MA 137 — Calculus 1 with Life Science Applications
The Definite Integral
(Section 6.1)

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Some Properties of Definite Integrals

1. $\displaystyle\int_a^a f(x)\,dx = 0$

2. $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

3. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \left(\int_a^b f(x)\,dx\right) \pm \left(\int_a^b g(x)\,dx\right)$

4. $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

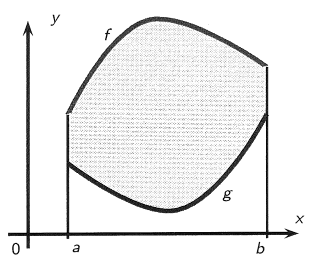
5. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

6. If $\quad m \le f(x) \le M \quad$ on $[a,b]$ $\quad$ then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Geometric Illustration of Some of the Properties

Property **3.** says that if f and g are two positive valued functions with f greater than g, then

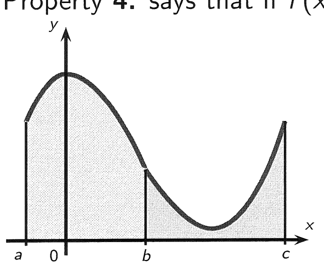


$$\int_a^b (f(x) - g(x))\,dx$$

gives the area between the graphs of f and g

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

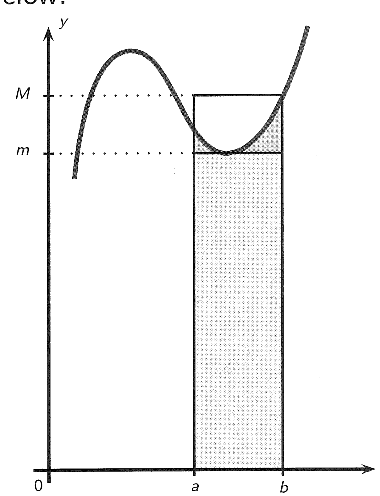
Property **4.** says that if $f(x)$ is a positive valued function then the area underneath the graph of $f(x)$ between a and b plus the area underneath the graph of $f(x)$ between b and c equals the area underneath the graph of $f(x)$ between a and c.



Property **5.** follows from Properties **4.** and **1.** by letting $c = a$.

$$0 = \int_a^a f(x)\,dx \;=\; \int_a^b f(x)\,dx + \int_b^a f(x)\,dx.$$

Property **6.** is illustrated in the picture below.



Example 1: (Online Homework, HW23, # 8)

The sum

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

can be written as a single definite integral of the form

$$\int_{a}^{b} f(x)\,dx$$

for appropriate a and b. Determine these values.

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx \underbrace{- \int_{-2}^{-1} f(x)\,dx}_{+\int_{-1}^{-2} f(x)\,dx}$$

because of property 5.

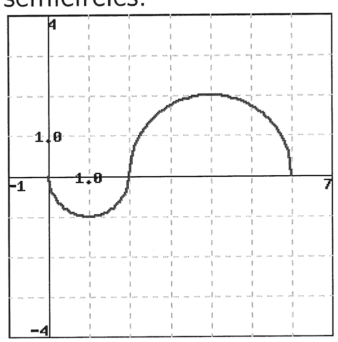
$$= + \int_{-1}^{-2} f(x)\,dx + \int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx$$

$$= \int_{-1}^{5} f(x)\,dx$$

because of property 4.

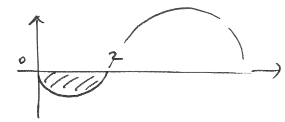
Example 2: (Online Homework, HW23, # 5)

Evaluate the integrals for $f(x)$ shown in the figure below. The two parts of the graph are semicircles.



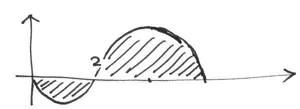
$$\int_{0}^{2} f(x)\,dx \qquad \int_{0}^{6} f(x)\,dx \qquad \int_{1}^{4} f(x)\,dx \qquad \int_{1}^{6} |f(x)|\,dx.$$

$$\int_{0}^{2} f(x)\,dx =$$



$$= -\frac{\pi(1)^2}{2} = \boxed{-\pi/2} \cong -1.5708$$

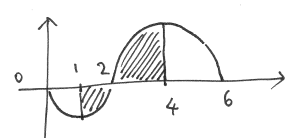
$$\int_{0}^{6} f(x)\,dx =$$



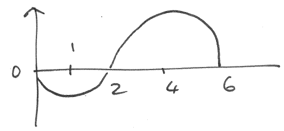
$$= -\frac{\pi(1)^2}{2} + \frac{\pi(2)^2}{2} = -\pi/2 + 2\pi = \boxed{\frac{3}{2}\pi}$$

$$\cong 4.7124$$

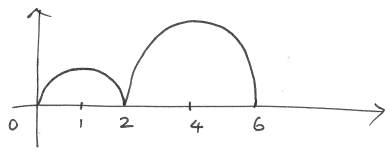
$$\int_{1}^{4} f(x)\,dx =$$



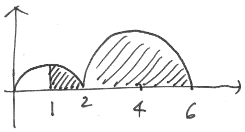
$$= -\frac{\pi(1)^2}{4} + \frac{\pi(2)^2}{4} = -\frac{\pi}{4} + \pi = \boxed{\frac{3}{4}\pi} \cong 2.3562$$

If f has the graph 

then $|f|$ has the following graph



Hence

$$\int_1^6 |f(x)|\,dx =$$ 

$$= +\frac{\pi(1)^2}{4} + \frac{\pi(2)^2}{2} = \frac{\pi}{4} + 2\pi = \boxed{\frac{9}{4}\pi}$$

$$\cong 7.0686$$

Example 3: (Neuhauser, Problem # 61, p. 293)

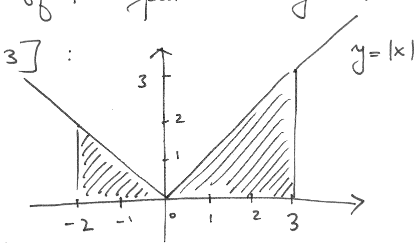
Use an area formula from geometry to find the value of the integral below

$$\int_{-2}^{3} |x|\,dx$$

by interpreting it as the (signed) area under the graph of an appropriately chosen function.

$$\int_{-2}^{3} |x|\,dx$$

Let's look at the graph of the function $y = |x|$ over the interval $[-2, 3]$:



Hence $\int_{-2}^{3} |x|\,dx$ gives the area of the 2 shaded regions (which are triangles) :

$$\int_{-2}^{3} |x|\,dx = \frac{2\cdot 2}{2} + \frac{3\cdot 3}{2} = \frac{13}{2} = \underline{\underline{6.5}}$$

Example 4: (Neuhauser, Problem # 65, p. 293)

Use an area formula from geometry to find the value of the integral below

$$\int_{-2}^{2} \left(\sqrt{4-x^2} - 2\right) dx$$

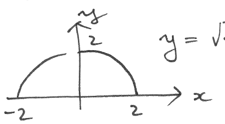
by interpreting it as the (signed) area under the graph of an appropriately chosen function.

$$\int_{-2}^{2}\left(\sqrt{4-x^2}-2\right)dx$$

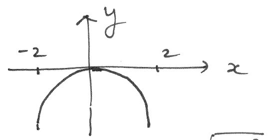
Let's graph the function

$y=\sqrt{4-x^2}-2$ on $[-2,2]$

Notice that $y=\sqrt{4-x^2}$ by itself is the graph of the upper half of the circle of radius 2 centered at the origin:

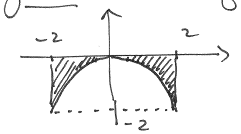


hence



$y=\sqrt{4-x^2}-2$

$$\int_{-2}^{2}\left(\sqrt{4-x^2}-2\right)dx = \text{"signed" area of the region}$$



$\cong \boxed{-1.7168}$

$= -\left[\text{area rectangle} - \text{area semicircle}\right] = -\left[4\cdot 2 - \dfrac{\pi 2^2}{2}\right]$

Example 5: (Neuhauser, Problem # 68(c),(f), p. 293)

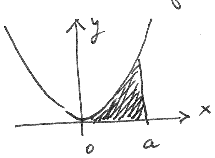
Given that

$$\int_0^a x^2\,dx = \frac{1}{3}a^3$$

evaluate the following

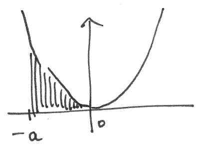
$$\int_{-1}^{3}\frac{1}{3}x^2\,dx \qquad \int_2^4 (x-2)^2\,dx.$$

$$\int_0^a x^2\,dx = \frac{1}{3}a^3 = \text{area of the shaded region}$$



By symmetry of the function we also have that

$$\int_{-a}^{0} x^2\,dx =$$



$= \dfrac{1}{3}a^3$ positive sign

Hence:

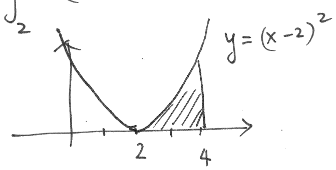
$$\int_{-1}^{3}\frac{1}{3}x^2\,dx = \frac{1}{3}\int_{-1}^{3}x^2\,dx = \frac{1}{3}\left[\int_{-1}^{0}x^2dx + \int_0^3 x^2dx\right]$$

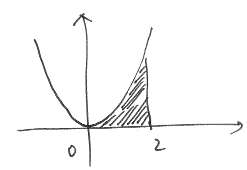
$$= \frac{1}{3}\left[\frac{1}{3}\cdot 1^2 + \frac{1}{3}\cdot 3^2\right] = \frac{1}{3}\left[\frac{1}{3}+3\right] = \boxed{\frac{10}{9}}$$

the graph of $y=(x-2)^2$ is obtained from the graph of $y=x^2$ by shifting it of 2 units to the right;

hence:

$\displaystyle\int_2^4 (x-2)^2\,dx$ must be the same as $\displaystyle\int_0^2 x^2\,dx$





Hence $$\int_2^4 (x-2)^2dx = \int_0^2 x^2\,dx = \frac{1}{3}2^3 = \boxed{\frac{8}{3}}$$