

# MA137 – Calculus 1 with Life Science Applications

## Exponential and Logarithmic Functions

(Sections 1.1 and 1.2)

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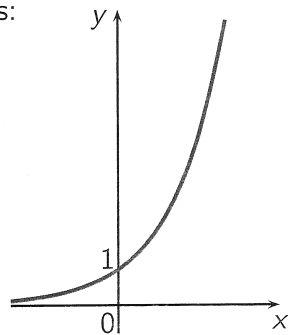
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## Exponential Functions

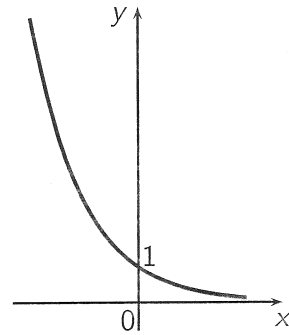
The **exponential function**

$$f(x) = a^x \quad (a > 0, a \neq 1)$$

has domain  $\mathbb{R}$  and range  $(0, \infty)$ . The graph of  $f(x)$  has one of these shapes:



$f(x) = a^x$  for  $a > 1$



$f(x) = a^x$   
 for  $0 < a < 1$

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## Laws of Exponents

Let  $a$  and  $b$  be real numbers so that  $a, b > 0$  and  $a, b \neq 1$ .

- $a^0 = 1$
- $a^u a^v = a^{u+v}$
- $\frac{a^u}{a^v} = a^{u-v}$
- $(a^u)^v = a^{uv}$
- $(ab)^u = a^u b^u$

In particular,  $\frac{1}{a^v} = \frac{a^0}{a^v} = a^{0-v} = a^{-v}$

In particular,  $a^{1/n} = \sqrt[n]{a}$

$$\left(\frac{a}{b}\right)^u = \frac{a^u}{b^u}$$

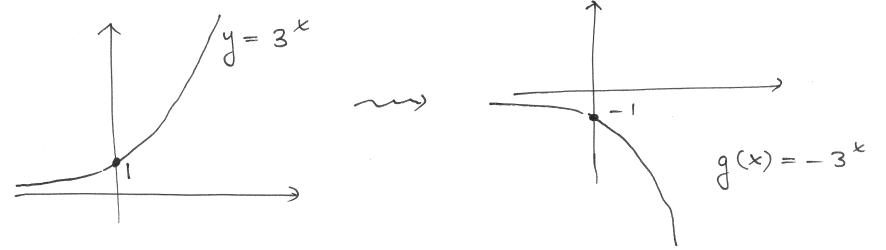
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**Example 1:**

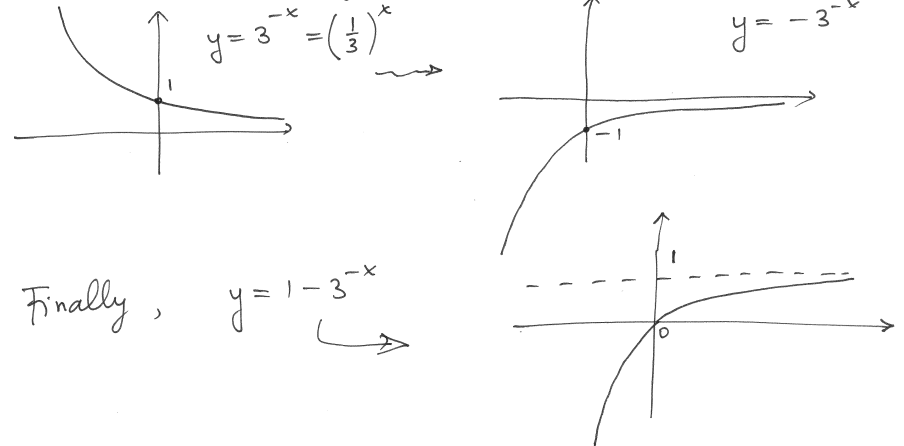
Use the graph of  $f(x) = 3^x$  to sketch the graph of each function:

$g(x) = -3^x$

$h(x) = 1 - 3^{-x}$



Let's construct the graph of  $h(x) = 1 - 3^{-x}$  in steps:



Why are exponential functions of interest?  
Suppose that we study a population of 100 individuals and suppose that it grows annually at a 3% rate. Describe the population growth at time  $t$ .  
(You can also consider \$100 in a bank growing annually at a 3%.)

$P_0 = P(0) = 100$  ; growth rate = 3% or  $r = 0.03$

$P(1) = 100 + \underbrace{0.03 \cdot 100}_{\text{growth}} = 100 + 3 = 103 = 100(1.03)$

$P(2) = 103 + \underbrace{0.03 \cdot 103}_{\text{growth}} = 103(1 + 0.03) = 100(1.03)(1.03)$   
 $= 100(1.03)^2$

In general  $P(t) = 100(1.03)^t$  OR  $P(t) = P_0(1+r)^t$

The formula for compounded interest is  $P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt}$  (see earlier discussion with  $n=1$ )

where  $P_0$  = initial principal  
 $r$  = interest rate per year  
 $n$  = number of times interest is compounded per year  
 $t$  = number of years.

If  $n$  becomes very large ( $\equiv$  interest is compounded continuously) the above formula becomes

$P(t) = P_0 e^{rt}$

## The Number 'e' (Euler's constant)

The most important base is the number denoted by the letter  $e$ .

The number  $e$  is defined as the value that  $\left(1 + \frac{1}{n}\right)^n$  approaches as  $n$  becomes very large.

Correct to five decimal places (note that  $e$  is an irrational number),  $e \approx 2.71828$ .

$n$	$\left(1 + \frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

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## The Natural Exponential Function

The Natural Exponential Function  
The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base  $e$ . It is often referred to as the exponential function.

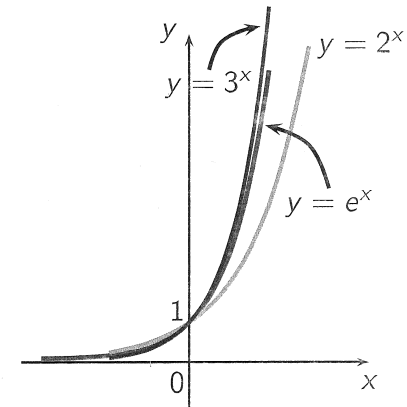
**Note:**

Sometimes we write

$$f(x) = \exp(x)$$

to denote the exponential function.

Since  $2 < e < 3$ , the graph of  $y = e^x$  lies between the graphs of  $y = 2^x$  and  $y = 3^x$ .



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### Example 2:

When a certain drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after  $t$  hours is modeled by

$$D(t) = 50e^{-0.2t}$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

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(\*) Notice that  $D(0) = 50e^{-0.2 \cdot 0} = 50e^0 = 50$

Thus 50 is the initial amount of drug administered to the patient.

(\*\*)  $D(3) = 50e^{-0.2 \cdot 3} = 50e^{-0.6} \approx 27.44$  mg

(\*\*\*) It would have been more interesting to ask: How long do we need to wait so that the bloodstream of the patient only has 25 mg left of drug?

$$25 = D(\bar{t}) = 50e^{-0.2\bar{t}} \iff$$

$$\frac{1}{2} = e^{-0.2\bar{t}}$$

How do we solve for  $\bar{t}$ ?

## Logarithmic Functions

Every exponential function  $f(x) = a^x$ , with  $0 < a \neq 1$ , is a one-to-one function (Horizontal Line Test). Thus, it has an inverse function, called the *logarithmic function with base a* and denoted by  $\log_a x$ .

### Definition

Let  $a$  be a positive number with  $a \neq 1$ . The **logarithmic function** with base  $a$ , denoted by  $\log_a$ , is defined by

$$y = \log_a x \iff a^y = x.$$

That is,  $\log_a x$  is the exponent to which  $a$  must be raised to give  $x$ .

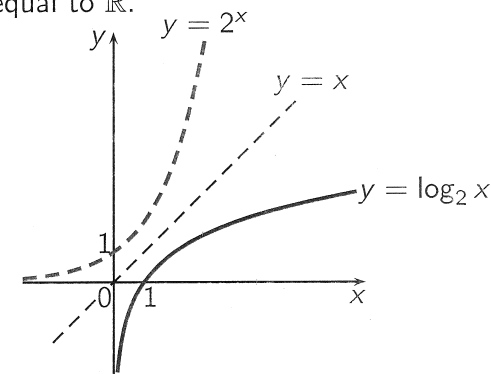
### Properties of Logarithms

1.  $\log_a 1 = 0$
2.  $\log_a a = 1$
3.  $\log_a a^x = x$
4.  $a^{\log_a x} = x$

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## Graphs of Logarithmic Functions

The graph of  $f^{-1}(x) = \log_a x$  is obtained by reflecting the graph of  $f(x) = a^x$  in the line  $y = x$ . Thus, the function  $y = \log_a x$  is defined for  $x > 0$  and has range equal to  $\mathbb{R}$ .



The point  $(1, 0)$  is on the graph of  $y = \log_a x$  (as  $\log_a 1 = 0$ ) and the  $y$ -axis is a vertical asymptote.

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## Natural Logarithms

Of all possible bases  $a$  for logarithms, it turns out that the most convenient choice for the purposes of Calculus is the number  $e$ .

### Definition

The logarithm with base  $e$  is called the **natural logarithm** and denoted:

$$\ln x := \log_e x.$$

We recall again that, by the definition of inverse functions, we have

$$y = \ln x \iff e^y = x.$$

### Properties of Natural Logarithms

1.  $\ln 1 = 0$
2.  $\ln e = 1$
3.  $\ln e^x = x$
4.  $e^{\ln x} = x$

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## Common Logarithms

Another convenient choice of base for the purposes of the Life Sciences is the number 10.

### Definition

The logarithm with base 10 is called the **common logarithm** and denoted:

$$\log x := \log_{10} x.$$

We recall again that, by the definition of inverse functions, we have

$$y = \log x \iff 10^y = x.$$

### Properties of Natural Logarithms

1.  $\log 1 = 0$
2.  $\log 10 = 1$
3.  $\log 10^x = x$
4.  $10^{\log x} = x$

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## Laws of Logarithms

Since logarithms are 'exponents', the Laws of Exponents give rise to the Laws of Logarithms:

### Laws of Logarithms

Let  $a$  be a positive number, with  $a \neq 1$ . Let  $A$ ,  $B$  and  $C$  be any real numbers with  $A > 0$  and  $B > 0$ .

1.  $\log_a(AB) = \log_a A + \log_a B$ ;
2.  $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$ ;
3.  $\log_a(A^C) = C \log_a A$ .

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## Proof of Law 1.: $\log_a(AB) = \log_a A + \log_a B$

Let us set

$$\log_a A = u \quad \text{and} \quad \log_a B = v.$$

When written in exponential form, they become

$$a^u = A \quad \text{and} \quad a^v = B.$$

$$\begin{aligned} \text{Thus: } \log_a(AB) &= \log_a(a^u a^v) \\ &= \log_a(a^{u+v}) \\ &\stackrel{\text{why?}}{=} u + v \\ &= \log_a A + \log_a B. \end{aligned}$$

In a similar fashion, one can prove **2.** and **3.**

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## Expanding and Combining Logarithmic Expressions

### Example 3:

Use the Laws of Logarithms to combine the expression

$$\log_a b + c \log_a d - r \log_a s - \log_a t$$

into a single logarithm.

$$\begin{aligned} &\log_a b + c \log_a d - r \log_a s - \log_a t \\ &= [\log_a b + \log_a(d^c)] - [\log_a(s^r) + \log_a t] \\ &= \log_a(bd^c) - \log_a(s^r t) \\ &= \log_a\left(\frac{bd^c}{s^r t}\right) \end{aligned}$$

We used properties 1.-3. of Logarithms.

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## Change of Base

For some purposes, we find it useful to change from logarithms in one base to logarithms in another base. One can prove that:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

**Proof:** Set  $y = \log_b x$ . By definition, this means that  $b^y = x$ . Apply now  $\log_a(\cdot)$  to  $b^y = x$ . We obtain

$$\log_a(b^y) = \log_a x \quad \rightsquigarrow \quad y \log_a b = \log_a x.$$

Thus

$$\log_b x = y = \frac{\log_a x}{\log_a b}$$

**Example:**  $\log_5 2 = \frac{\log 2}{\log 5} = \frac{\ln 2}{\ln 5} \approx 0.43068.$

## Exponential Equations

An exponential equation is one in which the variable occurs in the exponent. For example,

$$3^{x+2} = 7.$$

We take the (either common or natural) logarithm of each side and then use the Laws of Logarithms to 'bring down the variable' from the exponent:

$$\log(3^{x+2}) = \log 7$$

$$\rightsquigarrow (x+2) \log 3 = \log 7$$

$$\rightsquigarrow x+2 = \frac{\log 7}{\log 3}$$

$$\rightsquigarrow x = \frac{\log 7}{\log 3} - 2 \approx -0.228756$$

### Example 4: (Online Homework HW03, # 6)

Solve the given equation for x:

$$2^{5x-4} = 3^{10x-10}$$

$$2^{5x-4} = 3^{10x-10}$$

Take log of both sides (OR ln)

$$\log(2^{5x-4}) = \log(3^{10x-10})$$

$$\Leftrightarrow (5x-4) \log 2 = (10x-10) \log 3$$

$$\Leftrightarrow (5 \log 2) x - 4 \log 2 = (10 \log 3) x - 10 \log 3$$

$$\Leftrightarrow (10 \log 3) x - (5 \log 2) x = 10 \log 3 - 4 \log 2$$

$$\Leftrightarrow [10 \log 3 - 5 \log 2] x = 10 \log 3 - 4 \log 2$$

$$\Leftrightarrow x = \frac{(10 \log 3 - 4 \log 2)}{(10 \log 3 - 5 \log 2)} = \frac{\log\left(\frac{3^{10}}{2^4}\right)}{\log\left(\frac{3^{10}}{2^5}\right)} \approx \underline{\underline{1.09216}}$$

## Logarithmic Equations

A logarithmic equation is one in which a logarithm of the variable occurs. For example,

$$\log_2(25 - x) = 3.$$

To solve for  $x$ , we write the equation in exponential form, and then solve for the variable:

$$25 - x = 2^3 \rightsquigarrow 25 - x = 8 \rightsquigarrow x = 17.$$

Alternatively, we raise the base, 2, to each side of the equation; we then use the Laws of Logarithms:

$$2^{\log_2(25-x)} = 2^3 \rightsquigarrow 25 - x = 2^3 \rightsquigarrow x = 17.$$

## Example 5: (Online Homework HW03, # 5)

Solve the given equation for  $x$ :

$$\log_{10} x + \log_{10}(x + 21) = 2$$

$$\log_{10} x + \log_{10} (x+21) = 2$$

$$\Leftrightarrow \log_{10} [x(x+21)] = 2$$

$$\Leftrightarrow \log_{10} [x(x+21)] = 2$$

$$\Leftrightarrow \boxed{x(x+21) = 100}$$

$$\Leftrightarrow x^2 + 21x - 100 = 0$$

$$\Leftrightarrow (x+25)(x-4) = 0$$

$$\Leftrightarrow \underline{x = -25, 4}$$

HOWEVER,  $\log_{10}(-25) + \log_{10}(-25+21) = 2$   
 does not make any sense! So  $x = 4$  is the  
 only solution.