

FastTrack 2015 — MA 137 — BioCalculus

Functions (3): The Algebra of Functions

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Goal: We learn how two functions can be combined to form new functions. We then define one-to-one functions, which allows us to introduce the notion of inverse of a one-to-one function. These topics are of importance when we study exponential and logarithmic functions.

Combining functions

Let f and g be functions with domains A and B . We define new functions $f + g$, $f - g$, fg , and f/g as follows:

$$(f + g)(x) = f(x) + g(x)$$

Domain $A \cap B$

$$(f - g)(x) = f(x) - g(x)$$

Domain $A \cap B$

$$(fg)(x) = f(x)g(x)$$

Domain $A \cap B$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

Note

Consider the above definition $(f+g)(x) = f(x)+g(x)$.

The $+$ on the left hand side stands for the operation of addition of functions.

The $+$ on the right hand side, however, stands for addition of the *numbers* $f(x)$ and $g(x)$.

Similar remarks hold true for the other definitions.

Example 1:

Let us consider the functions $f(x) = x^2 - 2x$ and $g(x) = 3x - 1$.

Find $f + g$, $f - g$, fg , and f/g and their domains.

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) = (x^2 - 2x) + (3x - 1) \\ &= x^2 + x - 1\end{aligned}$$

$$\begin{aligned}(f-g)(x) &= f(x) - g(x) = (x^2 - 2x) - (3x - 1) \\ &= x^2 - 5x + 1\end{aligned}$$

$$\begin{aligned}(fg)(x) &= f(x) \cdot g(x) = (x^2 - 2x)(3x - 1) \\ &= 3x^3 - 7x^2 + 2x\end{aligned}$$

domain: all real numbers

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 2x}{3x - 1}$$


$$\text{domain : } \{ x \in \mathbb{R} \mid x \neq 1/3 \}$$

Example 2:

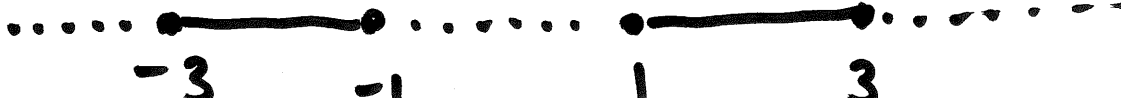
Let us consider the functions $f(x) = \sqrt{9 - x^2}$ and $g(x) = \sqrt{x^2 - 1}$.

Find $f + g$, $f - g$, fg , and f/g and their domains.

$$(f + g)(x) = f(x) + g(x) = \sqrt{9 - x^2} + \sqrt{x^2 - 1}$$

domain: 

$$(f - g)(x) = f(x) - g(x) = \sqrt{9 - x^2} - \sqrt{x^2 - 1}$$

domain: 

$$(f \circ g)(x) = \sqrt{9-x^2} \cdot \sqrt{x^2-1} = \sqrt{(9-x^2)(x^2-1)}$$

$$9-x^2: \quad \begin{array}{ccccccc} \cdots & + & + & + & + & + & \cdots \\ \hline & -3 & & & & 3 & \end{array}$$

$$x^2-1: \quad \begin{array}{ccccccc} + & + & + & + & - & - & - \\ \hline & -1 & & & & 1 & \end{array}$$

$$(9-x^2)(x^2-1) \quad \begin{array}{ccccccc} \cdots & + & - & - & + & \cdots \\ \hline & -3 & -1 & 1 & 3 & \end{array}$$

$$\therefore \text{domain:} \quad -3 \leq x \leq -1 \quad \text{and} \quad 1 \leq x \leq 3$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{9-x^2}}{\sqrt{x^2-1}} = \sqrt{\frac{9-x^2}{x^2-1}}$$

$$\text{domain:} \quad \cdots \cdots \bullet \text{---} \circ \cdots \cdots \circ \text{---} \bullet \cdots \cdots$$

-3 -1 1 3

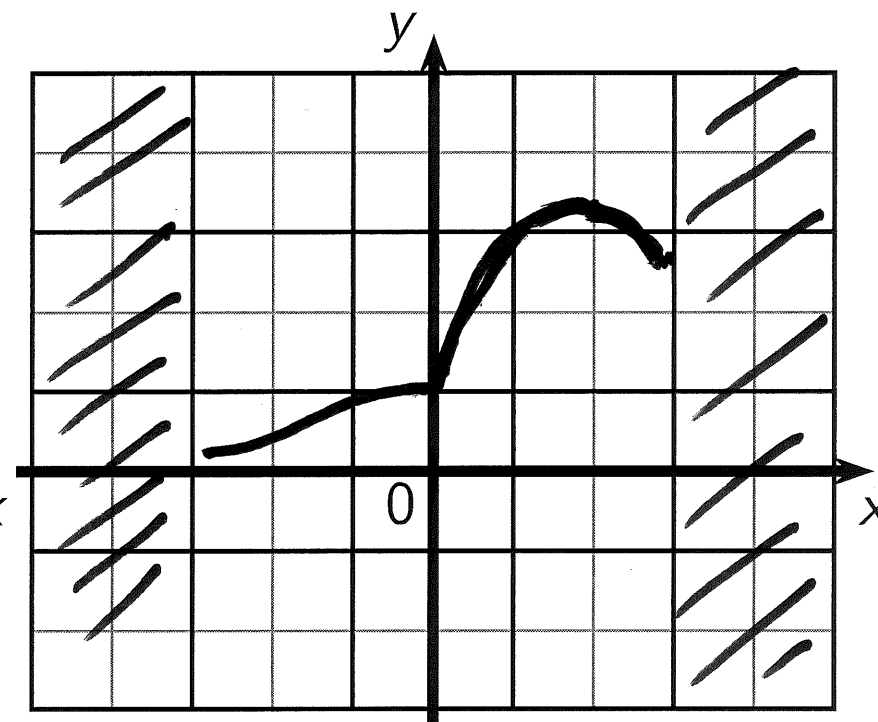
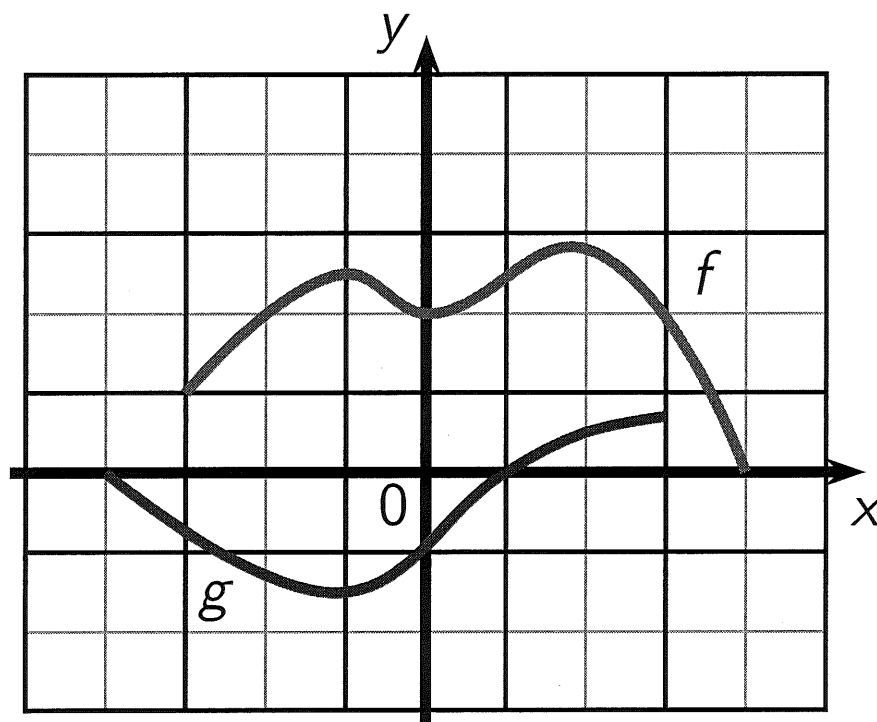
The graph of the function $f + g$ can be obtained from the graphs of f and g by **graphical addition**.

This means that to obtain the value of $f + g$ at any point x we add the corresponding values of $f(x)$ and $g(x)$, that is, the corresponding y -coordinates.

Similar statements can be made for the other operations on functions.

Example 3:

Use graphical addition to sketch the graph of $f + g$.



graph of $f + g$

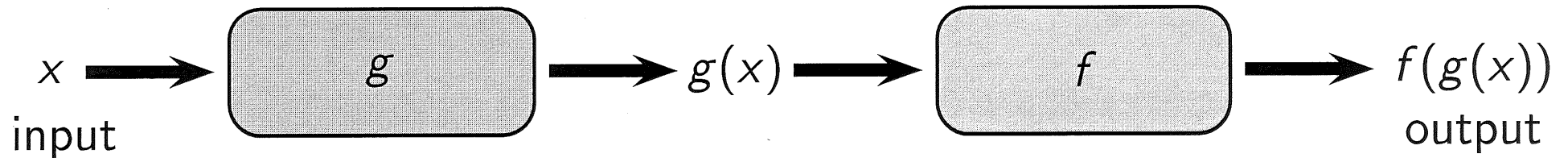
Composition of Functions

Given any two functions f and g , we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f , we can then calculate the value of $f(g(x))$.

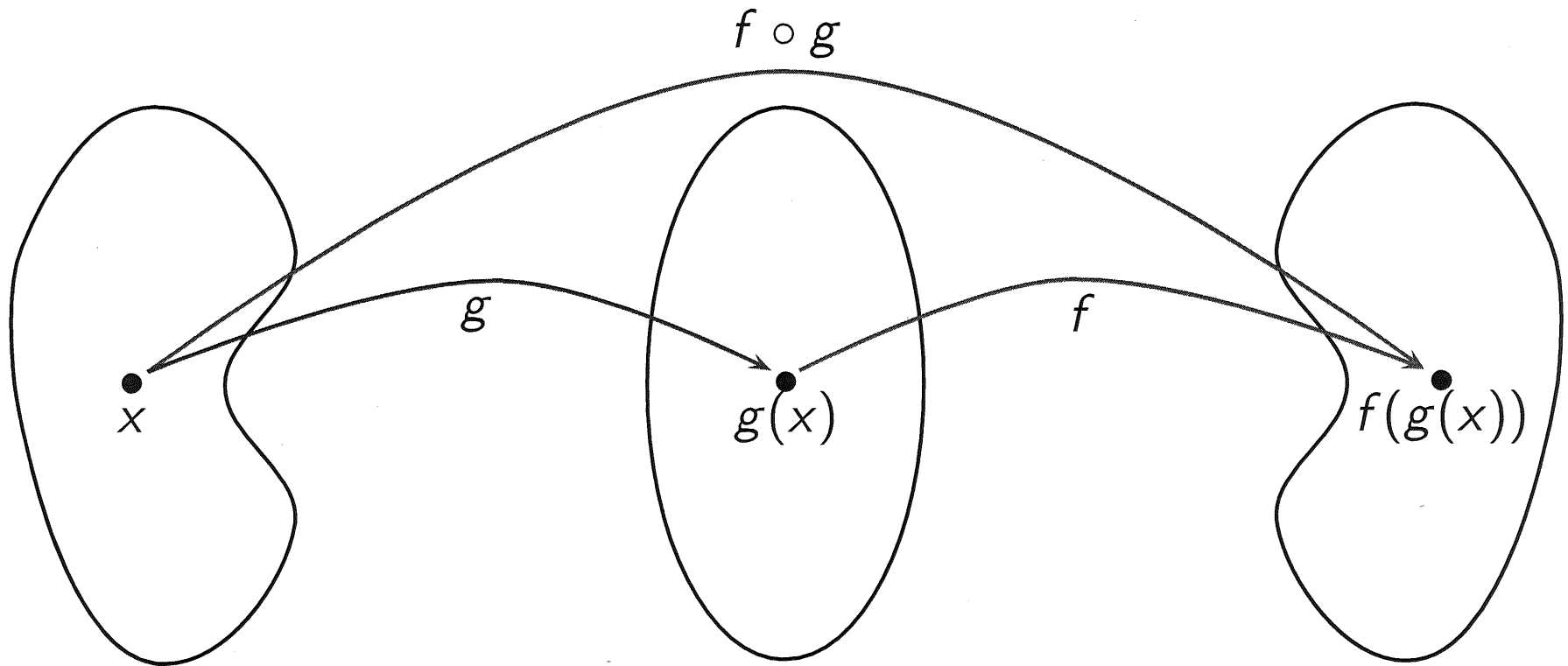
The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f . It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (read: ' f composed with g ' or ' f after g ')

$$(f \circ g)(x) \stackrel{\text{def}}{=} f(g(x)).$$

WARNING: $f \circ g \neq g \circ f$.



Machine diagram of $f \circ g$



Arrow diagram of $f \circ g$

Example 4:

Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate:

$$f(g(0)) = 3g(0) - 5 = 3 \cdot 2 - 5$$

$$\quad \quad \quad \downarrow$$

$$\quad \quad \quad = \textcircled{1}$$

$$f(f(4)) = 3(f(4)) - 5$$

$$\quad \quad \quad \downarrow$$

$$\quad \quad \quad = 3(3 \cdot 4 - 5) - 5 = \textcircled{16}$$

$$(f \circ g)(x) = f(g(x))$$

$$= 3g(x) - 5 =$$

$$= 3(2 - x^2) - 5 = \boxed{1 - 3x^2}$$

$$g(f(0)) = 2 - [f(0)]^2$$

$$= 2 - [-5]^2 = \textcircled{-23}$$

$$(g \circ g)(2) = 2 - [g(2)]^2$$

$$= 2 - [-2]^2 = \textcircled{-2}$$

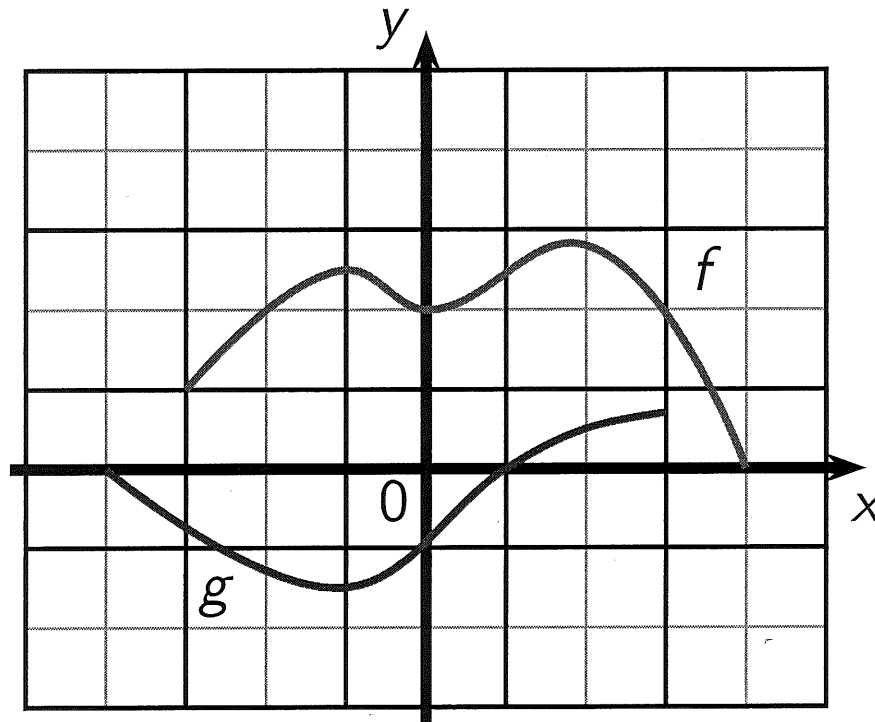
$$(g \circ f)(x) = 2 - [f(x)]^2$$

$$= 2 - [3x - 5]^2$$

$$= \boxed{-9x^2 + 30x - 23}$$

Example 5:

Let f and g be the functions considered in Example 3. Use the information provided by the graphs of f and g to find $f(g(1))$, $g(f(0))$, $f(g(0))$, and $g(f(4))$.



$$g(1) = 0 \Rightarrow f(g(1)) = \underline{2}$$

$$f(0) = 2 \Rightarrow g(f(0)) = \underline{0.5}$$

$$g(0) = -1 \Rightarrow f(g(0)) = \underline{2.5}$$

$$g(f(4)) = g(0) \\ = \underline{-1}$$

Example 6:

Let $f(x) = \frac{x}{x+1}$ and $g(x) = 2x - 1$.

Find the functions $f \circ g$, $g \circ f$, and $f \circ f$ and their domains.

$$(f \circ g)(x) = f(g(x)) = \frac{g(x)}{g(x)+1} = \frac{2x-1}{(2x-1)+1} = \frac{2x-1}{2x}$$

domain: $x \neq 0$

$$(g \circ f)(x) = g(f(x)) = 2f(x) - 1 = 2 \cdot \frac{x}{x+1} - 1$$
$$= \frac{2x}{x+1} - 1 = \frac{2x - (x+1)}{x+1} = \frac{x-1}{x+1}$$

domain: $x \neq -1$

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$$(f \circ f)(x) = f(f(x)) = \frac{f(x)}{f(x)+1} =$$

$$= \frac{x/x+1}{\frac{x}{x+1} + 1} = \frac{x/x+1}{\frac{x + (x+1)}{x+1}} =$$

$$= \frac{\frac{x}{x+1}}{\frac{2x+1}{x+1}} = \frac{x}{\cancel{x+1}} \cdot \frac{\cancel{x+1}}{2x+1}$$

$$= \frac{x}{2x+1} //$$

domain: $x \neq -\frac{1}{2}$

Example 7:

Express the function $F(x) = \frac{x^2}{x^2 + 4}$ in the form $F(x) = f(g(x))$.

$$x \xrightarrow{g} x^2 \xrightarrow{f} \frac{x^2}{x^2 + 4}$$

thus : $g(x) = x^2$

$$f(x) = \frac{x}{x + 4}$$

Example 8:

Find functions f and g so that $f \circ g = H$ if $H(x) = \sqrt[3]{2 + \sqrt{x}}$.

$$x \xrightarrow{g} 2 + \sqrt{x} \xrightarrow{f} \sqrt[3]{2 + \sqrt{x}}$$

thus : $g(x) = 2 + \sqrt{x}$

$$f(x) = \sqrt[3]{x}$$

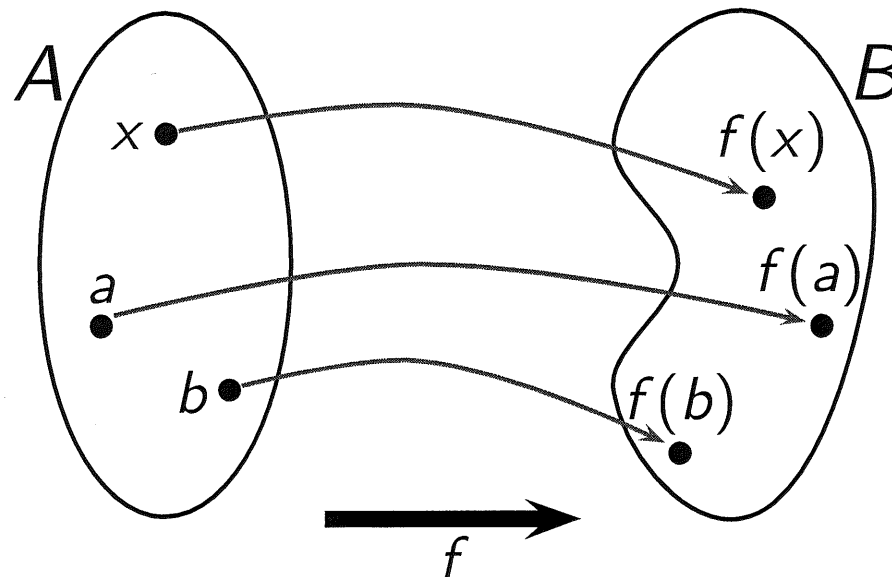
Definition of a One-One Function

A function f with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

An equivalent way of writing the above condition is:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

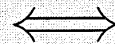


Horizontal Line Test

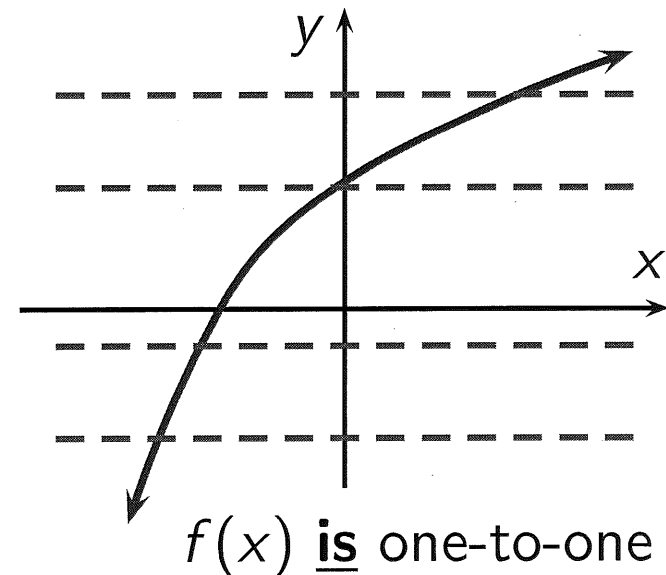
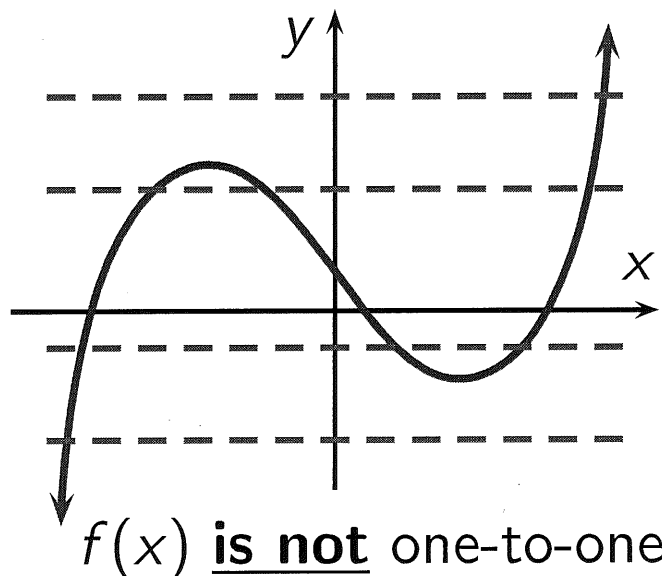
For functions that can be graphed in the coordinate plane, there is a useful criterion to determine whether a function is one-to-one or not.

Horizontal Line Test

A function is one-to-one



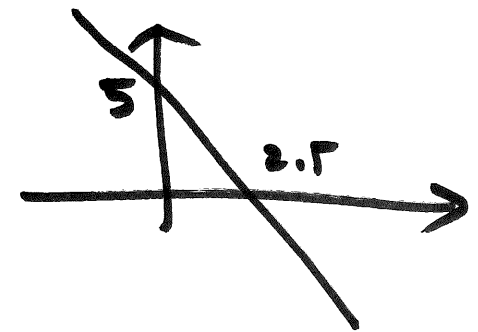
no horizontal line intersects its graph more than once.



Example 9:

Show that the function $f(x) = 5 - 2x$ is one-to-one.

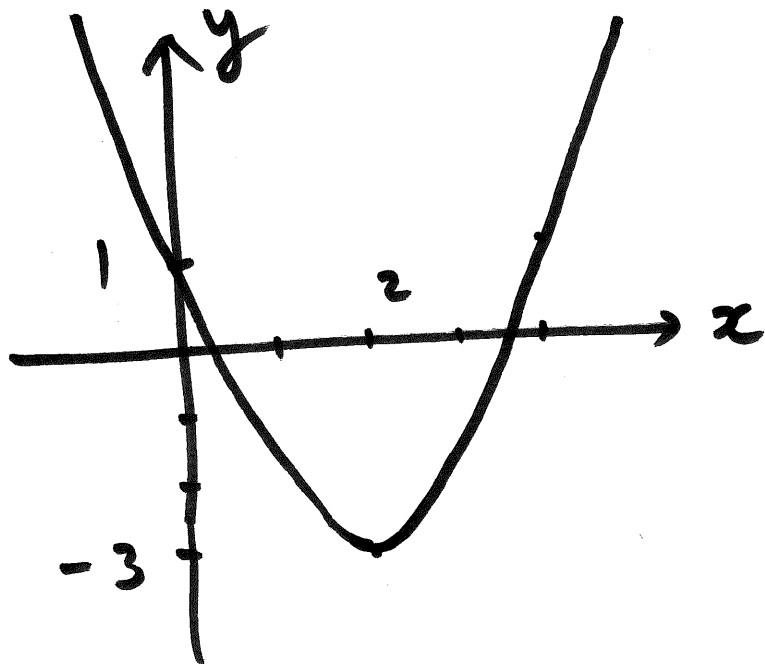
1st method: Use the horizontal line test. The graph of f is a straight line and any ^{hor.} line intersects the graph in one point.



2nd method: Use the definition. Let $f(x_1) = f(x_2)$
i.e. $5 - 2x_1 = 5 - 2x_2$. Simplify "5".
We get $-2x_1 = -2x_2$. Now cancel "-2"
 $\Rightarrow x_1 = x_2$.

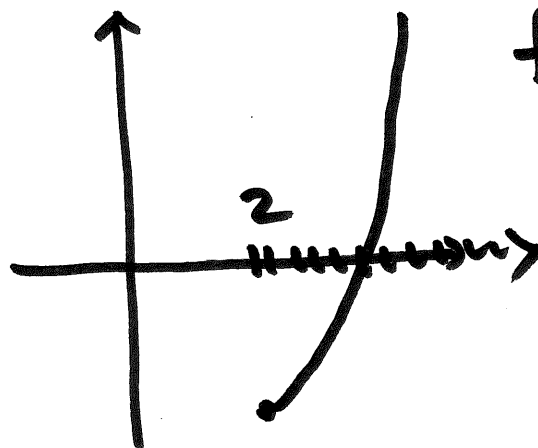
Example 10:

Graph the function $f(x) = (x - 2)^2 - 3$. The function is not one-to-one: Why? Can you restrict its domain so that the resulting function is one-to-one? (There is more than one correct answer.)

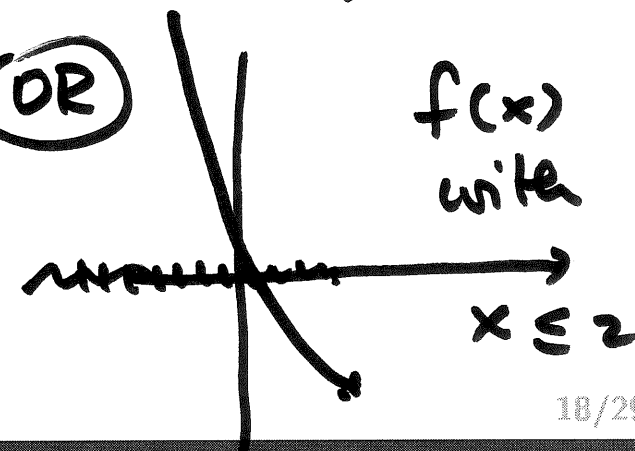


it is not one-to-one as it fails the horizontal line test.

To make it one-one
 $f(x)$ with $x \geq 2$



(OR)



The Inverse of a Function

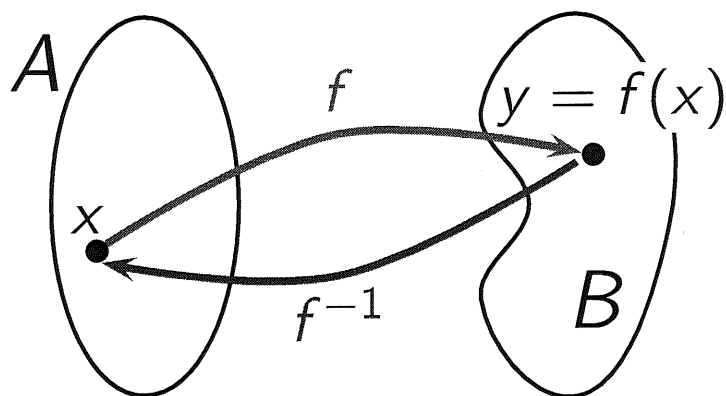
One-to-one functions are precisely those for which one can define a (unique) **inverse function** according to the following definition.

Definition of the Inverse of a Function

Let f be a one-to-one function with domain A and range B . Its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y,$$

for any $y \in B$.



If f takes x to y ,
then f^{-1} takes y back to x .
I.e., f^{-1} undoes what f does.

NOTE:

f^{-1} does NOT mean $\frac{1}{f}$.

Example 11:

Suppose $f(x)$ is a one-to-one function.

If $f(2) = 7$, $f(3) = -1$, $f(5) = 18$, $f^{-1}(2) = 6$ find:

$$f^{-1}(7) = 2$$

$$f(6) = 2$$

$$f^{-1}(-1) = 3$$

$$f(f^{-1}(18)) = 18$$

If $g(x) = 9 - 3x$, then $g^{-1}(3) = 2$

Suppose $9 - 3x = g(x) = 3$ then

$$-3x = -6 \Rightarrow$$

$$x = 2$$

Properties of Inverse Functions

Let $f(x)$ be a one-to-one function with domain A and range B .
The inverse function $f^{-1}(x)$ satisfies the following “cancellation” properties:

$$f^{-1}(f(x)) = x \text{ for every } x \in A$$

$$f(f^{-1}(x)) = x \text{ for every } x \in B$$

Conversely, any function $f^{-1}(x)$ satisfying the above conditions is the inverse of $f(x)$.

Example 12:

Show that the functions $f(x) = x^5$ and $g(x) = x^{1/5}$ are inverses of each other.

$$f(g(x)) = [g(x)]^5 = [x^{1/5}]^5 = x$$

$$g(f(x)) = [f(x)]^{1/5} = [x^5]^{1/5} = x$$

Example 13:

Show that the functions $f(x) = \frac{1+3x}{5-2x}$ and $g(x) = \frac{5x-1}{2x+3}$ are inverses of each other.

we do one of the verifications : $f(g(x)) = x \dots$

$$f(g(x)) = \frac{1+3g(x)}{5-2g(x)} = \frac{1+3\left(\frac{5x-1}{2x+3}\right)}{5-2\left(\frac{5x-1}{2x+3}\right)} =$$

$$= \frac{\frac{(2x+3) + 3(5x-1)}{2x+3}}{\frac{5(2x+3) - 2(5x-1)}{2x+3}} = \frac{\cancel{17}x}{\cancel{2x+3}} \cdot \frac{\cancel{2x+3}}{\cancel{17}} = \underline{x}$$

How to find the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y . The resulting equation is $y = f^{-1}(x)$.

Example 14:

Find the inverse of $y = 4x - 7$.

① $y = 4x - 7$

② $4x = y + 7 \longrightarrow x = \frac{1}{4}y + \frac{7}{4}$

③ $y = \frac{1}{4}x + \frac{7}{4}$

Example 15:

Find the inverse of $y = \frac{1}{x+2}$.

① $y = \frac{1}{x+2}$

② $x+2 = \frac{1}{y} \rightsquigarrow x = \frac{1}{y} - 2$

③ $y = \frac{1}{x} - 2$ or $y = \frac{1-2x}{x}$

Example 16:

Find the inverse of $y = \frac{2-x}{x+2}$.

$$\textcircled{1} \quad y = \frac{2-x}{x+2}$$

$$\begin{aligned} \textcircled{2} \quad y(x+2) &= 2-x \rightsquigarrow xy + 2y = 2-x \\ &\rightsquigarrow x(y+1) = 2-2y \rightsquigarrow x = \frac{2-2y}{y+1} \end{aligned}$$

$$\textcircled{3} \quad \boxed{y = \frac{2-2x}{x+1}}$$

Graph of the Inverse Function

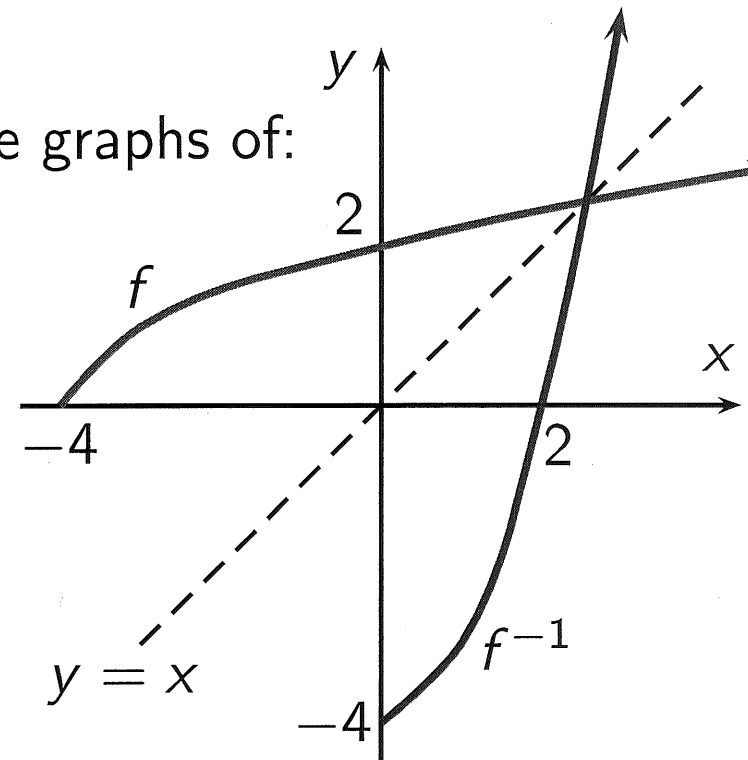
The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f . **The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.**

The picture on the right hand side shows the graphs of:

$$f(x) = \sqrt{x+4}$$

and

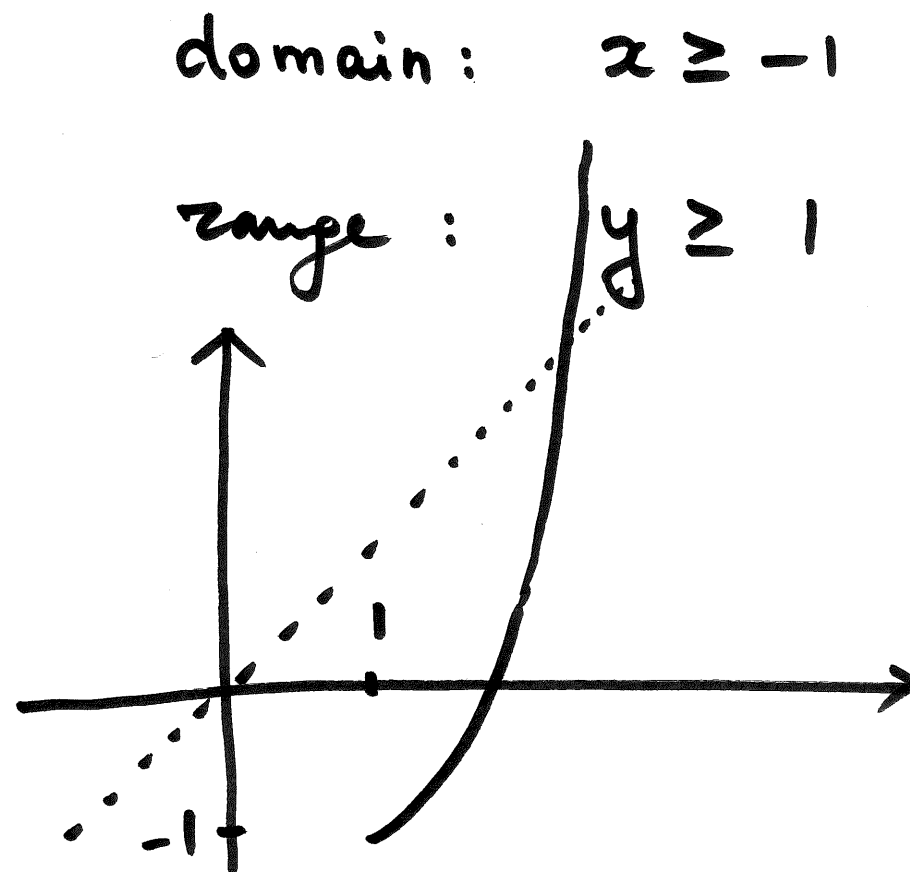
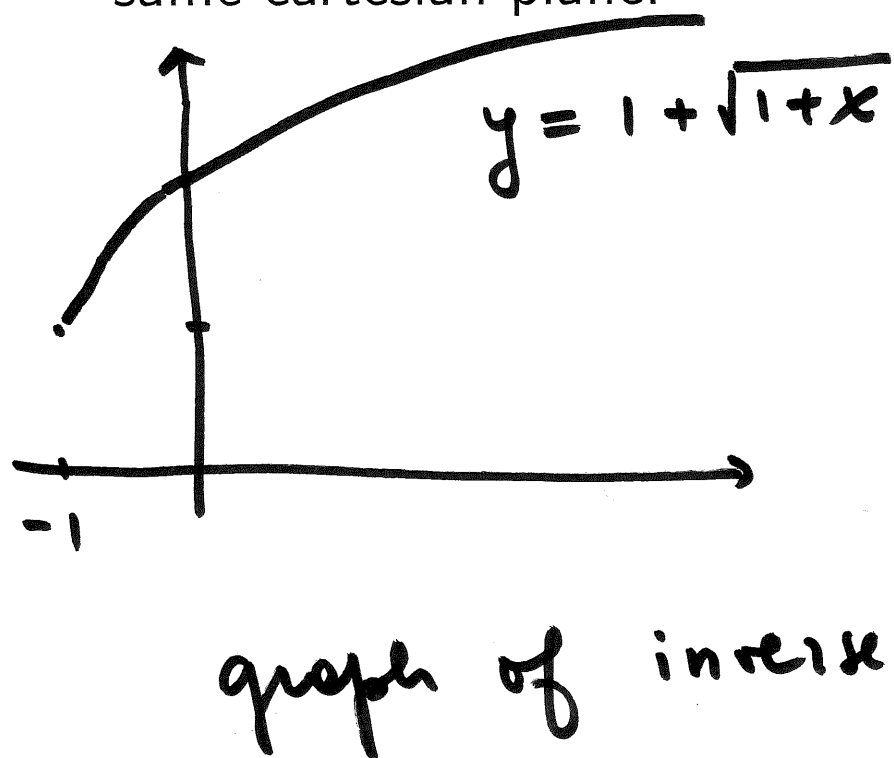
$$f^{-1}(x) = x^2 - 4, \quad x \geq 0.$$



Example 17:

Find the inverse of the function $f(x) = 1 + \sqrt{1+x}$.

Find the domain and range of f and f^{-1} . Graph f and f^{-1} on the same cartesian plane.



the domain of the inverse is:

$$\underline{\underline{x \geq 1}}$$

the range is: $y \geq -1$

To get the expression of the inverse

$$\textcircled{1} \quad y = 1 + \sqrt{1+x}$$

$$\textcircled{2} \quad y-1 = \sqrt{1+x} \quad \rightsquigarrow \quad (y-1)^2 = (\sqrt{1+x})^2$$

$$y^2 - 2y + 1 = 1 + x \quad \rightsquigarrow \quad x = y^2 - 2y$$

$$\textcircled{3} \quad \boxed{y = x^2 - 2x}$$

with $x \geq 1$