

FastTrack 2015 — MA 137 — BioCalculus

Functions (3): The Algebra of Functions

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Goal: We learn how two functions can be combined to form new functions. We then define one-to-one functions, which allows us to introduce the notion of inverse of a one-to-one function. These topics are of importance when we study exponential and logarithmic functions.

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Combining functions

Let f and g be functions with domains A and B . We define new functions $f + g$, $f - g$, fg , and f/g as follows:

$$(f + g)(x) = f(x) + g(x) \quad \text{Domain } A \cap B$$

$$(f - g)(x) = f(x) - g(x) \quad \text{Domain } A \cap B$$

$$(fg)(x) = f(x)g(x) \quad \text{Domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{Domain } \{x \in A \cap B \mid g(x) \neq 0\}$$

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Note

Consider the above definition $(f+g)(x) = f(x)+g(x)$.

The $+$ on the left hand side stands for the operation of addition of functions.

The $+$ on the right hand side, however, stands for addition of the numbers $f(x)$ and $g(x)$.

Similar remarks hold true for the other definitions.

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Example 1:

Let us consider the functions $f(x) = x^2 - 2x$ and $g(x) = 3x - 1$.

Find $f + g$, $f - g$, fg , and f/g and their domains.

$$(f+g)(x) = f(x) + g(x) = (x^2 - 2x) + (3x - 1) = x^2 + x - 1$$

$$(f-g)(x) = f(x) - g(x) = (x^2 - 2x) - (3x - 1) = x^2 - 5x + 1$$

$$(fg)(x) = f(x) \cdot g(x) = (x^2 - 2x)(3x - 1) = 3x^3 - 7x^2 + 2x$$

domain: all real numbers

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$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 2x}{3x - 1}$$

$$\text{domain: } \{x \in \mathbb{R} \mid x \neq \frac{1}{3}\}$$

$$(f \cdot g)(x) = \sqrt{9-x^2} \cdot \sqrt{x^2-1} = \sqrt{(9-x^2)(x^2-1)}$$

$$9-x^2: \begin{array}{ccccccc} & & + & + & + & + & \\ - & \cdots & & & & & \cdots - \\ & & -3 & & & & 3 \end{array}$$

$$x^2-1: \begin{array}{ccccccc} & & + & + & + & + & \\ - & \cdots & & & & & \cdots - \\ & & -3 & & & & 3 \end{array}$$

$$(9-x^2)(x^2-1): \begin{array}{ccccccc} & & + & - & - & + & \\ - & \cdots & & & & & \cdots - \\ & & -3 & & -1 & & 1 & & 3 \end{array}$$

$$\therefore \text{domain: } -3 \leq x \leq -1 \text{ and } 1 \leq x \leq 3$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{9-x^2}}{\sqrt{x^2-1}} = \sqrt{\frac{9-x^2}{x^2-1}}$$

$$\text{domain: } \begin{array}{ccccccc} & & + & - & - & + & \\ - & \cdots & & & & & \cdots - \\ & & -3 & & -1 & & 1 & & 3 \end{array}$$

Example 2:

Let us consider the functions $f(x) = \sqrt{9-x^2}$ and $g(x) = \sqrt{x^2-1}$.

Find $f+g$, $f-g$, fg , and f/g and their domains.

$$(f+g)(x) = f(x) + g(x) = \sqrt{9-x^2} + \sqrt{x^2-1}$$

$$\text{domain: } \begin{array}{ccccccc} & & + & - & - & + & \\ - & \cdots & & & & & \cdots - \\ & & -3 & & -1 & & 1 & & 3 \end{array}$$

$$(f-g)(x) = f(x) - g(x) = \sqrt{9-x^2} - \sqrt{x^2-1}$$

$$\text{domain: } \begin{array}{ccccccc} & & + & - & - & + & \\ - & \cdots & & & & & \cdots - \\ & & -3 & & -1 & & 1 & & 3 \end{array}$$

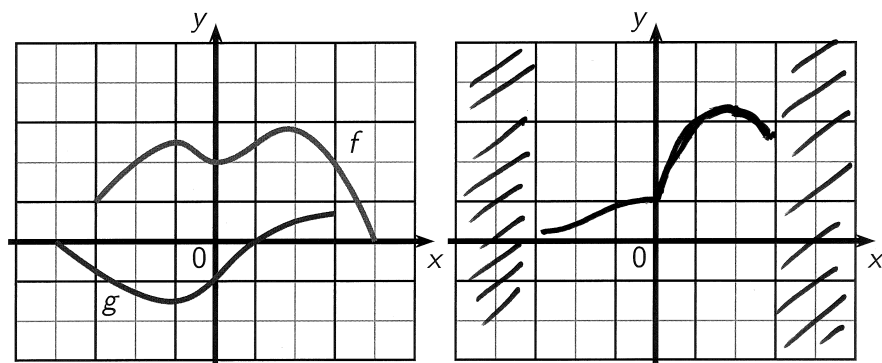
The graph of the function $f+g$ can be obtained from the graphs of f and g by **graphical addition**.

This means that to obtain the value of $f+g$ at any point x we add the corresponding values of $f(x)$ and $g(x)$, that is, the corresponding y -coordinates.

Similar statements can be made for the other operations on functions.

Example 3:

Use graphical addition to sketch the graph of $f + g$.



graph of $f + g$

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Composition of Functions

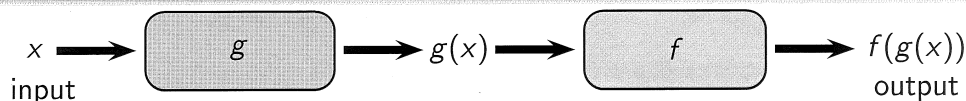
Given any two functions f and g , we start with a number x in the domain of g and find its image $g(x)$. If this number $g(x)$ is in the domain of f , we can then calculate the value of $f(g(x))$.

The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f . It is called the *composition* (or *composite*) of f and g and is denoted by $f \circ g$ (read: 'f composed with g' or 'f after g')

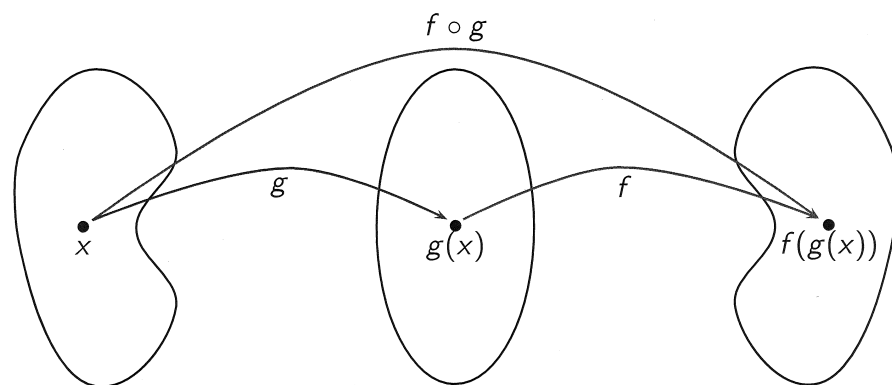
$$(f \circ g)(x) \stackrel{\text{def}}{=} f(g(x)).$$

WARNING: $f \circ g \neq g \circ f$.

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Machine diagram of $f \circ g$



Arrow diagram of $f \circ g$

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Example 4:

Use $f(x) = 3x - 5$ and $g(x) = 2 - x^2$ to evaluate:

$$f(g(0)) = 3g(0) - 5 = 3 \cdot 2 - 5 = 1$$

$$f(f(4)) = 3(f(4)) - 5 = 3(3 \cdot 4 - 5) - 5 = 16$$

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= 3g(x) - 5 = \\ &= 3(2 - x^2) - 5 = 1 - 3x^2 \end{aligned}$$

$$g(f(0)) = 2 - [f(0)]^2 = 2 - [-5]^2 = -23$$

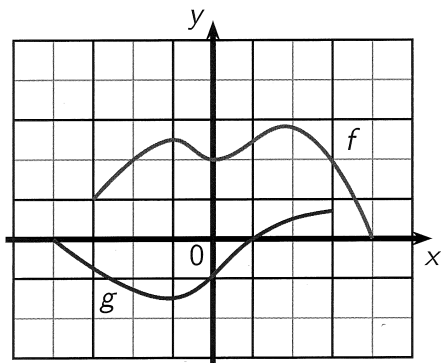
$$(g \circ g)(2) = 2 - [g(2)]^2 = 2 - [-2]^2 = -2$$

$$\begin{aligned} (g \circ f)(x) &= 2 - [f(x)]^2 \\ &= 2 - [3x - 5]^2 \\ &= -9x^2 + 30x - 23 \end{aligned}$$

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Example 5:

Let f and g be the functions considered in Example 3. Use the information provided by the graphs of f and g to find $f(g(1))$, $g(f(0))$, $f(g(0))$, and $g(f(4))$.



$$\begin{aligned} g(1) &= 0 \Rightarrow f(g(1)) = 2 \\ f(0) &= 2 \Rightarrow g(f(0)) = 0.5 \\ g(0) &= -1 \Rightarrow f(g(0)) = 2.5 \\ g(f(4)) &= g(0) \\ &= -1 \end{aligned}$$

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Example 6:

Let $f(x) = \frac{x}{x+1}$ and $g(x) = 2x - 1$.

Find the functions $f \circ g$, $g \circ f$, and $f \circ f$ and their domains.

$$(f \circ g)(x) = f(g(x)) = \frac{g(x)}{g(x)+1} = \frac{2x-1}{(2x-1)+1} = \frac{2x-1}{2x}$$

domain: $x \neq 0$

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) = 2f(x) - 1 = 2 \cdot \frac{x}{x+1} - 1 \\ &= \frac{2x}{x+1} - 1 = \frac{2x - (x+1)}{x+1} = \frac{x-1}{x+1} \end{aligned}$$

domain: $x \neq -1$

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$$(f \circ f)(x) = f(f(x)) = \frac{f(x)}{f(x)+1} =$$

$$= \frac{\frac{x}{x+1}}{\frac{x}{x+1} + 1} = \frac{\frac{x}{x+1}}{\frac{x + (x+1)}{x+1}} =$$

$$= \frac{\frac{x}{x+1}}{\frac{2x+1}{x+1}} = \frac{x}{2x+1} \cdot \frac{x+1}{2x+1}$$

$$= \frac{x}{2x+1} \quad \text{domain: } x \neq -\frac{1}{2}$$

Example 7:

Express the function $F(x) = \frac{x^2}{x^2+4}$ in the form $F(x) = f(g(x))$.

$$x \xrightarrow{g} x^2 \xrightarrow{f} \frac{x^2}{x^2+4}$$

thus: $g(x) = x^2$

$$f(x) = \frac{x}{x+4}$$

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Example 8:

Find functions f and g so that $f \circ g = H$ if $H(x) = \sqrt[3]{2 + \sqrt{x}}$.

$$x \xrightarrow{g} 2 + \sqrt{x} \xrightarrow{f} \sqrt[3]{2 + \sqrt{x}}$$

thus: $g(x) = 2 + \sqrt{x}$
 $f(x) = \sqrt[3]{x}$

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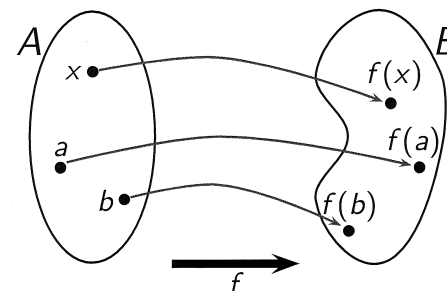
Definition of a One-One Function

A function f with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2.$$

An equivalent way of writing the above condition is:

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$



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Horizontal Line Test

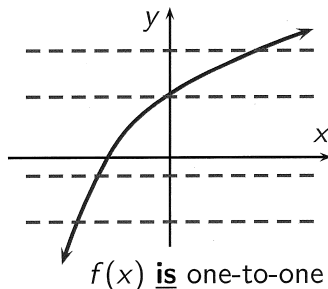
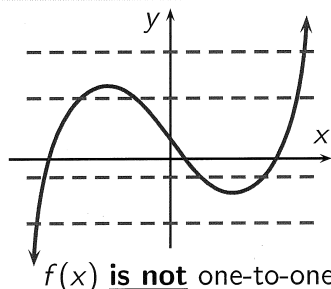
For functions that can be graphed in the coordinate plane, there is a useful criterion to determine whether a function is one-to-one or not.

Horizontal Line Test

A function is one-to-one



no horizontal line intersects its graph more than once.

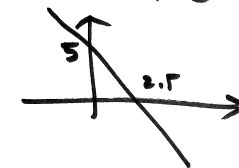


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Example 9:

Show that the function $f(x) = 5 - 2x$ is one-to-one.

1st method: Use the horizontal line test. The graph of f is a straight line and any horizontal line intersects the graph in one point.

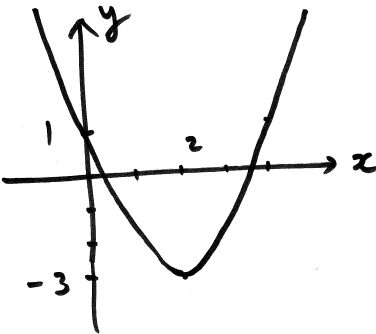


2nd method: use the definition. Let $f(x_1) = f(x_2)$ i.e. $5 - 2x_1 = 5 - 2x_2$. Simplify "5". We get $-2x_1 = -2x_2$. Now cancel "-2" $\implies x_1 = x_2$.

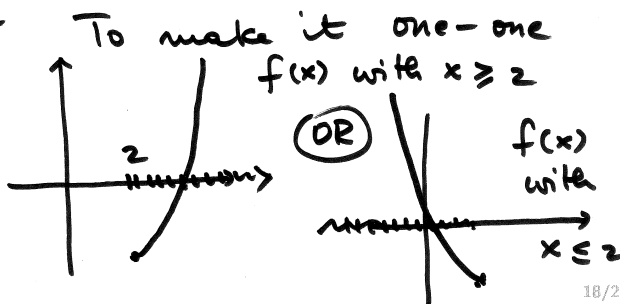
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Example 10:

Graph the function $f(x) = (x - 2)^2 - 3$. The function is not one-to-one: Why? Can you restrict its domain so that the resulting function is one-to-one? (There is more than one correct answer.)



it is not one-to-one as it fails the horizontal line test.



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The Inverse of a Function

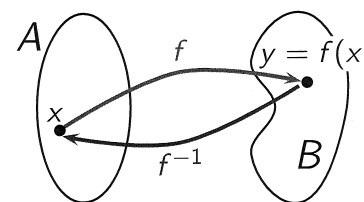
One-to-one functions are precisely those for which one can define a (unique) **inverse function** according to the following definition.

Definition of the Inverse of a Function

Let f be a one-to-one function with domain A and range B . Its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y,$$

for any $y \in B$.



If f takes x to y , then f^{-1} takes y back to x . I.e., f^{-1} undoes what f does.

NOTE:

f^{-1} does NOT mean $\frac{1}{f}$.

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Example 11:

Suppose $f(x)$ is a one-to-one function.

If $f(2) = 7$, $f(3) = -1$, $f(5) = 18$, $f^{-1}(2) = 6$ find:

$$f^{-1}(7) = 2$$

$$f(6) = 2$$

$$f^{-1}(-1) = 3$$

$$f(f^{-1}(18)) = 18$$

If $g(x) = 9 - 3x$, then $g^{-1}(3) = 2$

Suppose $9 - 3x = g(x) = 3$ then

$$-3x = -6 \implies$$

$$x = 2$$

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Properties of Inverse Functions

Let $f(x)$ be a one-to-one function with domain A and range B . The inverse function $f^{-1}(x)$ satisfies the following "cancellation" properties:

$$f^{-1}(f(x)) = x \text{ for every } x \in A$$

$$f(f^{-1}(x)) = x \text{ for every } x \in B$$

Conversely, any function $f^{-1}(x)$ satisfying the above conditions is the inverse of $f(x)$.

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Example 12:

Show that the functions $f(x) = x^5$ and $g(x) = x^{1/5}$ are inverses of each other.

$$f(g(x)) = [g(x)]^5 = [x^{1/5}]^5 = x$$

$$g(f(x)) = [f(x)]^{1/5} = [x^5]^{1/5} = x$$

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Example 13:

Show that the functions $f(x) = \frac{1+3x}{5-2x}$ and $g(x) = \frac{5x-1}{2x+3}$ are inverses of each other.

we do one of the verifications : $f(g(x)) = x \dots$

$$f(g(x)) = \frac{1+3g(x)}{5-2g(x)} = \frac{1+3\left(\frac{5x-1}{2x+3}\right)}{5-2\left(\frac{5x-1}{2x+3}\right)}$$

$$= \frac{(2x+3) + 3(5x-1)}{5(2x+3) - 2(5x-1)} = \frac{2x+3}{2x+3} \cdot \frac{2x+3}{17} = x$$

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How to find the Inverse of a One-to-One Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y . The resulting equation is $y = f^{-1}(x)$.

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Example 14:

Find the inverse of $y = 4x - 7$.

$$\begin{aligned} \textcircled{1} \quad y &= 4x - 7 \\ \textcircled{2} \quad 4x &= y + 7 \longrightarrow x = \frac{1}{4}y + \frac{7}{4} \\ \textcircled{3} \quad y &= \frac{1}{4}x + \frac{7}{4} \end{aligned}$$

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Example 15:

Find the inverse of $y = \frac{1}{x+2}$.

$$\textcircled{1} \quad y = \frac{1}{x+2}$$

$$\textcircled{2} \quad x+2 = \frac{1}{y} \rightsquigarrow x = \frac{1}{y} - 2$$

$$\textcircled{3} \quad \boxed{y = \frac{1}{x} - 2} \quad \text{or} \quad \boxed{y = \frac{1-2x}{x}}$$

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Example 16:

Find the inverse of $y = \frac{2-x}{x+2}$.

$$\textcircled{1} \quad y = \frac{2-x}{x+2}$$

$$\textcircled{2} \quad y(x+2) = 2-x \rightsquigarrow xy + 2y = 2-x \rightsquigarrow x(y+1) = 2-2y \rightsquigarrow x = \frac{2-2y}{y+1}$$

$$\textcircled{3} \quad \boxed{y = \frac{2-2x}{x+1}}$$

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Graph of the Inverse Function

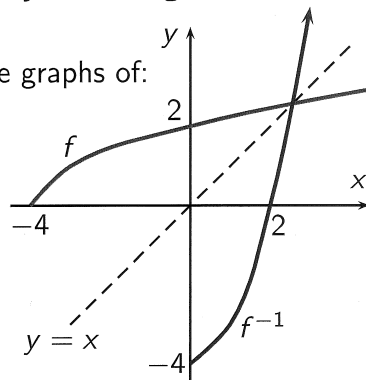
The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f . **The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.**

The picture on the right hand side shows the graphs of:

$$f(x) = \sqrt{x+4}$$

and

$$f^{-1}(x) = x^2 - 4, \quad x \geq 0.$$

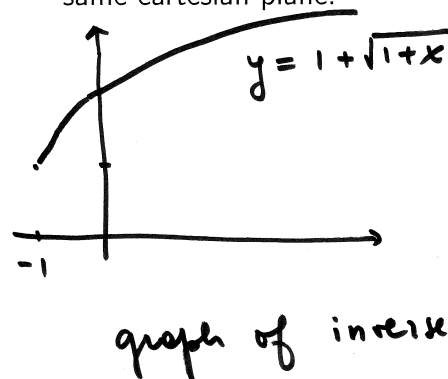


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Example 17:

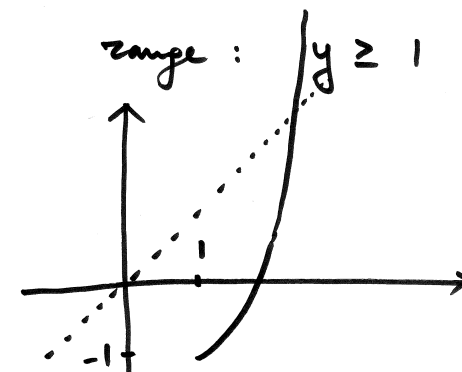
Find the inverse of the function $f(x) = 1 + \sqrt{1+x}$.

Find the domain and range of f and f^{-1} . Graph f and f^{-1} on the same cartesian plane.



domain: $x \geq -1$

range: $y \geq 1$



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the domain of the inverse is:

$$\boxed{x \geq 1}$$

the range is: $y \geq -1$

To get the expression of the inverse

$$\textcircled{1} \quad y = 1 + \sqrt{1+x}$$

$$\textcircled{2} \quad y-1 = \sqrt{1+x} \quad \rightsquigarrow \quad (y-1)^2 = (\sqrt{1+x})^2$$

$$y^2 - 2y + 1 = 1 + x \quad \rightsquigarrow \quad x = y^2 - 2y$$

$$\textcircled{3} \quad \boxed{y = x^2 - 2x} \quad \text{with } \underline{\underline{x \geq 1}} \quad \leftarrow$$