6. (Invasion of the Toads – Fitting an exponential model)

This comes from the Boston University Differential Equations Project. As is often the case in mathematical modeling, there is no “best answer” for this question. There are, however, “better answers:” the correctness of a mathematical model is determined by how well it agrees with the reality of experimental measurements.

From 1935-37, the American marine toad (*Bufo marinus*) was introduced into Queensland, Australia in eight coastal sugar cane districts. Due to lack of natural predators and an abundant food supply, the population grew and the poisonous toads began to be found far from the region in which they were originally introduced. Survey data presented by J. Covacevich and M. Archer (“The distribution of the cane toad, *Bufo marinus*, in Australia and its effects on indigenous vertebrates,” *Mem. Queensland Mus*, **17**: 305-310) shows how the toads expanded their territorial bounds within a forty-year period. This data is reproduced below; it was mathematically analyzed by M. Sabath, W. Boughton, and S. Easteal (“Cumulative Geographical Range of *Bufo marinus* in Queensland, Australia from 1935 to 1974”, *Copeia*, no. 3, 1981, pp. 676-680).

Year	Area Occupied (square km)
1939	32,800
1944	55,800
1949	73,600
1954	138,000
1959	202,000
1964	257,000
1969	301,000
1974	584,000

Cumulative geographical range of Bufo Marinus *in Queensland, Australia.*

Our goal is to construct a mathematical model that best fits the given data. Note that the data is not given to us as "number of toads at five year intervals," and, in fact, this is often the case. For the toads in question, this would be virtually impossible data to obtain, although statistical methods may be used to estimate this value.

- **a)** What assumptions could one make in order to convert the given data into population data? How realistic are these assumptions?
- **b)** Suppose you are writing a small grant to study the population of this toad. You want to ask for money to hire two research assistants to gather additional data for three months. What additional data would you propose to obtain that would give you additional insight into the toad's total population?
- **c)** For the ease of computation, we will assume that, on the average, there is one toad per square kilometer. (Of course, some fields are more densely populated, the middle of cities and lakes don't have any toads, and so on.) We will also count the toads in units of thousands, and time in units of years, beginning with 1939 as “time zero.”

In your report, please include sketches of four solution curves, for differing values of the birth rate, as described below. You can generate these solution curves by using the calculator and putting

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Y1=32.8 e^(kx)
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and plotting this along with the above data in L1 and L2.

- Start with $k = 0.1$ and vary the birth-rate k until the solution to the difference equation appears to fit the data well over the time period [0,35]. Record this value of k.
- It also is possible to solve analytically for a value of k that will guarantee that the curve passes through exactly two of the data points. If $P(0) = 32.8$ find and record a value of k so that $P(5) = 55.8$. Find a different value of k that will give $P(35) = 584$.
- Plot this data on a log-linear plot.

  If we suspect that data fits an exponential model, we can proceed as follows:

  - Take the natural log of the dependent quantity (in our case, the population, P) so that we get a new data set of the form $(t_i, \ln(P(t_i)))$. If you have put the population data into L2, then put L3=ln(L2).
  - Find the line of least squares that fits this data. This gives us an equation of the form $\ln(P(t)) = mt + b$ where m and b are the slope and intercept corresponding to the line of best fit. Record the value of the correlation coefficient that indicates how well the data is approximated by a line (a correlation coefficient of 1 or -1 means perfect correlation).
  - Exponentiating both sides allows us to find an initial population and value of the birth rate that best fits the given data.
- Do your four values of k agree closely with each other? Should they? Which of the four values do you think is the best model for the growth of *Bufo marinus* during the years for which we have data? Use this birth-rate to predict the toad's range in the year 2039. Given that the area of Australia is 7,619,000 square kilometers, how confident are you of this prediction? Explain your reasoning.

d) You may question the validity of the previous question's assumption that there is an average of one toad per square kilometer. Suppose we were wrong and there were actually an average of **two** toads per square kilometer.

As before, solve analytically for a value of k that will guarantee that the curve passes through exactly two of the data points. In particular, if we now assume that $P(0) = 65.6$, find and record a value of k so that $P(5) = 111.6$, and a different value of k so that $P(35) = 1168$. How do these values of k compare with the values you found in the previous question?

What does this tell us? Comment on the importance of knowing the exact average density of the toad population.

For more recent information about the ever growing population of *Bufo marinus,* see the December 1995 issue of FROGLOG (number 15), the Newsletter of the World Conservation Union (IUCN), Species Survival Commission Declining Amphibian Populations Task Force (DAPTF).