarithmic differentiation to find the first derivative of the functions

$$y = (\ln x)^{3x} \qquad y = x^{\cos x} \qquad y = (\cos x)^{x}$$

(a) $y = (\ln x)^{3x}$

Take $\ln y$ of both sides: $\ln y = \ln[(\ln x)^{3x}]$

So $\ln y = 3x \cdot \ln[\ln(x)]$. Take $\frac{d}{dx}$ of both sides:

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}\left[3x \cdot \ln[\ln(x)]\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 3 \cdot \ln(\ln(x)) + 3x \cdot \left(\frac{1}{\ln x} \cdot \frac{1}{x}\right)$$

$$\therefore \frac{dy}{dx} = y\left[3\ln(\ln(x)) + \frac{3}{\ln x}\right]$$

$$= (\ln x)^{3x} \cdot \left[3\ln(\ln(x)) + \frac{3}{\ln x}\right]$$

(b) Consider $y = x^{\cos x}$. Take: $\ln y = \ln[x^{\cos x}]$

$$\iff \ln y = \cos x \cdot \ln x$$

Take $\frac{d}{dx}$ of both sides: $\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\cos x \cdot \ln x\right]$

$$\underbrace{\frac{1}{y}\frac{dy}{dx}}_{\text{chain rule}} = -\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = y \cdot \left[-\sin x \ln x + \frac{\cos x}{x}\right]$$

OR

$$\boxed{\frac{dy}{dx} = x^{\cos x} \cdot \left[-\sin x \ln x + \frac{\cos x}{x}\right]}$$

(c) $y = (\cos x)^x \iff \ln y = x \cdot \ln(\cos x)$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln(\cos x) + x \cdot \frac{1}{\cos x}(-\sin x)$$

$$\Rightarrow \boxed{\frac{dy}{dx} = (\cos x)^x \cdot \left[\ln(\cos x) - x\tan x\right]}$$

Example 6: (Neuhauser, Problem # 75, p. 193)

Use logarithmic differentiation to find the first derivative of the function

$$y = \frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}}$$

$$y = \frac{e^{2x} \cdot (9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}}$$

If we try to do the derivative with the quotient rule, it will be complicated.

Let's take "ln" of both sides:

$$\Longleftrightarrow \quad \ln y = \ln\left[\frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}}\right]$$

$$\Longleftrightarrow \quad \ln y = \ln\left[e^{2x}(9x-2)^3\right] - \ln\left[(x^2+1)(3x^3-7)\right]^{1/4}$$

$$\Longleftrightarrow \quad \ln y = \ln(e^{2x}) + \ln\left[(9x-2)^3\right] - \frac{1}{4}\left[\ln\left[(x^2+1)(3x^3-7)\right]\right]$$

$$\Longleftrightarrow \quad \ln y = 2x + 3\ln(9x-2) - \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(3x^3-7)$$

Hence, after expanding, we have that

$$y = \frac{e^{2x} \cdot (9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}} \iff$$

$$\boxed{\ln y = 2x + 3\ln(9x-2) - \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(3x^3-7)}$$

Take the derivative w.r.t x of both sides:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 + 3\frac{1}{9x-2} \cdot (9) - \frac{1}{4}\frac{1}{x^2+1} \cdot (2x) - \frac{1}{4}\frac{1}{3x^3-7} \cdot (9x^2)$$

hence $$\frac{dy}{dx} = y\left[2 + \frac{27}{9x-2} - \frac{x}{2(x^2+1)} - \frac{9x^2}{4(3x^3-7)}\right]$$

or $$\frac{dy}{dx} = \frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}} \cdot \left[2 + \frac{27}{9x-2} - \frac{x}{2(x^2+1)} - \frac{9x^2}{4(3x^3-7)}\right]$$

Power Rule (General Form)

Theorem

Let $f(x) = x^r$, where r is any real number. Then

$$\frac{d}{dx} x^r = r\, x^{r-1}$$

Proof: We set $y = x^r$ and use logarithmic differentiation to obtain

$$\begin{aligned} \frac{d}{dx}\left[\ln y\right] &= \frac{d}{dx}\left[\ln x^r\right] \\ \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}\left[r \ln x\right] \\ \frac{1}{y}\frac{dy}{dx} &= r\frac{1}{x} \end{aligned}$$

Solving for dy/dx yields

$$\frac{dy}{dx} = r\frac{1}{x} y = r\frac{1}{x} x^r = r\, x^{r-1}$$