MA 137 — Calculus 1 with Life Science Applications
The Definite Integral
(Section 6.1)

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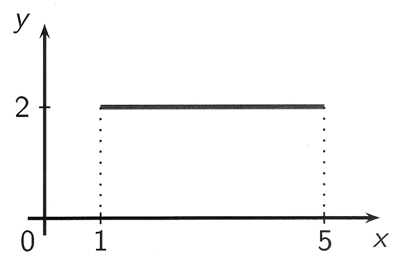
The Area Problem

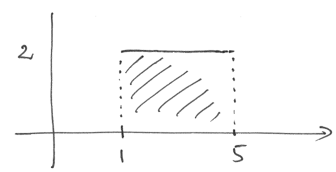
- We start with the area and distance problems and use them to formulate the idea of a definite integral, which is the basic concept of integral calculus. We discover that in trying to find the area under a curve or the distance traveled by a car, we end up with the same special type of limit.
- We will then see that there is a connection between integral calculus and differential calculus: The Fundamental Theorem of Calculus relates the integral to the derivative,
- Why would a biologist be interested in calculating an area? A botanist might want to know the area of a leaf and compare it with the leafs area at other stages of its development. An ecologist might want to know the area of a lake and compare it with the area in previous years. An oncologist might want to know the area of a tumor and compare it with the areas at prior times to see how quickly it is growing. But there are also indirect ways in which areas are of interest.

Example 1: (Easy Area Problem)

Find the area of the region in the xy-plane bounded above by the graph of the function $f(x) = 2$, below by the x-axis, on the left by the line $x = 1$, and on the right by the line $x = 5$.



$f(x) = 2$ defined on $[1,5]$

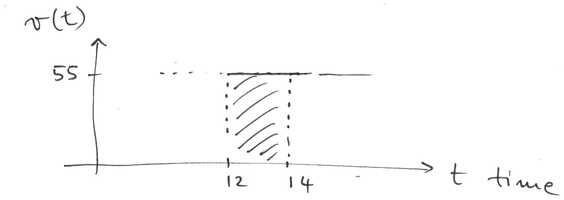


Area of the shaded region $= 2 \cdot \underbrace{(5-1)}_{\text{length of interval}} = 8$

Example 2: (Easy distance traveled problem)

Suppose a car is traveling due east at a constant velocity of 55 miles per hour. How far does the car travel between noon and 2:00 pm?

$v(t) = 55$ mph on the interval $[12, 14]$



distance traveled = area of the shaded region

$= 55 \cdot \underbrace{(14-12)}_{2 \text{ hours}} = \underline{110 \text{ miles}}$

General Philosophy

By means of the integral, problems similar to the previous ones can be solved when the ingredients of the problem are no longer constant but rather changing or variable.

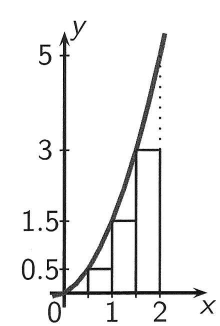
- We first learn how to *estimate* a solution to these more complex problems. The key idea is to notice that the value of the function does not vary very much over a small interval, and so it is approximatively constant over a small interval;
- We will then be able to solve these problems *exactly*;
- Finally, in Section 6.2, we will be able to solve them both exactly and *easily*.

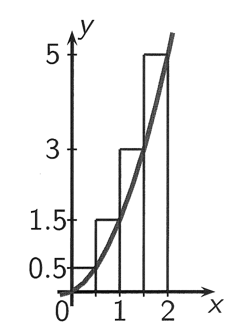
Example 3:

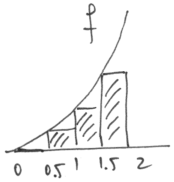
Estimate the area under the graph of $y = x^2 + \frac{1}{2}x$ for x between 0 and 2 in two different ways:

(*a*) Subdivide the interval $[0, 2]$ into four equal subintervals and use the left endpoint of each subinterval as "sample point".

(*b*) Subdivide the interval $[0, 2]$ into four equal subintervals and use the right endpoint of each subinterval as "sample point".





(a) 

area of those 4 rectangles

$$= 0.5\cdot\underbrace{f(0)}_{\text{height}} + 0.5\cdot\underbrace{f(0.5)}_{\text{height}} + 0.5\cdot\underbrace{f(1)}_{\text{height}} + 0.5\cdot\underbrace{f(1.5)}_{\text{height}}$$

$$= 0.5\,[0 + 0.5 + 1.5 + 3]$$

$$= 0.5\,(5) = \underline{\underline{2.5}}$$

(b) 

area of those 4 rectangles

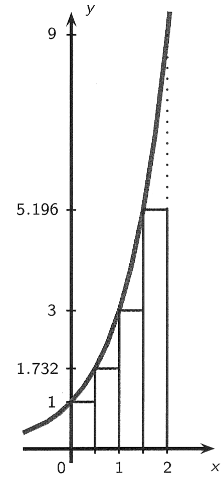
$$= 0.5\cdot f(0.5) + 0.5\cdot f(1) + 0.5\cdot f(1.5) + 0.5\cdot f(2)$$

$$= 0.5\,[0.5 + 1.5 + 3 + 5]$$

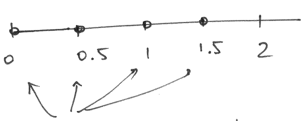
$$= 0.5\,[10] = \underline{\underline{5}}$$

Example 4:

Estimate the area under the graph of $y = 3^x$ for x between 0 and 2. Use a partition that consists of four equal subintervals of $[0, 2]$ and use the left endpoint of each subinterval as a sample point.



$f(x) = 3^x$ $\qquad [0, 2]$



left end points

Area of the four rectangles using the left-end point

$$= \underbrace{0.5\cdot f(0)}_{1^{st}\text{ rectangle}} + \underbrace{0.5\cdot f(0.5)}_{2^{nd}\text{ rect.}} + \underbrace{0.5\cdot f(1)}_{3^{rd}\text{ rect.}} + \underbrace{0.5\cdot f(1.5)}_{4^{th}\text{ rect}}$$

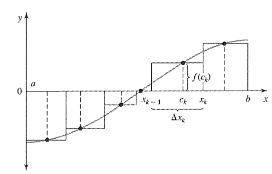
$$= 0.5\cdot 1 + 0.5\cdot 1.732 + 0.5\cdot 3 + 0.5\cdot 5.196$$

$$= 0.5\,(1 + 1.732 + 3 + 5.196) = \underline{\underline{5.464}}$$

Note:

- In the previous two examples we systematically chose the value of the function at one of the endpoints of each subinterval.
- However, since the guiding idea is that we are assuming that the values of the function over a small subinterval do not change by very much, we could take the value of the function at <u>any</u> point of the subinterval as a good sample or representative value of the function.

  We could also have chosen small subintervals of different lengths.



- However, we are trying to establish a systematic, general procedure.
- We can only expect the previous answers to be approximations of the correct answers. This is because the values of the function do change on each subinterval, even though they do not change by much.

Getting better estimates:

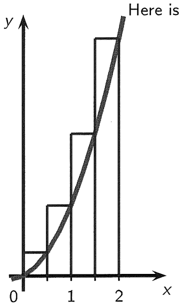
If we replace the subintervals we used by "smaller" subintervals we can reasonably expect the values of the function to vary by much less on each thinner subinterval. Thus, we can reasonably expect that the area of each thinner vertical strip under the graph of the function to be more accurately approximated by the area of these thinner rectangles. Then if we add up the areas of all these thinner rectangles, we should get a much more accurate estimate for the area of the original region.

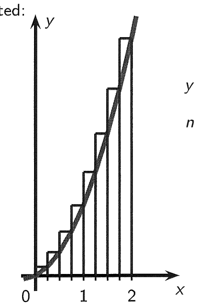
Here is Example 3(b), revisited:

$y = x^2 + \frac{1}{2}x$ on $[0, 2]$

$n = 4$ equal subintervals

Area ≈ 5





$y = x^2 + \frac{1}{2}x$ on $[0, 2]$

$n = 8$ equal subintervals

Area ≈ 4.3125

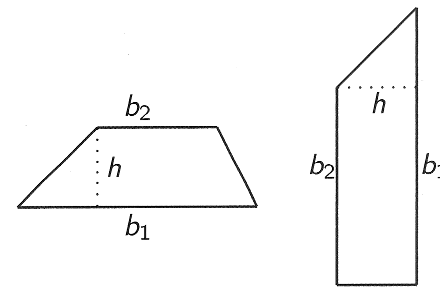
We will see later that the exact value of the area under consideration in Example 3 is $11/3 \approx 3.66$.

Trapezoids versus rectangles

We could use trapezoids instead of rectangles to obtain better estimates, even though the calculations get a little bit more complicated. (This will occur in Example 5.)

We recall that the area of a trapezoid is

$$\text{Area of a trapezoid} = \frac{(b_1 + b_2) \cdot h}{2}.$$

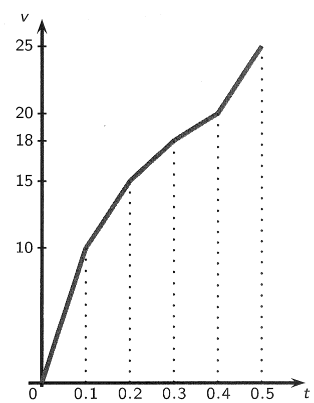


Example 5:

A train travels in a straight westward direction along a track. The velocity of the train varies, but it is measured at regular time intervals of 1/10 hour. The measurements for the first half hour are

time	0	0.1	0.2	0.3	0.4	0.5
velocity	0	10	15	18	20	25

The total distance traveled by the train is equal to the area underneath the graph of the velocity function and lying above the t-axis. Compute the total distance traveled by the train during the first half hour by assuming the velocity is a linear function of t on the subintervals. The velocity in the table is given in miles per hour.



the area under the graph consists of the areas of 5 trapezoids :

$$\text{Area} = 0.1\cdot\left(\frac{0+10}{2}\right) + 0.1\cdot\left(\frac{10+15}{2}\right) + 0.1\cdot\left(\frac{15+18}{2}\right) + 0.1\cdot\left(\frac{18+20}{2}\right) + 0.1\cdot\left(\frac{20+25}{2}\right)$$

$$= 0.1\cdot 5 + 0.1\cdot 12.5 + 0.1\cdot 16.5 + 0.1\cdot 19 + 0.1\cdot 22.5$$

$$= 0.1\cdot\left[5 + 12.5 + 16.5 + 19 + 22.5\right]$$

$$= 0.1\,(75.5)$$

$$= \underline{7.55 \text{ miles}}$$

Example 6: (Neuhauser, Example # 1, p. 277/8)

We will try to find the area of the region below the parabola $f(x) = x^2$ and above the x-axis between 0 and a.

To do this, we divide the interval $[0, a]$ into n subintervals of equal length and approximate the area of interest by a sum of the areas of rectangles, the widths of whose bases are equal to the lengths of the subintervals and whose heights are the values of the function at the left endpoints of these subintervals.

On page 278, Neuhauser uses left-end point. We will use right endpoints instead:

$0 \qquad\qquad a$

n equal subintervals of length $\frac{a-0}{n} = \frac{a}{n}$

hence the end points are:

$0 \quad \frac{a}{n} \quad \frac{2a}{n} \quad \frac{3a}{n} \quad \cdots \quad \frac{(n-1)a}{n} \quad \frac{na}{n} = a$

Area of the n-rectangles is:

$$\frac{a}{n}\cdot f\left(\frac{a}{n}\right) + \frac{a}{n}\cdot f\left(\frac{2a}{n}\right) + \frac{a}{n}\cdot f\left(\frac{3a}{n}\right) + \cdots + \frac{a}{n}\cdot f\left(\frac{na}{n}\right)$$

$$= \frac{a}{n}\cdot\left(\frac{a}{n}\right)^2 + \frac{a}{n}\cdot\left(\frac{2a}{n}\right)^2 + \frac{a}{n}\cdot\left(\frac{3a}{n}\right)^2 + \cdots + \frac{a}{n}\cdot\left(\frac{(n-1)a}{n}\right)^2 + \frac{a}{n}\left(\frac{na}{n}\right)^2$$

$$= \frac{a^3}{n^3} + \frac{a^3\cdot 2^2}{n^3} + \frac{a^3\cdot 3^2}{n^3} + \cdots + \frac{a^3}{n^3}\cdot(n-1)^2 + \frac{a^3}{n^3}\cdot n^2$$

$$= \frac{a^3}{n^3}\left[1 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2\right]$$

$$= \frac{a^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{a^3}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right]$$

$$= \frac{a^3(2n^3+3n^2+n)}{6n^3}$$

Now, if n goes to infinity this should be exactly the area under the parabola $y = x^2$

$$\lim_{n\to\infty} \frac{a^3(2n^3+3n^2+n)}{6n^3} = \lim_{n\to\infty} \frac{a^3(2n^3+3n^2+n)/n^3}{6n^3/n^3}$$

$$= \lim_{n\to\infty} \frac{a^3}{6}\left[2 + 3\frac{1}{n} + \frac{1}{n^2}\right] = \boxed{\frac{a^3}{3}}$$