MA 137 — Calculus 1 with Life Science Applications
Antiderivatives
(Section 5.8)

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From Differential to Integral Calculus

Roughly speaking, Calculus has two parts:

differential calculus and integral calculus

At the core of **differential calculus** (which we have been studying so far) is the concept of the instantaneous rate of change of a function. We have seen how this concept can be used to locally approximate functions, to identify maxima and minima, to decide stability of equilibria, etc.

Integral calculus, on the other hand, deals with accumulated change, and, thereby, recovering a function from a mathematical description of its instantaneous rate of change. This recovery process, interestingly enough, is related to the concept of finding the area enclosed by a curve. This will be studied in Chapter 6 (and in the follow up course, MA 138).

Antiderivatives

Many mathematical operations have an inverse. For example, to undo addition we use subtraction. To undo undo exponentiation we take logarithms. The process of differentiation can be undone by a process called *antidifferentiation*.
To motivate antidifferentiation, suppose we know the rate at which a bacteria population is growing and want to know the size of the population at some future time. The problem is to find a function F whose derivative is a known function f.

Definition

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Warning: Although we will learn rules that allow us to compute antiderivatives, this process is typically **much more** difficult than finding derivatives; in addition, there are even cases where it is impossible to find an expression for an antiderivative.

Corollaries of MVT

Two corollaries of the Mean Value Theorem will help us in finding antiderivatives. The first one is Corollary 2 from Section 5.1 (p. 212 of Neuhauser's textbook):

Corollary 2

If f is continuous on $[a, b]$ and differentiable on (a, b), with $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

Corollary 2 is the converse of the fact that $f'(x) = 0$ whenever $f(x)$ is a constant function. Corollary 2 tells us that all antiderivatives of a function that is identically 0 are constant functions.

Corollary 3 says that functions with identical derivative differ only by a constant; that is, to find all antiderivatives of a given function, we need only find one.

Corollary 3

If $F(x)$ and $G(x)$ are antiderivatives of the continuous function $f(x)$ on an interval I, then there exists a constant c such that $G(x) = F(x) + c$ for all $x \in I$.

Proof: Since $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, it follows that $F'(x) = f(x) = G'(x)$ for all $x \in I$. Thus

$$[F(x) - G(x)]' = F'(x) - G'(x) = f(x) - f(x) = 0.$$

It follows from Corollary 2, applied to the function $F - G$, that $F(x) - G(x) = c$, where c is a constant.

The Indefinite Integral

Notation

The *indefinite integral* of $f(x)$, denoted by

$$\int f(x)\,dx$$

represents the *general* antiderivative of $f(x)$.

For example, $\int 3x^2\,dx = x^3 + c$, where c is any constant.

Rules for Indefinite Integrals

A. $\int k\,f(x)\,dx = k \int f(x)\,dx$ $\quad$ k any constant

B. $\int [f(x) \pm g(x)]\,dx = \left[\int f(x)\,dx\right] \pm \left[\int g(x)\,dx\right]$

Basic Indefinite Integrals

The formulas below can be verified by differentiating the righthand side of each expression. The quantities a and c below denote (nonzero) constants.

1. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \qquad n \neq -1$
2. $\int \frac{1}{x}\,dx = \ln|x| + c$
3. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
4. $\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
5. $\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$

Warning: We do not have simple derivative rules for products and quotients, so we should not expect simple integral rules for products and quotients.

Example 1: (Online Homework HW22, # 2)

Find the antiderivative F of $f(x) = 5x^4 - 2x^5$ that satisfies $F(0) = -10$.

In other words:

$$F(x) = \int (5x^4 - 2x^5)\,dx =$$

$$= 5\int x^4\,dx - 2\int x^5\,dx$$

$$= 5 \cdot \left[\tfrac{1}{5}x^5\right] - 2\left[\tfrac{1}{6}x^6\right] + C$$

$$= x^5 - \tfrac{1}{3}x^6 + C$$

describes all antiderivatives of $f(x)$.

We want the one such that $F(0) = -10$. Thus

$$-10 = F(0) = 0^5 - \tfrac{1}{3}0^6 + C \quad \therefore \boxed{C = -10}$$

$$\therefore \boxed{F(x) = x^5 - \tfrac{1}{3}x^6 - 10}$$

Example 2:

Evaluate the indefinite integral $\int (t^3 + 3t^2 + 4t + 9)\, dt$.

$$\int (t^3 + 3t^2 + 4t + 9)\, dt$$

$$= \int t^3\, dt + 3\int t^2\, dt + 4\int t\, dt + 9\int 1 \cdot dt$$

$$= \frac{1}{4} t^4 + 3 \cdot \left(\frac{1}{3} t^3\right) + 4\left(\frac{1}{2} t^2\right) + 9 \cdot t + C$$

$$= \underline{\frac{1}{4} t^4 + t^3 + 2t^2 + 9t + C}$$

Example 3: (Online Homework HW22, # 5)

Evaluate the indefinite integral $\int x(10 - x^4)\, dx$.

$$\int \underbrace{x \cdot (10 - x^4)}\, dx$$

there are no rules for the antiderivative of a product

$$= \int (10x - x^5)\, dx = 10 \int x\, dx - \int x^5\, dx$$

$$= 10\left(\frac{1}{2} x^2\right) - \left(\frac{1}{6} x^6\right) + C$$

$$= 5x^2 - \frac{1}{6} x^6 + C$$

Example 4: (Online Homework HW22, # 7)

Evaluate the indefinite integral $\int \frac{9u^4 + 7\sqrt{u}}{u^2}\, du$.

$$\int \frac{9u^4 + 7\sqrt{u}}{u^2}\, du$$

there are no rules for the antiderivative of a quotient ....

$$= \int \left(\frac{9u^4}{u^2} + \frac{7\sqrt{u}}{u^2}\right) du = \int \left(9u^2 + 7u^{\frac{1}{2}-2}\right) du$$

$$= \int \left(9u^2 + 7u^{-\frac{3}{2}}\right) du = 9\int u^2\, du + 7\int u^{-\frac{3}{2}}\, du$$

$$= 9\left(\frac{1}{3}u^3\right) + 7\left(\frac{1}{-\frac{3}{2}+1} u^{-\frac{3}{2}+1}\right) + C =$$

$$= 3u^3 + 7\left(\frac{1}{-\frac{1}{2}} u^{-\frac{1}{2}}\right) + C = 3u^3 - 14\frac{1}{\sqrt{u}} + C$$

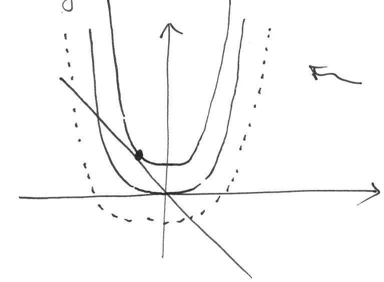
$$= \boxed{\frac{3u^3\sqrt{u} - 14}{\sqrt{u}} + C}$$

Example 4: (Online Homework HW22, # 10)

Find a function f such that $f'(x) = 4x^3$ and the line $x + y = 0$ is tangent to the graph of f.

$$f'(x) = 4x^3 \Longrightarrow f(x) = x^4 + C$$

We also know that "somewhere" this function is tangent to the line $y = -x$.



← all vertical translates of $y = x^4$

I.e. at some point of $f(x) = x^4 + C$ the tangent line has slope -1.

So $f'(x) = 4x^3 = -1 \iff x = -\sqrt[3]{\frac{1}{4}} \cong -0.62996$

What is the value of f at $x = -0.62996$?

It must be the value of the tangent line at that point!

Hence $f(-0.62996) = 0.62996$

since the tangent line is $y = -x$.

Hence $f(x) = x^4 + C$ is such that

$$0.62996 = (-0.62996)^4 + C$$

$\therefore\ C = 0.47247$

Hence: $\boxed{f(x) = x^4 + 0.47247}$

Example 5: (Online Homework HW22, # 13)

Find f if $f'''(x) = \sin(x)$, $f(0) = 8$, $f'(0) = 4$, and $f''(0) = -10$.

$f'''(x) = \sin x$, $f(0) = 8$, $f'(0) = 4$, $f''(0) = -10$

Now: $f'''(x) = \sin x \Longrightarrow f''(x) = -\cos x + C_1$

Since $f''(0) = -10$ we have $-10 = -\underbrace{\cos(0)}_{1} + C_1$

$\therefore\ C_1 = -10 + 1 = -9$

So $f''(x) = -\cos x - 9$. Thus

$$f'(x) = -\sin x - 9x + C_2$$

Since $f'(0) = 4$ we have $4 = f'(0) = -\underbrace{\sin(0)}_{0} + C_2$

$\therefore\ C_2 = 4$. Hence

$$f'(x) = -\sin x - 9x + 4$$

Since $f'(x) = -\sin x - 9x + 4$

we have that

$$f(x) = \cos x - \frac{9}{2}x^2 + 4x + C_3$$

Since $f(0) = 8$ we have

$$8 = f(0) = \underbrace{\cos(0)}_{1} + C_3$$

$\therefore\ C_3 = 8 - 1 = 7$

Finally $\boxed{f(x) = \cos x - \frac{9}{2}x^2 + 4x + 7}$

Solving Simple Differential Equations

In this course, we have repeatedly encountered differential equations ($\equiv$ DEs). Occasionally, we showed that a certain function would solve a given differential equation.

What we learned so far translates into solving DEs of the form

$$\frac{dy}{dx} = f(x).$$

That is, the rate of change of y with respect to x depends only on x. We now know that if we can find one such function y such that $y' = f(x)$, then there is a whole family of functions with this property, all related by vertical translations.

If we want to pick out one of these functions, we need to specify an initial condition — a point (x_0, y_0) on the graph of the function. Such a function is called a solution of the **initial-value problem**

$$\frac{dy}{dx} = f(x) \quad \text{with} \quad y = y_0 \quad \text{when} \quad x = x_0.$$

Example 6: (Neuhauser, Example 5, p. 270)

Solve the initial-value problem $\dfrac{dy}{dx} = -2x^2 + 3$ with $y_0 = 10$ when $x_0 = 3$.

$$\frac{dy}{dx} = -2x^2 + 3 \iff y = -\frac{2}{3}x^3 + 3x + C$$

Now when $x_0 = 3$ $y_0 = 10$. So

$$10 = -\frac{2}{3}(3)^3 + 3(3) + C$$

$$\iff$$

$$10 = -2 \cdot 9 + 9 + C$$

$$10 = -9 + C$$

$$\therefore \quad C = 19$$

$$\therefore \quad \boxed{y = -\frac{2}{3}x^3 + 3x + 19}$$

Note that we could think of

$$\frac{dy}{dx} = -2x^2 + 3 \quad \text{as}$$

$$dy = (-2x^2 + 3)\, dx$$

Now take the indefinite integral of both sides

$$\int 1 \cdot dy = \int (-2x^2 + 3)\, dx$$

$$y + C_1 = -\tfrac{2}{3}x^3 + 3x + C_2$$

hence

$$y = -\frac{2}{3}x^3 + 3x + \underbrace{(C_2 - C_1)}_{C \text{ constant}}$$

Example 7:

What about finding the solution of the initial-value problem $\frac{dy}{dx} = r\,y$ with $y(0) = y_0$ and r a constant? How can we do it?

We have already solved this differential equation while studying Section 5.1.

Now we give a nicer proof:

$$\frac{dy}{dx} = r y$$

$$\Longleftrightarrow$$

$$\underbrace{\frac{1}{y}\cdot\frac{dy}{dx}}_{} = r$$

what is this? By the chain rule

$$\frac{d}{dx}(\ln y) = r$$

If the derivative of a function is a constant then the antiderivative is

$$\frac{d}{dx}\left[\ln y\right] = r \quad \Longleftrightarrow \quad \ln y = rx + C$$

Hence, by taking the exponential of both sides,

$$e^{\ln y} = e^{rx+C} \quad \Longleftrightarrow \quad y = e^{rx}\cdot e^{C}$$

When $x=0$ we have $y(0)=y_0$ so

$$y_0 = y(0) = \underbrace{e^{r\cdot 0}}_{1}\cdot e^{C} \qquad \therefore \quad e^{C} = y_0$$

Hence the solution is:

$$\boxed{y(x) = y_0\, e^{rx}}$$

As we did in Example 6, we could think of $\frac{dy}{dx} = ry$ as

$$\frac{1}{y}\,dy = r\,dx$$

That is we separated the variables; now we take the indefinite integral of both sides

$$\int \frac{1}{y}\,dy = \int r\,dx$$

hence $\boxed{\ln y = rx + C}$

now proceed as before....