# MA 137 — Calculus 1 with Life Science Applications
# L'Hôpital's Rule
(Section 5.5)

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## Heuristics

We have often encountered the situation in which we had to compute $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ and we had that both the following limits were zero

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0.$$

Using a linear approximation at $x = a$, we find that, for $x$ close to $a$

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)}$$

Since $f(a) = g(a) = 0$ and $x \neq a$, the right-hand side is equal to

$$\frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

provided that $f'(a)/g'(a)$ is defined. We therefore hope that something like

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

holds when $f(a)/g(a)$ is of the form $0/0$ and $f'(a)/g'(a)$ is defined. In fact, something like this does hold; it is called **l'Hôpital's rule**.





## L'Hôpital's Rule

**Theorem**

*Suppose that $f$ and $g$ are differentiable functions and that*

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x) \qquad \text{or} \qquad \lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$$

*Then*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

*provided the second limit exists.*

L'Hôpital's rule can actually be applied to calculate limits for seven kinds of indeterminate expressions

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 0^0 \qquad 1^\infty \qquad \infty^0.$$

(Note that l'Hôpitals rule works for $a = +\infty$ or $-\infty$ as well.)





## Reduction to $0/0$ or $\infty/\infty$ Form

$0\cdot\infty$ Suppose we have to compute $\lim_{x\to a} f(x)g(x)$ where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$. To apply l'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a}\frac{f(x)}{1/g(x)} = \lim_{x\to a}\frac{g(x)}{1/f(x)}$$

In the first case the ratio is $0/0$, whereas in the second case the ratio is $\infty/\infty$. Usually only one of the two expressions is easy to evaluate.

$\infty-\infty$ Suppose we have to compute $\lim_{x\to a}[f(x)-g(x)]$ where $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. To apply l'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x)\left(1 - \frac{g(x)}{f(x)}\right) = \lim_{x\to a} g(x)\left(\frac{f(x)}{g(x)} - 1\right)$$

and hope that the limit is of the form $0\cdot\infty$.

$0^0$ $1^\infty$ $\infty^0$ Suppose we have to compute $\lim_{x\to a}[f(x)]^{g(x)}$, which becomes of the form $0^0$, $1^\infty$ or $\infty^0$. The key to solving these limits is to write them as exponentials

$$\lim_{x\to a}[f(x)]^{g(x)} = \lim_{x\to a}\exp\left\{\ln\,[f(x)]^{g(x)}\right\} = \lim_{x\to a}\exp\left\{g(x)\cdot\ln f(x)\right\} = \exp\left[\lim_{x\to a}\left(g(x)\cdot\ln f(x)\right)\right].$$

The last step, in which we interchanged lim and exp, uses the fact that the exponential function is continuous.

## Example 1: (Nuehauser, p. 247)

Evaluate $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x-3}$.





## Example 2: (Nuehauser, p. 247)

Evaluate $\displaystyle\lim_{x\to 0} \frac{e^x-1}{x}$.





## Example 3:

Evaluate $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$.





## Example 4: (Neuhauser, Problem # 25, p. 252)

Evaluate $\displaystyle\lim_{x\to\infty} x\cdot e^{-x}$.

What about $\displaystyle\lim_{x\to\infty} x^{13}\cdot e^{-x}$ ? (Online Homework HW20, # 5)

## Example 5: (Online Homework HW20, # 3)

Evaluate $\lim_{x\to 0^+} 7\sqrt{x}\cdot \ln x$.





## Example 6: (Neuhauser, Example # 10, p. 250)

Evaluate $\lim_{x\to\infty} x - \sqrt{x^2+x}$.





## Example 7: (Online Homework HW20, # 4)

Evaluate $\lim_{x\to 0^+} x^x$.





## Example 8: (Neuhauser, Problem # 62, p. 253)

Use l'Hôpital's rule to find $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x$ where $c$ is a constant.

What about $\lim_{x\to\infty} 3x\big(\ln(x+3)-\ln x\big)$ ? (Online Homework HW20, #