MA 137 — Calculus 1 with Life Science Applications

Monotonicity and Concavity

(Section 5.2)

Extrema, Inflection Points, and Graphing

(Section 5.3)

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Increasing and Decreasing Functions

A function f is said to be increasing when its graph rises and decreasing when its graph falls. More precisely, we say that

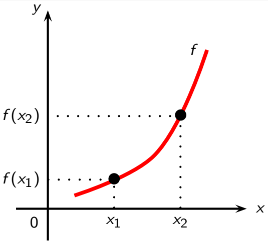
Definition

f is **(strictly) increasing** on an interval I if

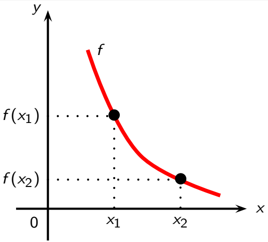
$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I$$

f is **(strictly) decreasing** on an interval I if

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I$$



f is increasing



f is decreasing

First Derivative Test for Monotonicity

Theorem (First Derivative Test for Monotonicity)

Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

(a) *If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.*

(b) *If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.*

Proof: Suppose $f'(x) > 0$ on an interval I. We wish to show that $f(x_1) < f(x_2)$ for any pair $x_1 < x_2$ in $[a, b]$.

Let x_1 and x_2 be any pair of point in $[a, b]$ satisfying $x_1 < x_2$. Then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). We can therefore apply the MVT to f defined on $[x_1, x_2]$: There exists a number $c \in (x_1, x_2)$ such that

$$\frac{f(x_1) - f(x_2)}{x_2 - x_1} = f'(c)$$

Now, $f'(c) > 0$ as $c \in [x_1, x_2] \subset [a, b]$; so

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

so $f(x_2) - f(x_1) > 0$, since $x_2 - x_1 > 0$. Therefore, $f(x_1) < f(x_2)$.
Because x_1 and x_2 are arbitrary numbers in $[a, b]$ satisfying $x_1 < x_2$, it follows that f is increasing on the whole interval.

The proof of part (b) is similar.

First DerivativeTest for (Local) Extrema

Theorem (First Derivative Test for (Local) Extrema)

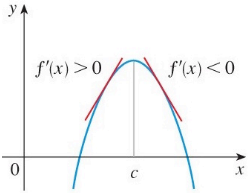
If f has a critical value at $x = c$, then

- *f has a local maximum at $x = c$ if the sign of f' around c is*

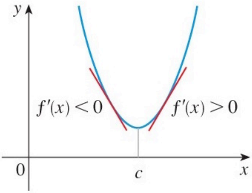
$$\begin{array}{cc} +++ & --- \\ \hline \multicolumn{2}{c}{c} \end{array}$$

- *f has a local minimum at $x = c$ if the sign of f' around c is*

$$\begin{array}{cc} --- & +++ \\ \hline \multicolumn{2}{c}{c} \end{array}$$



(a) Local maximum

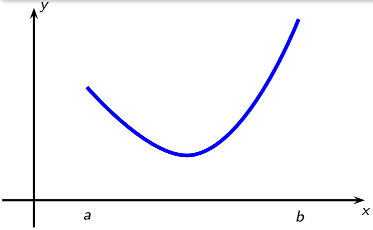


(b) Local minimum

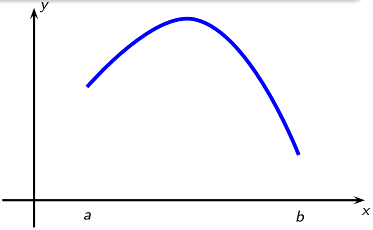
Concavity

The second derivative can also be used to help sketch the graph of a function. More precisely, the second derivative can be used to determine when the graph of a function is concave upward or concave downward.

The graph of a function $y = f(x)$ is **concave upward** on an interval $[a, b]$ if the graph lies above each of the tangent lines at every point in the interval $[a, b]$. The graph of a function $y = f(x)$ is **concave downward** on an interval $[a, b]$ if the graph lies below each of the tangent lines at every point in the interval $[a, b]$.



graph of function concave upward on $[a, b]$



graph of function concave downward on $[a, b]$

Second Derivative Test for Concavity

Consider a function $f(x)$.

If $f''(x) > 0$ over an interval $[a, b]$, then the derivative $f'(x)$ is increasing on the interval $[a, b]$. That means the slopes of the tangent lines to the graph of $y = f(x)$ are increasing on the interval $[a, b]$. From this it can be seen that the graph of the function $y = f(x)$ is concave upward.

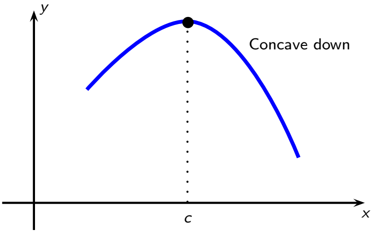
If $f''(x) < 0$ over an interval $[a, b]$. Then the derivative $f'(x)$ is decreasing on the interval $[a, b]$. That means the slopes of the tangent lines to the graph of $y = f(x)$ are decreasing on the interval $[a, b]$. From this it can be seen that the graph of the function $y = f(x)$ is concave downward.

Second Derivative Test for (Local) Extrema

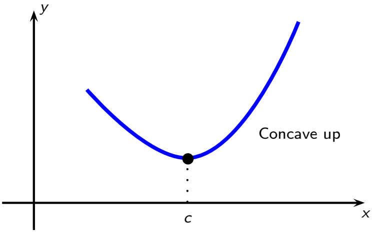
Theorem (Second Derivative Test for (Local) Extrema)

Suppose that f is twice differentiable on an open interval containing c.

- *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max. at $x = c$.*
- *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min. at $x = c$.*



f has a local max at c



f has a local min at c

Inflection Points

A point $(c, f(c))$ on the graph is called a **point of inflection** if the graph of $y = f(x)$ changes concavity at $x = c$. That is, if the graph goes from concave up to concave down, or from concave down to concave up.

If $(c, f(c))$ is a point of inflection on the graph of $y = f(x)$ and if the second derivative is defined at this point, then $f''(c) = 0$.

Thus, points of inflection on the graph of $y = f(x)$ are found where either $f''(x) = 0$ or the second derivative is not defined.

However, if either $f''(x) = 0$ or the second derivative is not defined at a point, it is not necessarily the case that the point is a point of inflection. Care must be taken.

About Graphing a Function

Using the first and the second derivatives of a twice-differentiable function, we can obtain a fair amount of information about the function.

We can determine intervals on which the function is increasing, decreasing, concave up, and concave down. We can identify local and global extrema and find inflection points.

To graph the function, we also need to know how the function behaves in the neighborhood of points where either the function or its derivative is not defined, and we need to know how the function behaves at the endpoints of its domain (or, if the function is defined for all $x \in \mathbb{R}$, how the function behaves for $x \to \pm\infty$).

A line $y = b$ is a horizontal asymptote if either

$$\lim_{x \to +\infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b$$

A line $x = c$ is a vertical asymptote if

$$\lim_{x \to c^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to c^-} f(x) = \pm\infty$$

Example 1:

Find the intervals where the function $f(x) = x^3 - 3x^2 + 1$ is increasing and the ones where it is decreasing. Use this information to sketch the graph of $f(x) = x^3 - 3x^2 + 1$.

Example 2:

Let $f(x) = \dfrac{x+4}{x+7}$. Find the intervals over which the function is increasing.

Example 3:

Let $h(x) = x^2 e^{-x}$.

(a) On what intervals is h increasing or decreasing?
(b) At what values of x does h have a local maximum or minimum?
(c) On what intervals is h concave upward or downward?
(d) State the x-coordinate of the inflection point(s) of h.
(e) Use the information in the above to sketch the graph of h.

Example 4

Find the inflection points of the function $g(x) = e^{-x^2}$.

Example 5:

Suppose $g(x) = \dfrac{\sqrt{x-3}}{x}$. Find the value of x in the interval $[3, +\infty)$ where $g(x)$ takes its maximum.

Example 6: (Exam 3, Fall 13, # 3)

Let $f(x) = \ln(x^2+1)$. You are given that

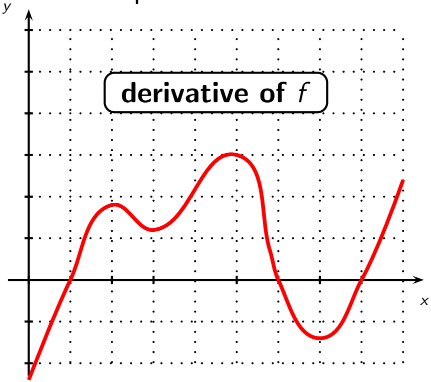
$$f'(x) = \frac{2x}{x^2+1} \qquad \text{and} \qquad f''(x) = \frac{2-2x^2}{(x^2+1)^2}.$$

(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upward or downward?
(d) State the x-coordinate of the inflection point(s) of f.
(e) Use the information in the above to sketch the graph of f.

Example 7:

The graph of the derivative f' of a function f is shown.

(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upward or downward?
(d) State the x-coordinate of the inflection points of f.



Example 8: (Online Homework HW18, #14)

Suppose that on the interval I, $f(x)$ is positive and concave up. Furthermore, assume that $f''(x)$ exists and let $g(x) = (f(x))^2$. Use this information to answer the following questions.

(a) $f''(x) >$ ____ on I.
(b) $g''(x) = 2(A^2 + Bf''(x))$, where $A =$ ____ and $B =$ ____
(c) $g''(x) >$ ____ on I.
(d) $g(x)$ is ____ on I.