1. (10 points) Consider the recursion

\[ x_{t+1} = \frac{1.5x_t}{1.5x_t + 2.0(1 - x_t)}. \]

(a) Find all equilibrium points for this recursion.

Solution: \( f(x) = \frac{1.5x}{1.5x + 2.0(1 - x)}. \) Solve \( f(x) = x. \)

\[
\begin{align*}
\frac{1.5x}{1.5x + 2.0(1 - x)} &= x \\
1.5x &= x(1.5x + 2.0(1 - x)) \\
3x &= 4x - x^2 \\
x^2 - x &= 0 \\
x &= 0 \text{ or } x = 1
\end{align*}
\]

(b) Classify the stability of each equilibrium point.

Solution: \( f(x) = \frac{1.5x}{1.5x + 2.0(1 - x)} = f(x) = \frac{1.5x}{2 - 0.5x}. \) So

\[
\begin{align*}
f'(x) &= \frac{1.5(2 - 0.5x) - (1.5x)(-0.5)}{(2 - 0.5x)^2} = \frac{3}{(2 - 0.5x)^2}.
\end{align*}
\]

Checking the derivative at the equilibrium points we see

\[ f'(0) = \frac{3}{4} < 1 \] so this equilibrium point is stable.

\[ f'(1) = \frac{12}{9} > 1 \] so this equilibrium point is unstable.

2. Find the following general antiderivatives.

(a) \( \int \left( x^3 + x + x^{-1} + x^{-3} \right) \, dx \)

Solution: \[ \int \left( x^3 + x + x^{-1} + x^{-3} \right) \, dx = \frac{1}{4}x^4 + \frac{1}{2}x^2 + \ln x - \frac{1}{2}x^{-2} + C. \]
(b) \[ \int \left( e^{3t} - e^{-2t} + \frac{1}{t} \right) \, dt \]

**Solution:**
\[ \int \left( e^{3t} - e^{-2t} + \frac{1}{t} \right) \, dt = \frac{1}{3} e^{3t} + \frac{1}{2} e^{-2t} + \ln t + C \]

(c) \[ \int (\cos w + (w - 2)(w + 1)) \, dw \]

**Solution:**
\[ \int (\cos w + (w - 2)(w + 1)) \, dw = \int (\cos w + w^2 - w - 2) \, dw = \sin w + \frac{1}{3} w^3 - \frac{1}{2} w^2 - 2w + C \]

(d) \[ \int \frac{3r^3 - 2r^2 + r - r^{-1} - 2r^{-2}}{r^2} \, dr \]

**Solution:**
\[ \int \frac{3r^3 - 2r^2 + r - r^{-1} - 2r^{-2}}{r^2} \, dr = \int \left( 3r - 2 + \frac{1}{r} - r^{-3} - 2r^{-4} \right) \, dr \\
= \frac{3}{2} r^2 - 2r + \ln r + \frac{1}{2} r^{-2} + \frac{2}{3} r^{-3} + C \]

3. Use the substitution \( u = 2x - 4 \) to find \( \int 8x\sqrt{2x-4} \, dx \).

**Solution:** Let \( u = 2x - 4 \), then \( du = 2dx \) and
\[ \int 8x\sqrt{2x-4} \, dx = \int 4x\sqrt{2x-4} \, 2dx \]
\[ = \int 4 \left( \frac{u + 4}{2} \right) \sqrt{u} \, du \]
\[ = \int (2u + 8) \sqrt{u} \, du \]
\[ = \int \left( 2u^{3/2} + 8u^{1/2} \right) \, du \]
\[ = \frac{4}{5} u^{5/2} + \frac{16}{3} u^{3/2} + C \]
\[ = \frac{4}{5} (2x - 4)^{5/2} + \frac{16}{3} (2x - 4)^{3/2} + C \]
4. (a) What is the value of $\int_{-3}^{1} |x| \, dx$?

**Solution:** Use geometry or analysis.
Geometry: The area under the curve is the area of a right triangle with base 3 and height 3 plus the area of a right triangle with base 1 and height 1, giving a total area of 5.

Analytic:

\[
\int_{-3}^{1} |x| \, dx = \int_{-3}^{0} (-x) \, dx + \int_{0}^{1} x \, dx = \left[ -\frac{1}{2} x^2 \right]_{-3}^{0} + \left[ \frac{1}{2} x^2 \right]_{0}^{1} = 5.
\]

(b) Find $\int_{0}^{6} f(t) \, dt$ where

\[
f(t) = \begin{cases} 
2 & \text{if } 0 \leq t \leq 1 \\
2t & \text{if } 1 \leq t \leq 2 \\
8 - 2t & \text{if } 2 \leq t \leq 4 \\
t - 4 & \text{if } 4 \leq t \leq 6 
\end{cases}
\]

**Solution:** This can be done geometrically if you draw a graph. Otherwise

\[
\int_{0}^{6} f(t) \, dt = \int_{0}^{1} 2 \, dt + \int_{1}^{2} 2t \, dt + \int_{2}^{4} (8 - 2t) \, dt + \int_{4}^{6} (t - 4) \, dt
\]

\[= 2t \bigg|_{0}^{1} + t^2 \bigg|_{1}^{2} + \left(8t - t^2\right) \bigg|_{2}^{4} + \left(\frac{1}{2} t^2 - 4t\right) \bigg|_{4}^{6}
\]

\[= (2 - 0) + (4 - 1) + (16 - 12) + (-6 - (-8)) = 11.
\]

5. In town A, the birth rate is given by $P_A(t) = 57e^{0.06t}$ (births per year) where $t$ is the number of years since 1990. In town B, the birth rate is given by $P_B(t) = 83e^{0.03t}$ births per year where $t$ is the number of years since 1990. How many more births are there in town B than in town A during the 1990s (from $t = 0$ to $t = 10$)?

**Solution:** The difference will be the area between the curves from $t = 0$ and $t = 10$.

\[
\int_{0}^{10} \left( 83e^{0.03t} - 57e^{0.06t} \right) \, dt = 186.93
\]
6. Find the following requested values.

(a) Find \( F'(2) \) if \( F(x) = \int_0^x \left( u^2 \ln(2 \cos(\pi u)) - e^{u^2-4} \right) \, du \).

**Solution:** By the Fundamental Theorem of Calculus,
\[
F'(x) = x^2 \ln(2 \cos(\pi x)) - e^{x^2-4}
\]
and thus
\[
F'(2) = 2^2 \ln(2 \cos(2\pi)) - e^{2^2-4} = 4 \ln 2 - 1 \approx 1.77259.
\]

(b) Find \( G'(x) \) if \( G(x) = \int_0^{3x^2} u^2 \ln(u^2 + 1)e^u \, du \).

**Solution:** Let \( F(t) = \int_0^t u^2 \ln(u^2 + 1)e^u \, du \), then \( G(x) = F(3x^2) \) and \( G'(x) = 6xF'(3x^2) \). By the FTC, \( F'(t) = t^2 \ln(t^2 + 1)e^t \). Thus,
\[
G'(x) = 6x \left( (3x^2)^2 \ln((3x^2)^2 + 1)e^{3x^2} \right) = 54x^5 \ln \left( 9x^4 + 1 \right) e^{3x^2}
\]

7. Evaluate \( \lim_{x \to 0} \frac{\int_0^x \sin u^2 \, du}{x^3} \).

**Solution:** We will use l’Hospital’s Rule since both the top and the bottom go to zero.
\[
\lim_{x \to 0} \frac{\int_0^x \sin u^2 \, du}{x^3} = \lim_{x \to 0} \frac{\sin x^2}{3x^2} = \frac{1}{3}.
\]

8. The blood alcohol concentration of a male adult subject after rapid consumption of 15 mL of ethanol (corresponding to one alcoholic drink) is given by the concentration function
\[
C(t) = 0.022te^{-0.0067t^2}
\]
where \( t \) is measured in minutes after consumption and \( C(t) \) is measured in mg/mL. What was the average blood alcohol concentration during the first hour?

**Solution:**
\[
\frac{1}{60 - 0} \int_0^{60} 0.022te^{-0.0067t^2} \, dt = \frac{1.64179}{60} = 0.02736.
\]
9. Find the following integrals.

(a) \[ \int_{2}^{a} 6x^5 \, dx \]

**Solution:**
\[ \int_{2}^{a} 6x^5 \, dx = x^6 \bigg|_{2}^{a} = a^6 - 64. \]

(b) \[ \int_{0}^{b} (x + 1)^3 \, dx \]

**Solution:**
\[ \int_{0}^{b} (x + 1)^3 \, dx = \frac{1}{4}(x + 1)^4 \bigg|_{0}^{b} = \frac{1}{4}(b + 1)^4 - 4. \]

10. Suppose that \( f \) and \( g \) are functions so that
\[ \int_{4}^{8} f(x) \, dx = -2 \quad \int_{4}^{8} g(x) \, dx = 10 \quad \int_{6}^{8} f(x) \, dx = 4 \quad \int_{4}^{6} g(x) \, dx = 5 \]

(a) Find \[ \int_{4}^{8} (4f(x) - 2g(x)) \, dx \]

**Solution:**
\[ \int_{4}^{8} (4f(x) - 2g(x)) \, dx = 4 \int_{4}^{8} f(x) \, dx - 2 \int_{4}^{8} g(x) \, dx = 4(-2) - 2(10) = -28. \]

(b) Find \[ \int_{4}^{6} (f(x) + 2g(x)) \, dx \]

**Solution:** Since \[ \int_{4}^{8} f(x) \, dx = -2 \] and \[ \int_{6}^{8} f(x) \, dx = 4, \] we then have that
\[ \int_{4}^{6} f(x) \, dx = -2 - 4 = -6. \] Therefore,
\[ \int_{4}^{6} (f(x) + 2g(x)) \, dx = \int_{4}^{6} f(x) \, dx + 2 \int_{4}^{6} g(x) \, dx = (-6) + 2(5) = 4. \]
(c) Find $\int_6^8 (f(x) - g(x)) \, dx$

**Solution:** Since $\int_4^8 g(x) \, dx = 10$ and $\int_4^6 g(x) \, dx = 5$, we then have that $\int_6^8 g(x) \, dx = 10 - 5 = 5$. Therefore,

$$\int_6^8 (f(x) - g(x)) \, dx = \int_6^8 f(x) \, dx - \int_6^8 g(x) \, dx = 4 - 5 = -1.$$  

11. Find the area of the region bounded by the graph of $y = x\sqrt{1 - x^2}$ and the x-axis in the first quadrant.

**Solution:**

$$\int_0^1 x\sqrt{1 - x^2} \, dx = -\left. \frac{1}{3} (1 - x^2)^{3/2} \right|_0^1 = \frac{1}{3}.$$

12. Let $f(x) = x^2 - 20$ and $g(x) = 12 - x^2$.

(a) Find the area between the curves $f(x)$ and $g(x)$ for $x \geq 0$.

**Solution:** The curves intersect where $x^2 - 20 = 12 - x^2$ or at $x = \pm 4$. Since we are interested only in $x \geq 0$, we have that the area between the curves will be

$$\int_0^4 \left( (12 - x^2) - (x^2 - 20) \right) \, dx = \int_0^4 32 - 2x^2 \, dx = 32x - \frac{2}{3}x^3 \bigg|_0^4 = \frac{256}{3}.$$

(b) The vertical line $x = c$ ($c > 0$) passes through this region and divides this region into two parts of equal area. Find the value of $c$. 

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Solution: Find $c$ so that

\[
\int_0^c \left( (12 - x^2) - (x^2 - 20) \right) dx = \frac{128}{3}
\]

\[
32x - \frac{2}{3}x^3 \bigg|_0^c = \frac{128}{3}
\]

\[
\frac{2}{3}c^3 + 32c - \frac{128}{3} = 0
\]

\[
2c^3 + 96c - 128 = 0
\]

\[
c \approx 1.38919
\]

13. Pollution enters a lake from a stream at rate $p(t) = 10\left(1 - e^{-t/2}\right)$ gal/hr. Find the total pollution in the lake after 10 hours.

Solution:

\[
P = \int_0^{10} 10\left(1 - e^{-t/2}\right) dt = 10x + 20e^{-t/2}\bigg|_0^{10} = 20\left(4 + e^{-5}\right) = 80.1348.
\]

14. In the case of a zombie apocalypse, do you feel qualified to work for the CDC to model the spread of zombies throughout the zombie nation?

Solution: It will be interesting to see what they have to say.