

# MA 137 — Calculus 1 with Life Science Applications

## L'Hôpital's Rule

(Section 5.5)

**Alberto Corso**

(alberto.corso@uky.edu)

Department of Mathematics  
University of Kentucky

November 15, 2017

1/12

## Heuristics

We have often encountered the situation in which we had to compute

$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  and we had that both the following limits were zero

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0.$$

Using a linear approximation at  $x = a$ , we find that, for  $x$  close to  $a$

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)}$$

Since  $f(a) = g(a) = 0$  and  $x \neq a$ , the right-hand side is equal to

$$\frac{f'(a)(x - a)}{g'(a)(x - a)} = \frac{f'(a)}{g'(a)}$$

provided that  $f'(a)/g'(a)$  is defined. We therefore hope that something like

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

holds when  $f(a)/g(a)$  is of the form  $0/0$  and  $f'(a)/g'(a)$  is defined. In fact, something like this does hold; it is called **L'Hôpital's rule**.

2/12

## L'Hôpital's Rule

### Theorem

Suppose that  $f$  and  $g$  are differentiable functions and that

$$\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x) \quad \text{or} \quad \lim_{x \rightarrow a} f(x) = \infty = \lim_{x \rightarrow a} g(x)$$

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

provided the second limit exists.

L'Hôpital's rule can actually be applied to calculate limits for seven kinds of indeterminate expressions

$$\frac{0}{0} \quad \frac{\infty}{\infty} \quad 0 \cdot \infty \quad \infty - \infty \quad 0^0 \quad 1^\infty \quad \infty^0.$$

(Note that L'Hôpital's rule works for  $a = +\infty$  or  $-\infty$  as well.)

3/12

## Reduction to $0/0$ or $\infty/\infty$ Form

$0 \cdot \infty$  Suppose we have to compute  $\lim_{x \rightarrow a} f(x)g(x)$  where  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = \infty$ . To apply L'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} \frac{f(x)}{1/g(x)} = \lim_{x \rightarrow a} \frac{g(x)}{1/f(x)}$$

In the first case the ratio is  $0/0$ , whereas in the second case the ratio is  $\infty/\infty$ . Usually only one of the two expressions is easy to evaluate.

$\infty - \infty$  Suppose we have to compute  $\lim_{x \rightarrow a} [f(x) - g(x)]$  where  $\lim_{x \rightarrow a} f(x) = \infty$  and  $\lim_{x \rightarrow a} g(x) = \infty$ . To apply L'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) \left( 1 - \frac{g(x)}{f(x)} \right) = \lim_{x \rightarrow a} g(x) \left( \frac{f(x)}{g(x)} - 1 \right)$$

and hope that the limit is of the form  $0 \cdot \infty$ .

$0^0$   $1^\infty$   $\infty^0$  Suppose we have to compute  $\lim_{x \rightarrow a} [f(x)]^{g(x)}$ , which becomes of the form  $0^0$ ,  $1^\infty$  or  $\infty^0$ . The key to solving these limits is to write them as exponentials

$$\lim_{x \rightarrow a} [f(x)]^{g(x)} = \lim_{x \rightarrow a} \exp \left\{ \ln [f(x)]^{g(x)} \right\} = \lim_{x \rightarrow a} \exp \left\{ g(x) \cdot \ln f(x) \right\} = \exp \left[ \lim_{x \rightarrow a} (g(x) \cdot \ln f(x)) \right].$$

The last step, in which we interchanged  $\lim$  and  $\exp$ , uses the fact that the exponential function is continuous.

4/12

**Example 1:** (Nuehauser, p. 247)

Evaluate  $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$ .

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \frac{0}{0} \quad \text{if we use direct evaluation}$$

Hence we can apply l'Hôpital's rule.

We obtain:

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{2x}{1} = \frac{6}{1} = \boxed{6}$$

Note: in Chapter 3 we solved the problem by factoring and simplifying the expression

$$\lim_{x \rightarrow 3} \frac{(x+3)(\cancel{x-3})}{(\cancel{x-3})} = \lim_{x \rightarrow 3} x+3 = 3+3 = 6$$

5/12

**Example 2:** (Nuehauser, p. 247)

Evaluate  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$ .

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \frac{e^0 - 1}{0} = \frac{0}{0}$$

Hence we can use l'Hôpital's rule:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \frac{e^x}{1} = e^0 = \boxed{1}$$

6/12

**Example 3:**

Evaluate  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$ .

7/12

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1 - 1}{0} = \frac{0}{0} \quad \text{by direct substitution}$$

Hence we can use l'Hôpital's rule

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{-(-\sin x)}{2x} = \lim_{x \rightarrow 0} \frac{\sin x}{2x}$$

 $= \frac{0}{0}$  again. Hence we use l'Hôpital's rule again

$$= \lim_{x \rightarrow 0} \frac{\cos x}{2} = \boxed{\frac{1}{2}}$$

Note that in Section 3.4 we gave a geometric argument for  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

**Example 4:** (Neuhauser, Problem # 25, p. 252)

Evaluate  $\lim_{x \rightarrow \infty} x \cdot e^{-x}$ .

What about  $\lim_{x \rightarrow \infty} x^{13} \cdot e^{-x}$ ? (Online Homework HW20, # 5)

8/12

$$\lim_{x \rightarrow \infty} x e^{-x} = \lim_{x \rightarrow \infty} \frac{x}{e^x} = \frac{\infty}{\infty}$$

We can use l'Hôpital rule. Hence

$$\lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = \frac{1}{\infty} = \boxed{0}$$

$$\text{HW20, \#5: } \lim_{x \rightarrow \infty} \frac{x^{13}}{e^x} = \frac{\infty}{\infty} = \text{use l'Hôpital's}$$

$$\text{rule} = \lim_{x \rightarrow \infty} \frac{13x^{12}}{e^x} = \frac{\infty}{\infty} = \dots = \text{use}$$

$$\text{many more times l'Hôpital's rule to get}$$
$$= \lim_{x \rightarrow \infty} \frac{13!}{e^x} = \frac{13!}{\infty} = \boxed{0}$$

**Example 5:** (Online Homework HW20, # 3)Evaluate  $\lim_{x \rightarrow 0^+} 7\sqrt{x} \cdot \ln x$ .

$$\lim_{x \rightarrow 0^+} 7\sqrt{x} \cdot \ln x = 0 \cdot (-\infty)$$

Hence we can use l'Hôpital's rule, where we rewrite  $7\sqrt{x} \ln x$  as  $\frac{7 \ln x}{\frac{1}{\sqrt{x}}} = \frac{7 \ln x}{x^{-1/2}}$

$$\begin{aligned} \text{Hence:} \\ \lim_{x \rightarrow 0^+} 7\sqrt{x} \cdot \ln x &= \lim_{x \rightarrow 0^+} \frac{7 \ln x}{x^{-1/2}} = \frac{-\infty}{+\infty} = \\ &= \text{l'Hôpital} = \lim_{x \rightarrow 0^+} \frac{7 \cdot \frac{1}{x}}{-\frac{1}{2} x^{-3/2}} = \lim_{x \rightarrow 0^+} 7 \cdot \frac{1}{x} \cdot [-2x^{3/2}] \end{aligned}$$

$$= \lim_{x \rightarrow 0^+} -14 \frac{x \sqrt{x}}{x} = \lim_{x \rightarrow 0^+} -14 \sqrt{x} = 0$$

$$\text{Hence } \boxed{\lim_{x \rightarrow 0^+} 7\sqrt{x} \ln x = 0}$$

9/12

**Example 6:** (Neuhauser, Example # 10, p. 250)Evaluate  $\lim_{x \rightarrow \infty} x - \sqrt{x^2 + x}$ .

$$\lim_{x \rightarrow \infty} x - \sqrt{x^2 + x} = \infty - \infty$$

Let's rewrite the limit as follows:

$$\lim_{x \rightarrow \infty} x - \sqrt{x^2 + x} = \lim_{x \rightarrow \infty} x \left[ 1 - \frac{\sqrt{x^2 + x}}{x} \right] =$$

$$= \lim_{x \rightarrow \infty} \frac{1 - \sqrt{1 + \frac{1}{x}}}{\frac{1}{x}} = \frac{0}{0} = \text{hence we can apply l'Hôpital's rule}$$

$$= \lim_{x \rightarrow \infty} \frac{-\frac{1}{2} \left(1 + \frac{1}{x}\right)^{-\frac{1}{2}} \cdot \left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{-\frac{1}{2}}{\sqrt{1 + \frac{1}{x}}} = \underline{\underline{\underline{-\frac{1}{2}}}}$$

10/12

**Example 7:** (Online Homework HW20, # 4)Evaluate  $\lim_{x \rightarrow 0^+} x^x$ .

$$\lim_{x \rightarrow 0^+} x^x = 0^0$$

Hence we can rewrite the limit as:

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{\ln x^x} = \lim_{x \rightarrow 0^+} e^{x \ln x}$$

$$= e^{\lim_{x \rightarrow 0^+} x \ln x} = e^0 = \boxed{1}$$

But  $\lim_{x \rightarrow 0^+} x \ln x = 0 \cdot (-\infty) = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{\infty}$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{1}{x} \cdot (-x^2) = \lim_{x \rightarrow 0^+} (-x) = \boxed{0}$$

11/12

**Example 8:** (Neuhauser, Problem # 62, p. 253)Use l'Hôpital's rule to find  $\lim_{x \rightarrow \infty} \left(1 + \frac{c}{x}\right)^x$  where  $c$  is a constant.What about  $\lim_{x \rightarrow \infty} 3x(\ln(x+3) - \ln x)$ ? (Online Homework HW20, #

$$\lim_{x \rightarrow \infty} \left(1 + \frac{c}{x}\right)^x = 1^\infty$$

Hence we can rewrite the limit as

$$\lim_{x \rightarrow \infty} \left(1 + \frac{c}{x}\right)^x = \lim_{x \rightarrow \infty} e^{\ln \left(1 + \frac{c}{x}\right)^x} =$$

$$= \lim_{x \rightarrow \infty} e^{x \ln \left(1 + \frac{c}{x}\right)} = e^{\lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{c}{x}\right)}{\frac{1}{x}}} = \boxed{e^c}$$

Note that  $\lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{c}{x}\right)}{\frac{1}{x}} = \frac{0}{0} =$  by l'Hôpital's rule

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{c}{x}} \cdot \left(-\frac{c}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{c}{1 + \frac{c}{x}} = \lim_{x \rightarrow \infty} \frac{cx}{x + c} = \boxed{c}$$

12/12

$$\lim_{x \rightarrow \infty} 3x [\ln(x+3) - \ln x] = \infty(\infty - \infty)$$

$$= \lim_{x \rightarrow \infty} 3x \cdot \ln\left(\frac{x+3}{x}\right) = \lim_{x \rightarrow \infty} 3 \cdot \ln\left(\frac{x+3}{x}\right)^x =$$

$$= 3 \cdot \ln \left[ \underbrace{\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x}_{e^3} \right] = 3 \ln e^3 = 3 \cdot 3 = 9$$

(by the first part)