

MA 137 — Calculus 1 with Life Science Applications
The Definite Integral
(Section 6.1)

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Some Properties of Definite Integrals

$$1. \int_a^a f(x) dx = 0$$

$$2. \int_a^b k f(x) dx = k \int_a^b f(x) dx$$

$$3. \int_a^b (f(x) \pm g(x)) dx = \left(\int_a^b f(x) dx \right) \pm \left(\int_a^b g(x) dx \right)$$

$$4. \int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

$$5. \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$6. \text{ If } m \leq f(x) \leq M \text{ on } [a, b] \text{ then}$$
$$m(b - a) \leq \int_a^b f(x) dx \leq M(b - a)$$

Geometric Illustration of Some of the Properties

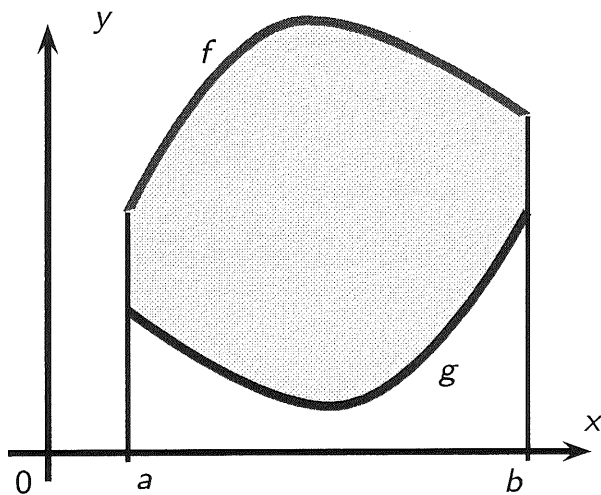
Property 3. says that if f and g are two positive valued functions with

f greater than g , then

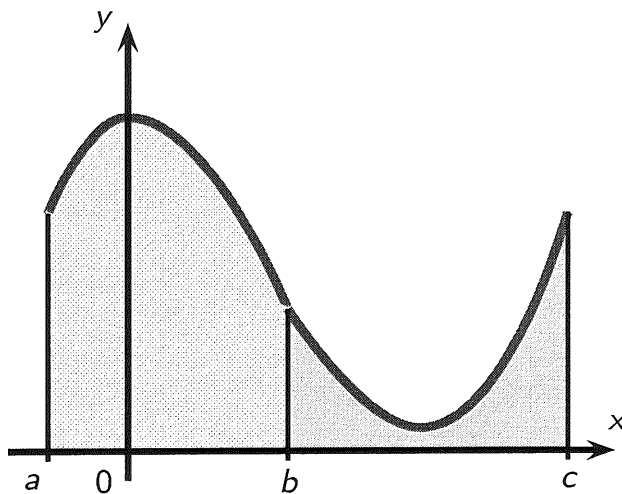
$$\int_a^b (f(x) - g(x)) dx$$

gives the area between the graphs of f and g

$$\int_a^b f(x) dx - \int_a^b g(x) dx$$



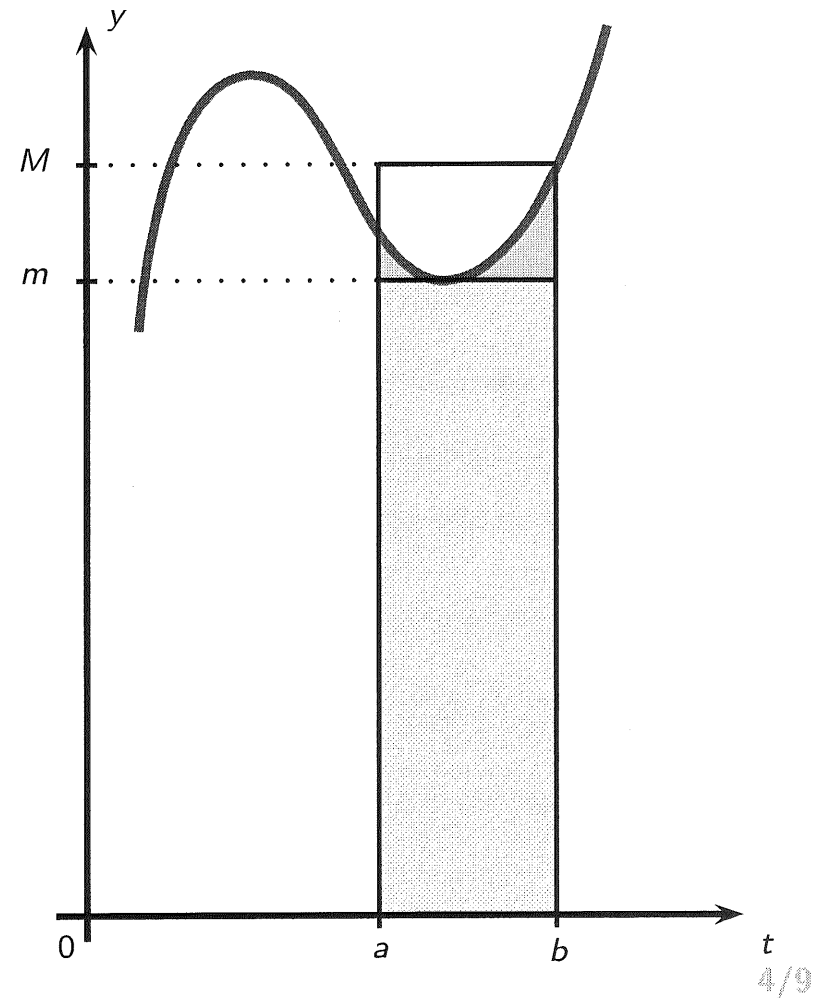
Property 4. says that if $f(x)$ is a positive valued function then the area underneath the graph of $f(x)$ between a and b plus the area underneath the graph of $f(x)$ between b and c equals the area underneath the graph of $f(x)$ between a and c .



Property **5.** follows from Properties **4.** and **1.** by letting $c = a$.

$$0 = \int_a^a f(x) dx = \int_a^b f(x) dx + \int_b^a f(x) dx.$$

Property **6.** is illustrated in the picture below.



Example 1: (Online Homework, HW23, # 8)

The sum

$$\int_{-2}^2 f(x) dx + \int_2^5 f(x) dx - \int_{-2}^{-1} f(x) dx$$

can be written as a single definite integral of the form

$$\int_a^b f(x) dx$$

for appropriate a and b . Determine these values.

$$\int_{-2}^2 f(x) dx + \int_2^5 f(x) dx - \underbrace{\int_{-2}^{-1} f(x) dx}_{-2} + \int_{-1}^2 f(x) dx$$

because of property 5.

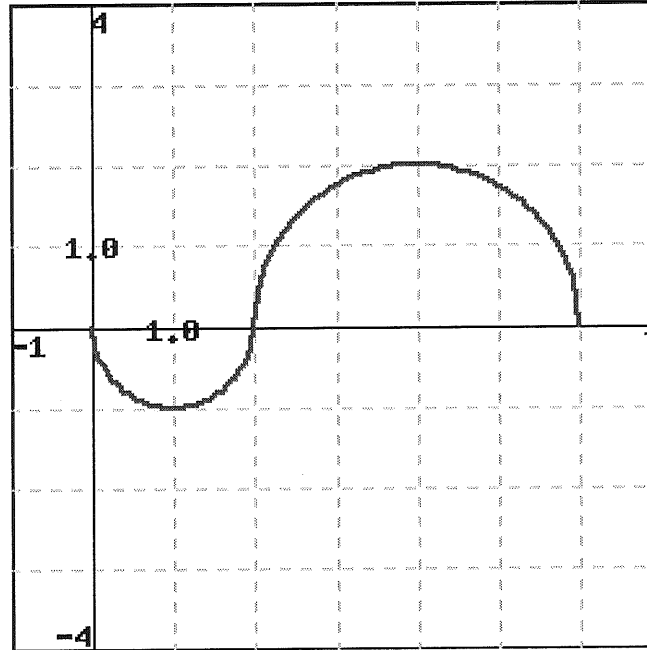
$$= + \int_{-1}^{\overset{-2}{\circ}} f(x) dx + \int_{\underset{-2}{\circ}}^{\overset{2}{\circ}} f(x) dx + \int_{\underset{2}{\circ}}^5 f(x) dx$$

$$= \int_{-1}^5 f(x) dx$$

because of property 4.

Example 2: (Online Homework, HW23, # 5)

Evaluate the integrals for $f(x)$ shown in the figure below. The two parts of the graph are semicircles.

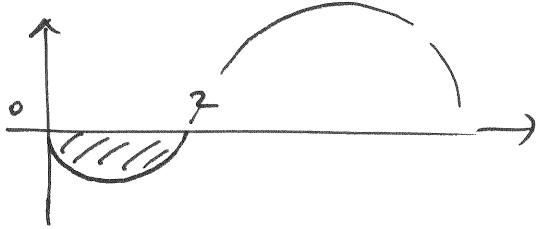


$$\int_0^2 f(x) dx$$

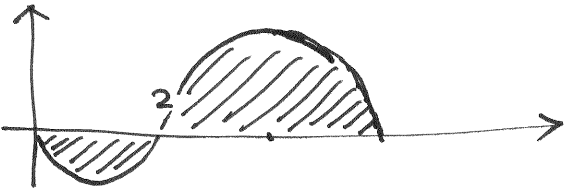
$$\int_0^6 f(x) dx$$

$$\int_1^4 f(x) dx$$

$$\int_1^6 |f(x)| dx.$$

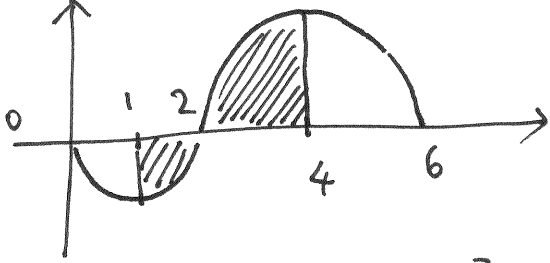
$$\int_0^2 f(x) dx =$$


$$= -\frac{\pi(1)^2}{2} = -\frac{\pi}{2} \approx -1.5708$$

$$\int_0^6 f(x) dx =$$


$$= -\frac{\pi(1)^2}{2} + \frac{\pi(2)^2}{2} = -\frac{\pi}{2} + 2\pi = \frac{3}{2}\pi$$

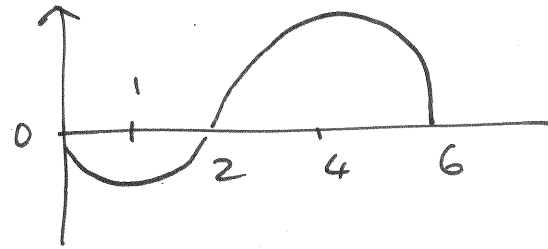
$$\approx 4.7124$$

$$\int_{-1}^4 f(x) dx =$$


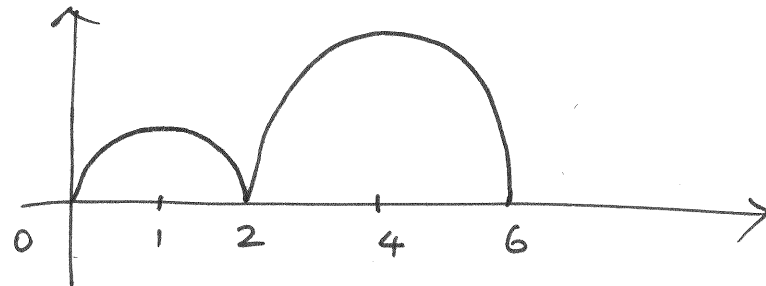
$$= -\frac{\pi(1)^2}{4} + \frac{\pi(2)^2}{4} = -\frac{\pi}{4} + \pi = \frac{3}{4}\pi$$

$$\approx 2.3562$$

If f has the graph



then $|f|$ has the following graph



Hence

$$\int_0^6 |f(x)| dx =$$

$$= + \frac{\pi(1)^2}{4} + \frac{\pi(2)^2}{2} = \frac{\pi}{4} + 2\pi = \boxed{\frac{9}{4}\pi}$$

$$\cong 7.0686$$

Example 3: (Neuhauser, Problem # 61, p. 293)

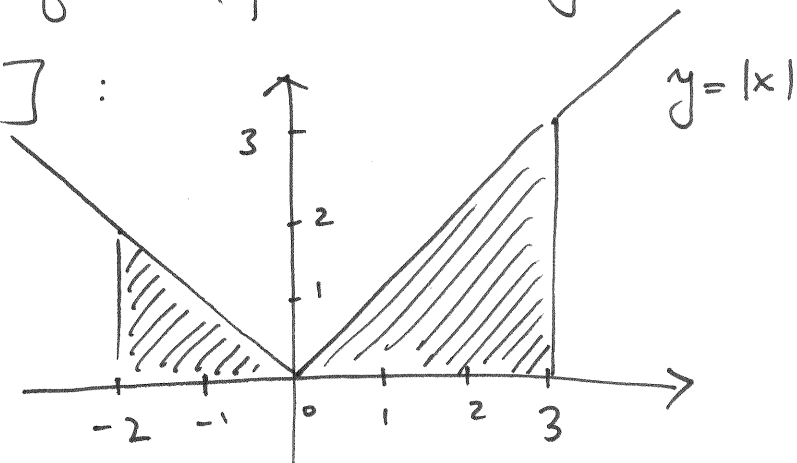
Use an area formula from geometry to find the value of the integral below

$$\int_{-2}^3 |x| dx$$

by interpreting it as the (signed) area under the graph of an appropriately chosen function.

$$\int_{-2}^3 |x| dx$$

Let's look at the graph of the function $y = |x|$ over the interval $[-2, 3]$:



Hence $\int_{-2}^3 |x| dx$ gives the area of the 2 shaded regions (which are triangles):

$$\int_{-2}^3 |x| dx = \frac{2 \cdot 2}{2} + \frac{3 \cdot 3}{2} = \frac{13}{2} = \underline{\underline{6.5}}$$

Example 4: (Neuhauser, Problem # 65, p. 293)

Use an area formula from geometry to find the value of the integral below

$$\int_{-2}^2 (\sqrt{4-x^2} - 2) dx$$

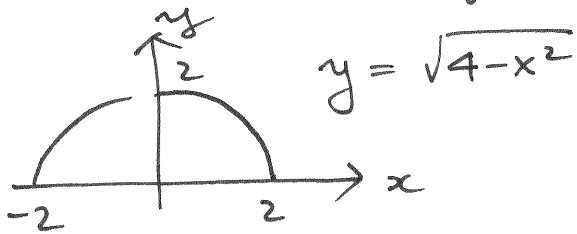
by interpreting it as the (signed) area under the graph of an appropriately chosen function.

$$\int_{-2}^2 (\sqrt{4-x^2} - 2) dx$$

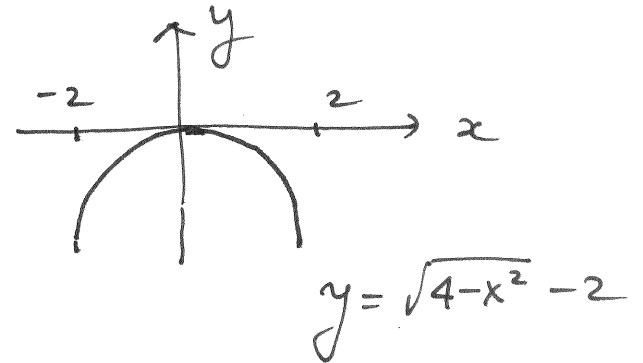
Let's graph the function

$$y = \sqrt{4-x^2} - 2 \quad \text{on } [-2, 2]$$

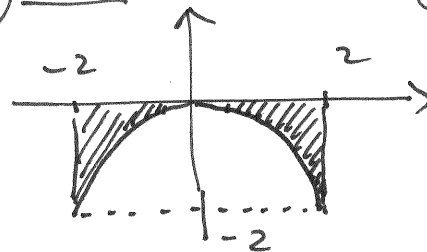
Notice that $y = \sqrt{4-x^2}$ by itself is the graph of the upper half of the circle of radius 2 centered at the origin:



hence



$$\int_{-2}^2 (\sqrt{4-x^2} - 2) dx = \text{"signed" area of the region}$$



$$\approx -1.7168$$

$$= - \left[\text{area rectangle} - \text{area semicircle} \right] = - \left[4 \cdot 2 - \frac{\pi \cdot 2^2}{2} \right]$$

Example 5: (Neuhauser, Problem # 68(c),(f), p. 293)

Given that

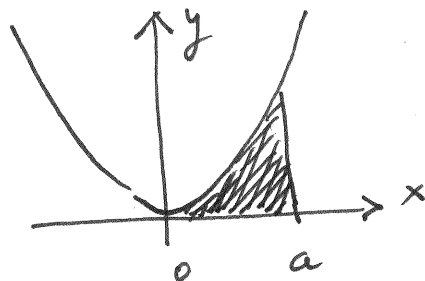
$$\int_0^a x^2 dx = \frac{1}{3}a^3$$

evaluate the following

$$\int_{-1}^3 \frac{1}{3}x^2 dx$$

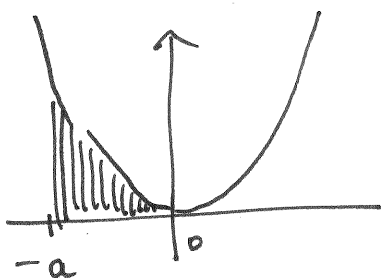
$$\int_2^4 (x - 2)^2 dx.$$

$$\int_0^a x^2 dx = \frac{1}{3} a^3 = \text{area of the shaded region}$$



By symmetry of the function we also have that

$$\int_{-a}^0 x^2 dx =$$



$$= \frac{1}{3} a^3$$

positive sign

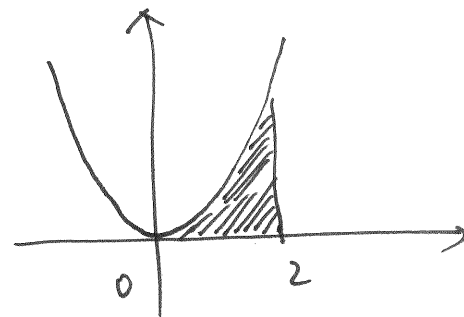
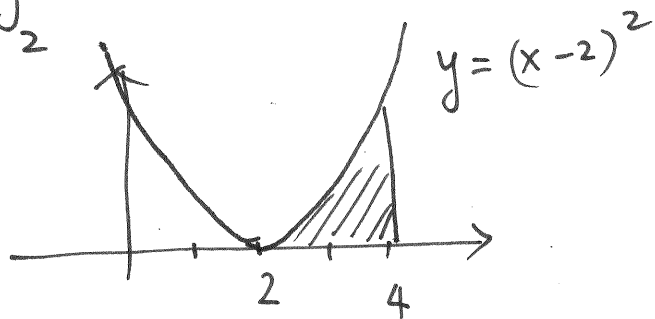
Hence :

$$\int_{-1}^3 \frac{1}{3} x^2 dx = \frac{1}{3} \int_{-1}^3 x^2 dx = \frac{1}{3} \left[\int_{-1}^0 x^2 dx + \int_0^3 x^2 dx \right]$$

$$= \frac{1}{3} \left[\frac{1}{3} \cdot 1^2 + \frac{1}{3} \cdot 3^2 \right] = \frac{1}{3} \left[\frac{1}{3} + 3 \right] = \boxed{\frac{10}{9}}$$

the graph of $y = (x-2)^2$ is obtained from the graph of $y = x^2$ by shifting it of 2 units to the right; hence:

$$\int_2^4 (x-2)^2 dx \quad \text{must be the same as} \quad \int_0^2 x^2 dx$$



Hence

$$\int_2^4 (x-2)^2 dx = \int_0^2 x^2 dx = \frac{1}{3} 2^3 = \boxed{\frac{8}{3}}$$