

MA 137 — Calculus 1 with Life Science Applications

Derivatives of Exponential Functions

(Section 4.6)

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The Derivative of the Natural Exponential Function

Theorem

The function e^x is differentiable for all x , and $\frac{d}{dx} e^x = e^x$.

In particular, if $g(x)$ is a differentiable function, it follows from the chain rule that

$$\frac{d}{dx} e^{g(x)} = e^{g(x)} \cdot g'(x).$$

We need to know the following limit to compute the derivative of the natural exponential function. Namely,

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1.$$

Although we cannot rigorously prove this result here, the table below should convince you of its validity

h	-0.1	-0.01	-0.001	...	0.001	0.01	0.1
$\frac{e^h - 1}{h}$	0.9516	0.9950	0.9995		1.0005	1.0050	1.0517

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Proof

We use the formal definition of the derivative. In the final step, we will be able to write the term e^x in front of the limit because e^x does not depend on h .

$$\begin{aligned} \frac{d}{dx} e^x &\stackrel{\text{def}}{=} \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} \\ &\stackrel{\text{exp. prop.}}{=} \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x (e^h - 1)}{h} \\ &\stackrel{\text{laws}}{=} e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h} \\ &\stackrel{\text{fund. lim.}}{=} e^x \cdot 1 \\ &= e^x \end{aligned}$$

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The Derivative of ANY Exponential Function

Theorem

The function a^x is differentiable for all x , and $\frac{d}{dx} a^x = a^x \cdot \ln a$.

In particular, if $g(x)$ is a differentiable function, it follows from the chain rule that

$$\frac{d}{dx} a^{g(x)} = a^{g(x)} \cdot \ln a \cdot g'(x).$$

We can prove the above result using the definition of the derivative and the limit

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \ln a,$$

in the same manner that we did for the natural exponential function.

Alternatively, we can use the following identity

$$a^x = e^{\ln a^x} = e^{x \ln a}$$

and the chain rule. Namely,

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \ln a = a^x \cdot \ln a.$$

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Example 1: (Nuehauser, Example # 1, p. 179)Find the derivative of $f(x) = e^{-x^2/2}$.

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$$f(x) = e^{-x^2/2}$$

We need to use the chain rule

$$\begin{aligned}
 f'(x) &= e^{-x^2/2} \cdot \frac{d}{dx} \left(-\frac{x^2}{2} \right) \\
 &= e^{-x^2/2} \cdot \left(-\frac{1}{2} \cdot 2x \right) = \boxed{-x e^{-x^2/2}}
 \end{aligned}$$

Example 2:Find the derivative with respect to x of $g(x) = xe^{-x}$.Evaluate $g'(x)$ at $x = 1$.

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$$g(x) = xe^{-x}$$

$$\begin{aligned}
 g'(x) &= 1 \cdot e^{-x} + x \cdot \frac{d}{dx} (e^{-x}) \\
 &\quad \uparrow \text{product rule} \\
 &= e^{-x} + x \left[e^{-x} (-1) \right] \\
 &\quad \quad \quad \uparrow \text{chain rule} \\
 &= e^{-x} - xe^{-x} \\
 &= \boxed{e^{-x}(1-x)} \quad || \text{u}
 \end{aligned}$$

$$g'(1) = e^{-1} \cdot (1-1) = \underline{\underline{0}}$$

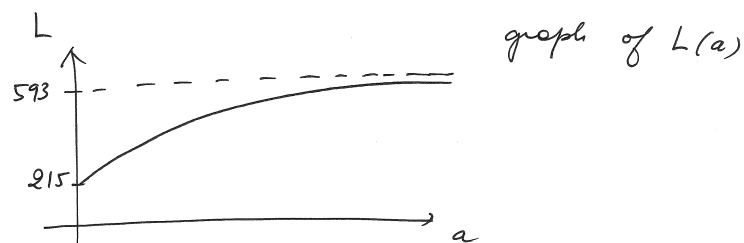
Example 3: (Online Homework HW15, # 14)

The cutlassfish is a valuable resource in the marine fishing industry in China. A von Bertalanffy model is fit to data for one species of this fish giving the length of the fish, $L(t)$ (in mm), as a function of the age, a (in yr). An estimate of the length of this fish is

$$L(a) = 593 - 378e^{-0.166a}$$

- (a) Find the L -intercept.
Find an equation for the horizontal asymptote of $L(a)$.
Find the maximum possible length of this fish.
- (b) Determine how long it takes for this fish to reach 90 percent of its maximum length.
- (c) Differentiate $L(a)$ with respect to a .

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- (b) We need to find the age such that
- $$\underbrace{0.9 \cdot 593}_{90\% \text{ of max. length}} = L(a) = 593 - 378e^{-0.166a}$$
- $$\Leftrightarrow 378e^{-0.166a} = 593 - 533.7$$
- $$\Leftrightarrow e^{-0.166a} = \frac{59.3}{378} \approx 0.15688$$
- $$\Leftrightarrow -0.166a = \ln(0.15688)$$

$$(a) \quad L(a) = 593 - 378e^{-0.166a}$$

To find the L -intercept we set $a=0$

$$L(0) = 593 - 378e^{-0.166 \cdot 0} = 593 - 378 \underbrace{e^0}_1 \\ = 593 - 378 = \underline{215}$$

To get the equation of the horizontal asymptote we need to evaluate $\lim_{a \rightarrow \infty} (593 - 378e^{-0.166a})$

$$= 593 - 378 \underbrace{\lim_{a \rightarrow \infty} e^{-0.166a}}_0 = 593 - 0 = \underline{\underline{593}}$$

Hence the maximum possible length of the fish is (close to) 593

$$\Leftrightarrow a = \frac{\ln(0.15688)}{-0.166} \approx \underline{\underline{11.1583}}$$

$$(c) \quad L(a) = 593 - 378e^{-0.166a}$$

$$\frac{dL}{da} = L'(a) = 0 - 378 \cdot \underbrace{e^{-0.166a} \cdot (-0.166)}_{\text{chain rule}} \\ = 378 \cdot (0.166) e^{-0.166a} \\ = \underline{\underline{62.748 e^{-0.166a}}}$$

notice that the derivative is always positive !!

Example 4: (Neuhauser, Example # 5, p. 180)

Radioactive Decay: Show that the function $W(t) = W_0 e^{-rt}$ satisfies the differential equation

$$\frac{dW}{dt} = -rW(t) \quad W(0) = W_0.$$

[W_0 is the amount of material at time $t = 0$ and r is called the radioactive decay rate.]

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$$W(t) = W_0 e^{-rt}$$

notice that at $t=0$

$$W(0) = W_0 \underbrace{e^{-r \cdot 0}}_{=1} = W_0 \checkmark$$

Let's compute the derivative of $W(t) = W_0 e^{-rt}$

$$\frac{dW}{dt} = W_0 \underbrace{e^{-rt}}_{\text{chain rule}} \cdot (-r)$$

Substitute in the D.E. $\frac{dW}{dt} = -rW$

$$\underbrace{W_0 e^{-rt}}_{(-r)} \stackrel{?}{=} -r \left(W_0 e^{-rt} \right)$$

the two sides are identical!

Example 5: (Neuhauser, Example # 6, p. 181)

Exponential Growth: Show that the function $N(t) = N_0 e^{rt}$ satisfies the differential equation

$$\frac{dN}{dt} = rN(t) \quad N(0) = N_0.$$

[N_0 is the population size at time $t = 0$ and r is called the growth rate.]

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$$N(t) = N_0 e^{rt}$$

notice that at $t=0$ we get

$$N(0) = N_0 \underbrace{e^{r \cdot 0}}_1 = N_0 \checkmark$$

Let's compute the derivative of $N(t) = N_0 e^{rt}$

$$\frac{dN}{dt} = N_0 \underbrace{e^{rt}}_{\text{chain rule}} \cdot (r)$$

Substitute in the D.E. $\frac{dN}{dt} = rN$

$$\underbrace{N_0 e^{rt}}_{(r)} \stackrel{?}{=} r \left(N_0 e^{rt} \right)$$

the two sides are identical!

Example 6: (Neuhauser, Problem # 63, p. 182)

- (a) Find the derivative of the logistic growth curve (Example 3, Section 3.3, p.112)

$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}}$$

- (b) Show that $N(t)$ satisfies the differential equation

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) \quad N(0) = N_0$$

- (c) Plot the per capita rate of growth $\frac{1}{N} \frac{dN}{dt}$ as a function of N , and note that it decreases with increasing population size.

(a)
$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}} = K \left[1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}\right]^{-1}$$

Let's compute the derivative using this form instead of the quotient rule:

$$\begin{aligned} \frac{dN}{dt} = N' &= K (-1) \cdot \left[1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}\right]^{-2} \cdot \left(\frac{K}{N_0} - 1\right) e^{-rt} (-r) \\ &= \frac{K r \left(\frac{K}{N_0} - 1\right) e^{-rt}}{\left[1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}\right]^2} \end{aligned}$$

chain rule

Let us substitute the derivative and the function into the D.E $\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right)$

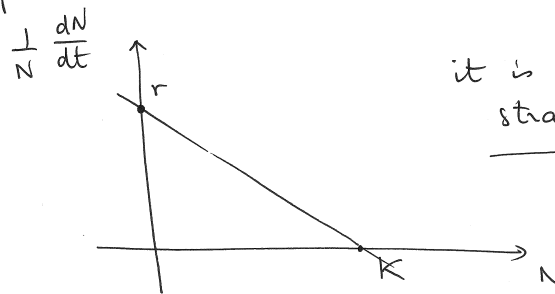
$$\begin{aligned} \frac{dN}{dt} & \stackrel{?}{=} rN \left(1 - \frac{N}{K}\right) \\ \frac{K r \left(\frac{K}{N_0} - 1\right) e^{-rt}}{\left[1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}\right]^2} & \stackrel{?}{=} r \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}} \cdot \left(1 - \frac{K}{K \left(1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}\right)}\right) \\ & \stackrel{?}{=} r \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}} \cdot \frac{\left[1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}\right] - 1}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}} \\ & \stackrel{?}{=} \frac{r K \cdot \left(\frac{K}{N_0} - 1\right) e^{-rt}}{\left[1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}\right]^2} \end{aligned}$$

Yes!!

- (c) As we have discussed in a previous lecture

$$\frac{1}{N} \frac{dN}{dt} = r - \frac{r}{K} N$$

as a function of N has the following graph:



it is a decreasing straight line!