

MA 137 – Calculus 1 with Life Science Applications

The Chain Rule and Higher Derivatives

(Section 4.4)

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The Chain Rule

Theorem

If g is differentiable at x and f is differentiable at $y = g(x)$, then the composite function $(f \circ g)(x) = f[g(x)]$ is differentiable at x , and the derivative is given by

$$(f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$

- The proof of the theorem is on p. 164 of the Neuhouser's textbook.
- The function g is the inner function; the function f is the outer function.
- The expression $f'[g(x)] \cdot g'(x)$ thus means that we need to find the derivative of the outer function, evaluated at $g(x)$, and the derivative of the inner function, evaluated at x , and then multiply the two together.
- A special case of the chain rule is called the **power chain rule**:

$$\text{If } y = [f(x)]^n \text{ then } \frac{dy}{dx} = n[f(x)]^{n-1} \cdot f'(x)$$

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The Chain Rule in Leibniz Notation

The derivative of $f \circ g$ can be written in Leibniz notation.

If we set $u = g(x)$, then

$$\begin{aligned} \frac{d}{dx}[(f \circ g)(x)] &= \frac{d}{dx}f[g(x)] \\ &\stackrel{u=g(x)}{=} \frac{d}{dx}f(u) \\ &= \frac{df}{du} \cdot \frac{du}{dx} \end{aligned}$$

This form of the chain rule emphasizes that, in order to differentiate $f \circ g$, we multiply the derivative of the outer function and the derivative of the inner function, the former evaluated at u , the latter at x .

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Example 1: (Online Homework HW13, # 3)

Let $F(x) = f(f(x))$ and $G(x) = (F(x))^2$ and suppose that

$$f(5) = 3 \quad f(7) = 5 \quad f'(5) = 8 \quad f'(7) = 13$$

Find $F'(7)$ and $G'(7)$.

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$$f(5) = 3 \quad f(7) = 5 \quad f'(5) = 8 \quad f'(7) = 13$$

$$(1) \quad \boxed{F(x) = f(f(x))}$$

By the chain rule $F'(x) = f'(f(x)) \cdot f'(x)$

$$\begin{aligned} \text{Thus } F'(7) &= f'(f(7)) \cdot f'(7) = f'(5) \cdot f'(7) \\ &= 8 \cdot 13 = \underline{\underline{104}} \end{aligned}$$

$$(2) \quad \boxed{G(x) = (F(x))^2}$$

By the (power) chain rule $G'(x) = 2 F(x) \cdot F'(x)$

$$\text{Thus } G'(7) = 2 F(7) \cdot F'(7) = 2 \cdot 3 \cdot 104 = \underline{\underline{624}}$$

$$F(7) = f(f(7)) = f(5) = 3$$

Example 2: (Online Homework HW13, # 6)

Let $f(x) = \frac{9}{(2x^2 - 3x + 6)^4}$. Find $f'(x)$.

$$f(x) = \frac{9}{(2x^2 - 3x + 6)^4} = 9(2x^2 - 3x + 6)^{-4}$$

Hence, by the (power) chain rule

$$f'(x) = 9 \cdot (-4) (2x^2 - 3x + 6)^{-4-1} \cdot (4x - 3)$$

$$= \boxed{\frac{-36(4x-3)}{(2x^2-3x+6)^5}}$$

The Quotient Rule Using the Chain Rule

We can prove quotient rule using the product and (power) chain rules. Treat the quotient f/g as a product of f and the reciprocal of g . I.e.,

$$\frac{f(x)}{g(x)} = f(x) \cdot g(x)^{-1}$$

Next, apply the product rule

$$\left(\frac{f(x)}{g(x)}\right)' = [f(x) \cdot g(x)^{-1}]' = f'(x) \cdot g(x)^{-1} + f(x) \cdot [g(x)^{-1}]'$$

and apply the (power) chain rule to find $[g(x)^{-1}]'$. We obtain

$$= f'(x) \cdot [g(x)^{-1}] + f(x) \cdot [(-1)g(x)^{-2} \cdot g'(x)].$$

Finish by writing the expression with a common denominator of $[g(x)]^2$

$$= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

Example 3: (Neuhauser, Example # 5, p. 161)

Find the derivative of $h(x) = \left(\frac{x}{x+1}\right)^2$.

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$$h(x) = \left(\frac{x}{x+1}\right)^2$$

By the (power) chain rule

$$h'(x) = 2 \left(\frac{x}{x+1}\right)^{2-1} \cdot \frac{d}{dx} \left(\frac{x}{x+1}\right)$$

$$= 2 \cdot \frac{x}{x+1} \cdot \frac{1 \cdot (x+1) - x(1)}{(x+1)^2}$$

$$= \boxed{2 \cdot \frac{x}{(x+1)^3}}$$

Example 4: (Neuhauser, Problem # 32, p. 172)

Differentiate $g(N) = \frac{N}{(k+bN)^3}$ with respect to N .

Assume that b and k are positive constants.

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$$g(N) = \frac{N}{(k+bN)^3}$$

Use the quotient rule and the chain rule:

$$g'(N) = \frac{1 \cdot (k+bN)^3 - N \cdot 3(k+bN)^{3-1} \cdot (b)}{[(k+bN)^3]^2}$$

$$= \frac{(k+bN)^3 - 3bN(k+b)^2}{(k+bN)^6}$$

$$= \frac{(k+bN)^2 [k+bN - 3bN]}{(k+bN)^6}$$

$$= \boxed{\frac{k - 2bN}{(k+bN)^4}}$$

Example 5: (Neuhauser, Problem # 39, p. 172)

Find the derivative of

$$\frac{[f(x)]^2}{g(2x) + 2x}$$

assuming that f and g are both differentiable functions.

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$$y = \frac{[f(x)]^2}{g(2x) + 2x}$$

We find y' using the quotient rule and the chain rule:

$$\begin{aligned} y' &= \frac{\{[f(x)]^2\}' \cdot (g(2x) + 2x) - [f(x)]^2 \cdot \{g(2x) + 2x\}'}{(g(2x) + 2x)^2} \\ &= \frac{2f(x) \cdot f'(x) \cdot (g(2x) + 2x) - [f(x)]^2 \cdot (g'(2x) \cdot 2 + 2)}{(g(2x) + 2x)^2} \\ &= \frac{2f(x) \cdot [f'(x)(g(2x) + 2x) - f(x)(g'(2x) + 1)]}{[g(2x) + 2x]^2} \end{aligned}$$

Higher Derivatives

- The derivative of a function f is itself a function. We refer to this derivative as the **first derivative**, denoted f' . If the first derivative exists, we say that the function is once differentiable.
- Given that the first derivative is a function, we can define its derivative (where it exists). This derivative is called the **second derivative** and is denoted f'' . If the second derivative exists, we say that the original function is twice differentiable.
- This second derivative is again a function; hence, we can define its derivative (where it exists). The result is the **third derivative**, denoted f''' . If the third derivative exists, we say that the original function is three times differentiable.
- We can continue in this manner; from the fourth derivative on, we denote the derivatives by $f^{(4)}$, $f^{(5)}$, and so on. If the n th derivative exists, we say that the original function is n times differentiable.

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- Polynomials are functions that can be differentiated as many times as desired. The reason is that the first derivative of a polynomial of degree n is a polynomial of degree $n - 1$. Since the derivative is a polynomial as well, we can find its derivative, and so on. Eventually, the derivative will be equal to 0.
- We can write higher-order derivatives in Leibniz notation: The n th derivative of $f(x)$ is denoted by

$$\frac{d^n f}{dx^n}$$

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Example 6: (Online Homework HW13, # 4)

Find the first and second derivatives of the following function

$$f(x) = (5 - 3x^2)^4$$

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$$f(x) = (5 - 3x^2)^4$$

Then:

$$\begin{aligned} f'(x) &= 4(5 - 3x^2)^3 \cdot (-6x) \\ &= \underline{-24x(5 - 3x^2)^3} \end{aligned}$$

$$\begin{aligned} f''(x) &= -24(5 - 3x^2)^3 - 24x \cdot [3(5 - 3x^2)^2 \cdot (-6x)] \\ &= -24(5 - 3x^2)^3 + 18 \cdot 24x^2(5 - 3x^2)^2 \\ &= 24(5 - 3x^2)^2 \cdot [-(5 - 3x^2) + 18x^2] \\ &= \boxed{24(5 - 3x^2)^2 \cdot (21x^2 - 5)} \end{aligned}$$

Example 7: (Online Homework HW13, # 16)

Find the first and second derivatives of the following function

$$y = \frac{1 - 4u}{1 + 3u}$$

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$$y = \frac{1 - 4u}{1 + 3u}$$

$$\begin{aligned} y' &= \frac{-4(1 + 3u) - (1 - 4u)(3)}{(1 + 3u)^2} \\ &= \frac{-4 - 12u - 3 + 12u}{(1 + 3u)^2} = \boxed{\frac{-7}{(1 + 3u)^2}} \end{aligned}$$

$$= -7(1 + 3u)^{-2}$$

$$\begin{aligned} y'' &= -7(-2)(1 + 3u)^{-2-1} \cdot (3) \\ &= \boxed{\frac{42}{(1 + 3u)^3}} \end{aligned}$$

Velocity and Acceleration

The velocity of an object that moves on a straight line is the derivative of the object's position. The derivative of the velocity is the acceleration.

If $s(t)$ denotes the position of an object moving on a straight line, $v(t)$ its velocity, and $a(t)$ its acceleration, then the three quantities are related as follows:

$$v(t) = \frac{ds}{dt} \quad \text{and} \quad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

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Example 8: (Neuhauser, Problem # 87, p. 173)

Neglecting air resistance, the height h (in meters) of an object thrown vertically from the ground with initial velocity v_0 is given by

$$h(t) = v_0 t - \frac{1}{2} g t^2$$

where $g = 9.81 \text{ m/s}^2$ is the earth's gravitational constant and t is the time (in seconds) elapsed since the object was released.

- Find the velocity and the acceleration of the object.
- Find the time when the velocity is equal to 0. In which direction is the object traveling right before this time? in which direction right after this time?

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$$h(t) = v_0 t - \frac{1}{2} g t^2$$

$$(a) \quad v(t) = h'(t) = \frac{dh}{dt} = v_0 - \frac{1}{2} g \cdot 2t$$

$$= \boxed{v_0 - g t}$$

$$a(t) = v'(t) = h''(t) = \frac{d^2h}{dt^2} = \boxed{-g}$$

$$(b) \quad v(t) = 0 \iff v_0 - g t = 0 \iff$$

$$\boxed{t = \frac{v_0}{g}}$$

Before $\frac{v_0}{g}$ we have that $v(t)$ is positive so the object goes up; after $\frac{v_0}{g}$ the velocity is negative so the object goes down.