MA 137 — Calculus 1 with Life Science Applications
L'Hôpital's Rule
(Section 5.5)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

November 15, 2017

Heuristics

We have often encountered the situation in which we had to compute $\lim_{x \to a} \frac{f(x)}{g(x)}$ and we had that both the following limits were zero

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0.$$

Using a linear approximation at $x = a$, we find that, for x close to a

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)}$$

Since $f(a) = g(a) = 0$ and $x \neq a$, the right-hand side is equal to

$$\frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

provided that $f'(a)/g'(a)$ is defined. We therefore hope that something like

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

holds when $f(a)/g(a)$ is of the form $0/0$ and $f'(a)/g'(a)$ is defined. In fact, something like this does hold; it is called **l'Hôpital's rule.**

L'Hôpital's Rule

Theorem

Suppose that f and g are differentiable functions and that

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x) \qquad \textit{or} \qquad \lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x)$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the second limit exists.

L'Hôpital's rule can actually be applied to calculate limits for seven kinds of indeterminate expressions

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty \qquad \infty - \infty \qquad 0^0 \qquad 1^\infty \qquad \infty^0.$$

(Note that l'Hôpitals rule works for $a = +\infty$ or $-\infty$ as well.)

Reduction to 0/0 or ∞/∞ Form

$0 \cdot \infty$ Suppose we have to compute $\lim_{x \to a} f(x)g(x)$ where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$. To apply l'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x \to a} f(x)g(x) = \lim_{x \to a} \frac{f(x)}{1/g(x)} = \lim_{x \to a} \frac{g(x)}{1/f(x)}$$

In the first case the ratio is $0/0$, whereas in the second case the ratio is ∞/∞. Usually only one of the two expressions is easy to evaluate.

$\infty - \infty$ Suppose we have to compute $\lim_{x \to a} [f(x) - g(x)]$ where $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$. To apply l'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) \left(1 - \frac{g(x)}{f(x)} \right) = \lim_{x \to a} g(x) \left(\frac{f(x)}{g(x)} - 1 \right)$$

and hope that the limit is of the form $0 \cdot \infty$.

$0^0 \ 1^\infty \ \infty^0$ Suppose we have to compute $\lim_{x \to a} [f(x)]^{g(x)}$, which becomes of the form 0^0, 1^∞ or ∞^0. The key to solving these limits is to write them as exponentials

$$\lim_{x \to a} [f(x)]^{g(x)} = \lim_{x \to a} \exp\left\{ \ln\,[f(x)]^{g(x)} \right\} = \lim_{x \to a} \exp\left\{ g(x) \cdot \ln f(x) \right\} = \exp\left[\lim_{x \to a} \left(g(x) \cdot \ln f(x) \right) \right].$$

The last step, in which we interchanged lim and exp, uses the fact that the exponential function is continuous.

Example 1: (Nuehauser, p. 247)

Evaluate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$.

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \frac{0}{0}$$ if we use direct evaluation

Hence we can apply l'Hôpital's rule.

We obtain:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{2x}{1} = \frac{6}{1} = \boxed{6}$$

Note: in Chapter 3 we solved the problem by factoring and simplifying the expression

$$\lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} = \lim_{x \to 3} x + 3 = 3 + 3 = 6$$

Example 2: (Nuehauser, p. 247)

Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x}$.

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \frac{e^0 - 1}{0} = \frac{0}{0}$$

Hence we can use l'Hôpital's rule:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = e^0 = \boxed{1}$$

Example 3:

Evaluate $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1-1}{0} = \frac{0}{0}$$ by direct substitution

Hence we can use l'Hôpital's rule

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{-(-\sin x)}{2x} = \lim_{x \to 0} \frac{\sin x}{2x}$$

$= \frac{0}{0}$ again. Hence we use l'Hôpital's rule again

$$= \lim_{x \to 0} \frac{\cos x}{2} = \boxed{\frac{1}{2}}$$

<u>Note</u> that in Section 3.4 we gave a geometric argument for $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Example 4: (Neuhauser, Problem # 25, p. 252)

Evaluate $\lim_{x\to\infty} x \cdot e^{-x}$.

What about $\lim_{x\to\infty} x^{13} \cdot e^{-x}$? (Online Homework HW20, # 5)

$$\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$$

We can use l'Hôpital rule. Hence

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = \boxed{0}$$

<u>HW20, #5</u>: $\lim_{x\to\infty} \frac{x^{13}}{e^x} = \frac{\infty}{\infty}$ = use l'Hôpital's rule $= \lim_{x\to\infty} \frac{13x^{12}}{e^x} = \frac{\infty}{\infty} = \dots =$ use <u>many</u> <u>more</u> <u>times</u> l'Hôpital's rule to get

$$= \lim_{x\to\infty} \frac{13!}{e^x} = \frac{13!}{\infty} = \boxed{0}$$

Example 5: (Online Homework HW20, # 3)

Evaluate $\lim_{x \to 0^+} 7\sqrt{x} \cdot \ln x$.

$$\lim_{x\to 0^+} 7\sqrt{x}\cdot \ln x = 0\cdot(-\infty)$$

Hence we can use l'Hôpital's rule, where we rewrite $7\sqrt{x}\ln x$ as $\dfrac{7\ln x}{1/\sqrt{x}} = \dfrac{7\ln x}{x^{-1/2}}$

Hence :

$$\lim_{x\to 0^+} 7\sqrt{x}\cdot \ln x = \lim_{x\to 0^+} \frac{7\ln x}{x^{-1/2}} = \frac{-\infty}{+\infty} =$$

$$= \text{l'Hôpital} = \lim_{x\to 0^+} \frac{7\,\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} 7\frac{1}{x}\left[-2x^{3/2}\right]$$

$$= \lim_{x\to 0^+} -14\frac{x\sqrt{x}}{x} = \lim_{x\to 0^+} -14\sqrt{x} = 0$$

Hence $\boxed{\lim_{x\to 0^+} 7\sqrt{x}\ln x = 0}$

Example 6: (Neuhauser, Example # 10, p. 250)

Evaluate $\lim_{x \to \infty} x - \sqrt{x^2 + x}$.

$$\lim_{x\to\infty} x - \sqrt{x^2+x} = \infty - \infty$$

Let's rewrite the limit as follows:

$$\lim_{x\to\infty} x - \sqrt{x^2+x} = \lim_{x\to\infty} x\left[1 - \frac{\sqrt{x^2+x}}{x}\right] =$$

$$= \lim_{x\to\infty} \frac{1-\sqrt{1+\frac{1}{x}}}{\frac{1}{x}} = \frac{0}{0} =$$ hence we can apply l'Hôpital's rule

$$= \lim_{x\to\infty} \frac{-\frac{1}{2}\left(1+\frac{1}{x}\right)^{-\frac{1}{2}} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-\frac{1}{2}}{\sqrt{1+\frac{1}{x}}} = \boxed{-\frac{1}{2}}$$

Example 7: (Online Homework HW20, # 4)

Evaluate $\lim_{x \to 0^+} x^x$.

$$\lim_{x\to 0^+} x^x = 0^0$$

Hence we can rewrite the limit as:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{x \ln x}$$

$$= e^{\lim_{x\to 0^+} x \ln x} = e^0 = \boxed{1}$$

<u>**But**</u> $\lim_{x\to 0^+} x \ln x = 0\cdot(-\infty) = \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \dfrac{-\infty}{\infty}$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \frac{1}{x}\cdot(-x^2) = \lim_{x\to 0^+} (-x) = \underline{\underline{0}}$$

Example 8: (Neuhauser, Problem # 62, p. 253)

Use l'Hôpital's rule to find $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x$ where c is a constant.

What about $\lim_{x\to\infty} 3x\big(\ln(x+3)-\ln x\big)$? (Online Homework HW20, #

$$\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = 1^\infty$$

Hence we can rewrite the limit as

$$\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{c}{x}\right)^x} =$$

$$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{c}{x}\right)} = e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}}} = \boxed{e^c}$$

Note that $\lim_{x\to\infty}\frac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}} = \frac{0}{0} =$ by l'Hôpital's rule

$$= \lim_{x\to\infty}\frac{\frac{1}{1+c/x}\cdot\left(-\frac{c}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{c}{1+c/x} = \lim_{x\to\infty}\frac{cx}{x+c} = \boxed{c}$$

$$\lim_{x\to\infty} 3x\left[\ln(x+3)-\ln x\right] = \infty(\infty-\infty)$$

$$= \lim_{x\to\infty} 3x\cdot \ln\left(\frac{x+3}{x}\right) = \lim_{x\to\infty} 3\cdot \ln\left(\frac{x+3}{x}\right)^x =$$

$$= 3\cdot \ln\Big[\underbrace{\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x}_{e^3}\Big] = 3\ln e^3 = 3\cdot 3 = \boxed{9}$$

(by the first part)