# MA 137: Calculus I for the Life Sciences

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# Section 3.4: Trigonometric Limits

The following two trigonometric limits are important for developing the differential calculus for trigonometric functions:

## Rule

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \textit{and} \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$

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- The proof of the first statement uses a nice geometric argument and the sandwich theorem.
- The second statement follows from the first.

# Trigonometric Functions

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What about $sin(\alpha/2)$ and $\cos(\alpha/2)$?

$$\cos(\alpha/2) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \quad \text{and} \quad \sin(\alpha/2) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

(the sign (+ or -) depends on the quadrant in which $\alpha/2$ lies.)

**Example (Online Homework HW10, #7)**

Evaluate

$$\lim_{\theta \to 0} \frac{\sin 4\theta \sin 8\theta}{\theta^2}.$$

**Example (Online Homework HW10, #10)**

Evaluate the limit:

$$\lim_{x \to 0} \frac{\tan 5x}{\tan 6x}$$

**Example (Neuhauser, Example 3(c), p. 118)**

Evaluate the limit:

$$\lim_{x \to 0} \frac{\sec x - 1}{x \sec x}$$

## Example (Online Homework HW10, #14)

A semicircle with diameter $PQ$ sits on an isosceles triangle $PQR$ to form a region shaped like an ice cream cone, as shown in the figure. If $A(\theta)$ is the area of the semicircle and $B(\theta)$ is the area of the triangle, find $\lim_{\theta \to 0^+} \dfrac{A(\theta)}{B(\theta)}$.

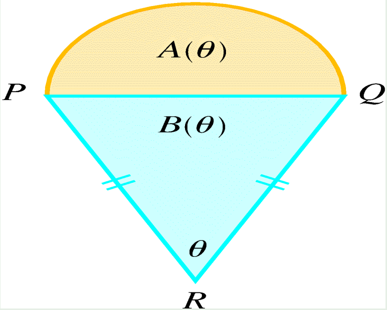


Figure: Ice cream cone.

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- By contrast, in populations that breed continuously, there is no natural time scale such as generations. Instead, we will look at how the population size changes over small time intervals.
- We denote the population size at time $t$ by $N(t)$, where t is now varying continuously over the interval $[0, \infty)$. We investigate how the population size changes during the interval $[t_0, t_0 + h]$, where $h > 0$. The absolute change during this interval, denoted by $\Delta N$, is $\Delta N = N(t_0 + h) - N(t_0)$.

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- To obtain the relative change during this interval, we divide $\Delta N$ by the length of the interval, denoted by $\Delta t$, which is $h$. We find that

$$\frac{\Delta N}{\Delta} = \frac{N(t_0 + h) - N(t_0)}{h}$$

This ratio is called the average growth rate.

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To obtain this quantity, we need to take a limit; that is, we need to shrink the length of the interval $[t_0, t_0 + h]$ to 0 by letting $h$ tend to 0. We express this operation as

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We denote the limiting value of $\Delta N / \Delta t$ as $\Delta t \to 0$ by $N'(t_0)$ (read "$N$ prime of $t_0$") and call this quantity **the derivative of** $N(t)$ **at** $t_0$, provided that this limit exists!

We formalize the previous discussion for any function $f$ . The average rate of change of the function $y = f(x)$ between $x = x_0$ and $x = x_1$ is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

By setting $h = x_1 - x_0$, i.e., $x_1 = x_0 + h$, the above expression becomes

$$\frac{\Delta f}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h}$$

Those quantities represent the slope of the secant line that passes through the points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$ [or $P(x_0, f(x_0))$ and $Q(x_0 + h, f(x_0 + h))$, respectively].

The instantaneous rate of change is defined as the result of computing the average rate of change over smaller and smaller intervals.

## Definition

The derivative of a function $f$ at $x_0$, denoted by $f'(x_0)$, is

$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that the limit exists. In this case we say that the function $f$ is differentiable at $x_0$.

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Geometrically $f'(x_0)$ represents the slope of the tangent line.

## Example (Online Homework HW11, #3)

Let $f(x)$ be the function $12x^2 - 2x + 11$. Then the quotient $\frac{f(1+h)-f(1)}{h}$ can be simplified to $ah + b$ for $a =$ ____ and $b =$ ____.

Compute

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$