

MA 137 — Calculus 1 with Life Science Applications  
**Linear Approximations**  
(Section 4.8)

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# Tangent Line Approximation

Assume that  $y = f(x)$  is differentiable at  $x = a$ ; then

$$L(x) = f(a) + f'(a)(x - a)$$

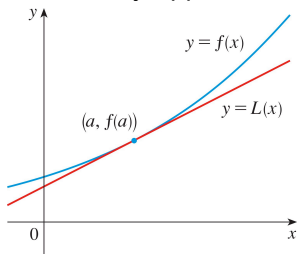
is the tangent line approximation, or **linearization**, of  $f$  at  $x = a$ .

Geometrically, the linearization of  $f$  at  $x = a$  is the equation of the tangent line to the graph of  $f(x)$  at the point  $(a, f(a))$ .

If  $|x - a|$  is sufficiently small, then  $f(x)$  can be linearly approximated by  $L(x)$ ; that is,

$$f(x) \approx f(a) + f'(a)(x - a).$$

This approximation is illustrated in the picture on the right:



**Example 1:** (Nuehauser, Example # 1, p. 194)

- (a) Find the linear approximation of  $f(x) = \sqrt{x}$  at  $x = a$ .
- (b) use your answer in (a) to find an approximate value of  $\sqrt{26}$ .

**Example 2:** (Online Homework HW16, # 18)

Find the linearization  $L(x)$  of the function  $g(x) = x f(x^2)$  at  $x = 2$  given the following information:

$$f(2) = 1 \quad f'(2) = 10 \quad f(4) = 5 \quad f'(4) = -2$$

**Example 3:** (Nuehauser, Problem # 34, p. 199)

**Plant Biomass:** Suppose that a certain plant is grown along a gradient ranging from nitrogen-poor to nitrogen-rich soil.

Experimental data show that the average mass per plant grown in a soil with a total nitrogen content of 1000 mg nitrogen per kg of soil is 2.7 g and the rate of change of the average mass per plant at this nitrogen level is  $1.05 \times 10^{-3}$  g per mg change in total nitrogen per kg soil.

Use a linear approximation to predict the average mass per plant grown in a soil with a total nitrogen content of 1100 mg nitrogen per kg of soil.

**Example 4:** (Nuehauser, Example # 3, p. 195)

Suppose  $N = N(t)$  represents a population size at time  $t$  and the rate of growth as a function of  $N$  is  $g(N)$ .

Find the linear approximation of the growth rate at  $N = 0$ .

[Hint: We can assume that  $g(0) = 0$ . Indeed, when the population has size  $N = 0$ , its grow rate will be zero.]

[Remark: Your answer should show that for small population sizes, the population grows approximately exponentially.]

**Example 5:** (Neuhauser, Problem # 33, p. 199)

**Plant Biomass:** Suppose that the specific growth rate of a plant is 1%; that is, if  $B(t)$  denotes the biomass at time  $t$ , then

$$\frac{1}{B(t)} \frac{dB}{dt} = 0.01$$

Suppose that the biomass at time  $t = 1$  is equal to 5 grams.

Use a linear approximation to compute the biomass at time  $t = 1.1$ .

# Higher Order Approximations

The tangent linear approximation  $L(x) = f(a) + f'(a)(x - a)$  is the best first-degree (linear) approximation to  $f(x)$  near  $x = a$  because  $f(x)$  and  $L(x)$  have the same value and the same rate of change at  $a$

$$L(a) = f(a) \quad L'(a) = f'(a).$$

For a better approximation than a linear one, let's try to find better approximations by looking for an  $n$ th-degree polynomial

$$T_n = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n$$

such that  $T_n$  and its first  $n$  derivatives have the same value at  $x = a$  as  $f$  and its first  $n$  derivatives at  $x = a$ .

We can show that the resulting polynomial is

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

It is called the  **$n$ th-degree Taylor polynomial** of  $f$  centered at  $x = a$ .

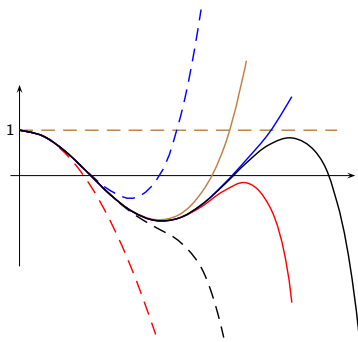


# Approximation of $\cos x$ centered at $a = 0$

Consider the graph of the polynomial

$$T_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^{n-1} \frac{x^{2(n-1)}}{(2n-2)!} + (-1)^n \frac{x^{2n}}{(2n)!}.$$

As  $n$  increases, the graph of  $T_{2n}(x)$  appears to approach the one of  $\cos x$ . This suggests that we can approximate  $\cos x$  with  $T_{2n}(x)$  as  $n \rightarrow \infty$ .



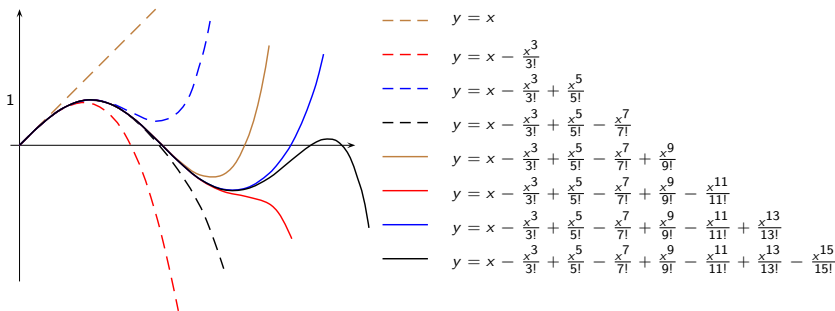
- $y = 1$
- - -  $y = 1 - \frac{x^2}{2!}$
- - -  $y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$
- - -  $y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$
- $y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$
- $y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$
- $y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!}$
- $y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!}$

# Approximation of $\sin x$ centered at $a = 0$

Consider the graph of the polynomial

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

As  $n$  increases, the graph of  $T_{2n+1}(x)$  appears to approach the one of  $\sin x$ . This suggests that we can approximate  $\sin x$  with  $T_{2n+1}(x)$  as  $n \rightarrow \infty$ .

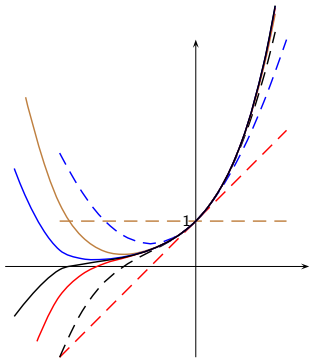


# Approximation of $e^x$ centered at $a = 0$

Consider the graph of the polynomial

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}.$$

As  $n$  increases, the graph of  $T_n(x)$  appears to approach the one of  $e^x$ . This suggests that we can approximate  $e^x$  with  $T_n(x)$  as  $n \rightarrow \infty$ .



- $y = 1$
- $y = 1 + x$
- $y = 1 + x + \frac{x^2}{2!}$
- $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$
- $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$
- $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$
- $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$
- $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

**Example 6:** (Bloomberg Business, 10/23/15)

Google parent Alphabet Inc. reached a record share price a day after reporting better-than-projected quarterly revenue and profit fueled by increased ad sales and a tighter lid on costs. [...] The actual figure that the company announced for the share buyback was unusually specific: \$5,099,019,513.59. **Turns out, those numbers correspond to the square root of 26, or the number of letters in the English alphabet.**

— . . . —

Let  $f(x) = \sqrt{x}$  and  $a = 25$ . The 5th-degree Taylor polynomial of  $f$  centered at 25 can be shown to be

$$T_5(x) = 5 + \frac{1}{10}(x-25) - \frac{1}{1,000}(x-25)^2 + \frac{1}{50,000}(x-25)^3 - \frac{1}{2,000,000}(x-25)^4 + \frac{1}{71,428,571.43}(x-25)^5$$

We can then check that

$$\sqrt{26} \approx T_5(26) = 5 + \frac{1}{10} - \frac{1}{1,000} + \frac{1}{50,000} - \frac{1}{2,000,000} + \frac{1}{71,428,571.43} = 5.099019514$$

This means that we overestimated Alphabet Inc. buyback by 41¢.