Record your answers to the multiple choice question using the bottom table on this page. For the free-response question, use the spaces provided. Clearly show all your work. You can lose points on free-response questions for correct answers with no supporting work.

Full Name: ______________________________ UKID: __________

Math 138 Final Exam. Circle Your Section:

001 (12:30 - 1:20) **002** (2:00 - 2:50)

Question	Points	Score
1	50	
2	10	
3	10	
4	10	
5	10	
6	10	
7	0	
Total:	100	

The first ten problems are multiple choice problems, each worth five points. Record your answers to these problems in the ten columns below by filling in the circle corresponding to your response. Leave the bottom row of this table blank.

Question:	1	2	3	4	5	6	7	8	9	10
	a	a	a	a	a	a	a	a	a	a
	b	b	b	b	b	b	b	b	b	b
	c	c	c	c	c	c	c	c	c	c
	d	d	d	d	d	d	d	d	d	d
	e	e	e	e	e	e	e	e	e	e
Score:										

1. Multiple Choice Questions:

   Remeber to record your answers using the table on the front of the exam.

   (i) (5 points) Evaluate the improper integral

   $$\int_e^{\infty} \frac{1}{x\ln(x)^4}\,dx$$

   A. The integral diverges.

   B. $\frac{1}{4}$

   C. $\frac{1}{3}$

   D. $\frac{1}{2}$

   E. 1

   (ii) (5 points) How many **stable** equilibrium solutions are there for the differential equation below?

   $$\frac{dy}{dx} = (y+2)(y-2)^2(y-4)(y-5).$$

   A. 0
   B. 1
   C. 2
   D. 3
   E. 4

(iii) (5 points) The augmented matrix below can be put in reduced row echelon form using just two row operations. Which row operations will accomplish this?

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

A. First $R_2 - R_3$, then $R_1 - 2R_2$.
B. First $R_1 + 2R_2$, then $R_2 - R_3$.
C. First $\frac{1}{7}R_3$, then $\frac{1}{3}R_2$.
D. First $R_3 - R_2$, then $R_2 + 2R_1$
E. First $R_2 - R_3$, then $R_1 + 2R_2$

(iv) (5 points) What is the linearization at the point $(1, 2)$ of the vector-valued function:

$$\vec{f}(x, y) = \begin{bmatrix} x^2 y \\ x + y \end{bmatrix}$$

A. $L(x, y) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix}$

B. $L(x, y) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix}$

C. $L(x, y) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix}$

D. $L(x, y) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix}$

E. $L(x, y) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix}$

(v) (5 points) Which of the following is the general solution to the system of differential equations.

$$\frac{dx}{dt} = x + 4y$$
$$\frac{dy}{dt} = 4x + y$$

A. $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$

B. $\begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{c_1 t} - 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{c_2 t}$

C. $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} 5e^t + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} - 3e^t$

D. $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$

E. $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$

(vi) (5 points) Suppose we have a linear system of differential equations

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

where A is a 2×2 matrix of real numbers. A plot of the trajectories near the origin is shown below. Which of the following could be the characteristic polynomial of A?

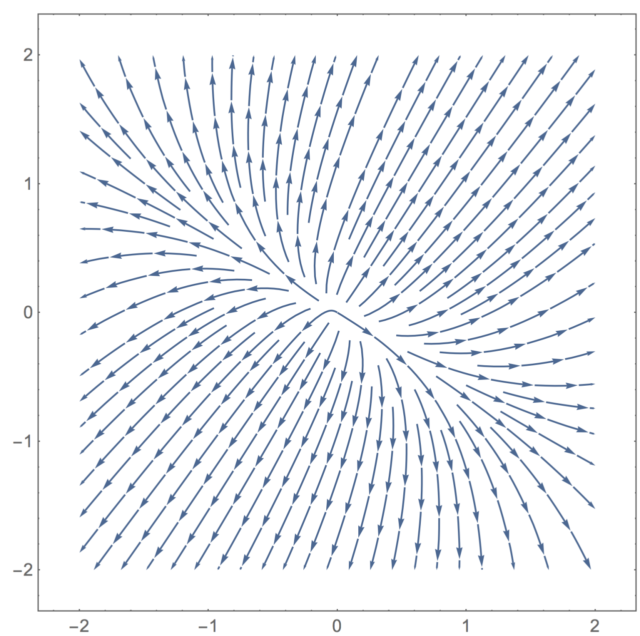
A. $\lambda^2 - 5\lambda + 5$

B. $\lambda^2 - 3\lambda + 3$

C. $\lambda^2 - 4\lambda + 10$

D. $\lambda^2 - 3\lambda - 4$

E. $\lambda^2 + 3\lambda + 20$



(vii) (5 points) What are the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 4 \\ 3 & 1 \end{bmatrix}$$

A. 4 and -3
B. 4 and 3
C. -4 and -3
D. ± 4 and ± 3
E. 0 and 1

(viii) (5 points) This linear system of differential equations has only one equilibrium point, at the origin. What type of equilibrium point is it?

$$\frac{dx}{dt} = x - y$$
$$\frac{dy}{dt} = x + y$$

A. It is globally unstable, a source.
B. It is globally stable, a sink.
C. It is a saddle point.
D. It is an unstable spiral.
E. It is a stable spiral.

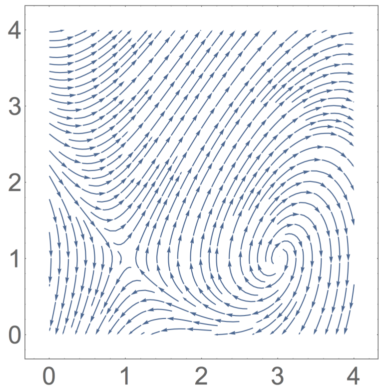
(ix) (5 points) How many equilibrium points does this non-linear system of differential equations have?

$$\frac{dx}{dt} = (y + x)(y - x)$$
$$\frac{dy}{dt} = (x - 2)(x - 3)$$

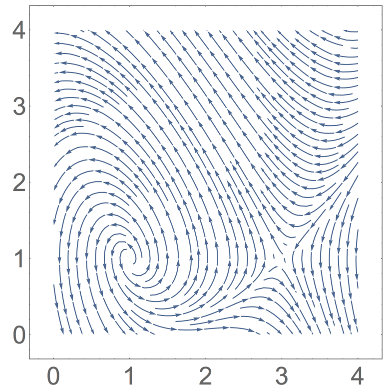
A. 1
B. 2
C. 3
D. 4
E. 5

(x) (5 points) Choose the the plot below which shows the trajectories of the given nonlinear system of differential equations in the region $0 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 4$: Larger versions of these pictures are at the back of the exam.

$$\frac{dx}{dt} = 1 - y$$
$$\frac{dy}{dt} = y - (x-2)^2$$



A



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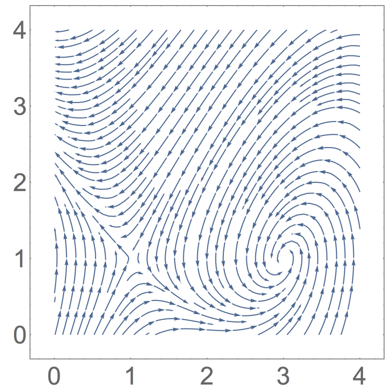

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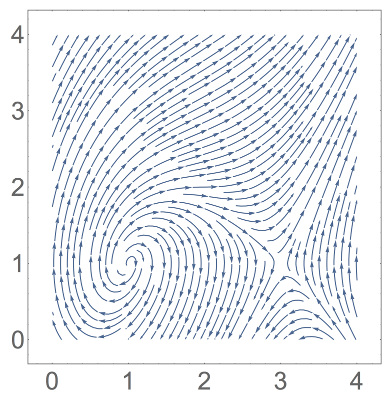
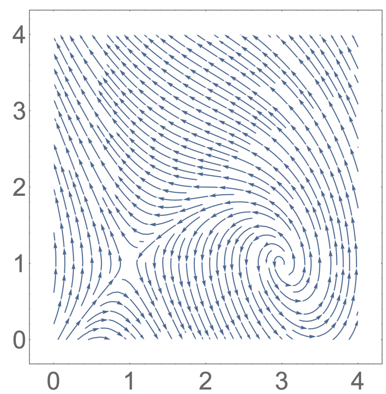


D



E

2. (10 points) a. Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{-2y}{x(x-2)}$$

b. Find the solution that satisfies the initial value condition $f(4) = 8$

3. (10 points) What is the linearization at the point $(0, \pi)$ of the vector-valued function

$$\vec{f}(x, y) = \begin{bmatrix} 2x + 4y \\ \sin(x + 2y) \end{bmatrix}$$

4. (10 points) Let A be the matrix

$$A = \begin{bmatrix} 7 & k \\ 1 & 9 \end{bmatrix}.$$

What are the values of k so that the differential equation $\dfrac{d\vec{x}}{dt} = A\vec{x}$ has a saddle point at $(0, 0)$? Express your answer in interval notation.

5. (10 points) a. Find the solution to the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= \ \ 12x - 2y \\ \frac{dy}{dt} &= -2x + 9y\end{aligned}$$

that satisfies the initial value conditions $x(0) = 15, y(0) = -10$

b. Classify the stability of the equilibrium point $(0, 0)$. Justify your answer.

6. (10 points) Consider the non-linear system of differential equations:

$$\frac{dx}{dt} = (y - x)(y - 2)$$
$$\frac{dy}{dt} = (y)(x - 1)$$

a. This system could be written in the form $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$, where $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector-valued function. What is the Jacobian matrix of this $\vec{f}$? (Your answer should depend on x and y.)

b. This system has three equilibrium points. One of them is the point $(0, 0)$, which is a saddle point. Find the other two equilibrium points of this system, and use the Hartman-Grobman Theorem to classify them (the classifications are source, sink, saddle point, stable spiral, and unstable spiral).

The equilibrium point __________ is a __________ .

The equilibrium point __________ is a __________ .

7. (0 points) **BONUS.** There is a way to use systems of first-order differential equations to solve higher-order differential equations. While we did not talk about this method in class, you actually already know all you need to know in order to solve such problems. This problem attempts to walk you through it step-by-step. Suppose y is a function of x and consider the second-order differential equation:

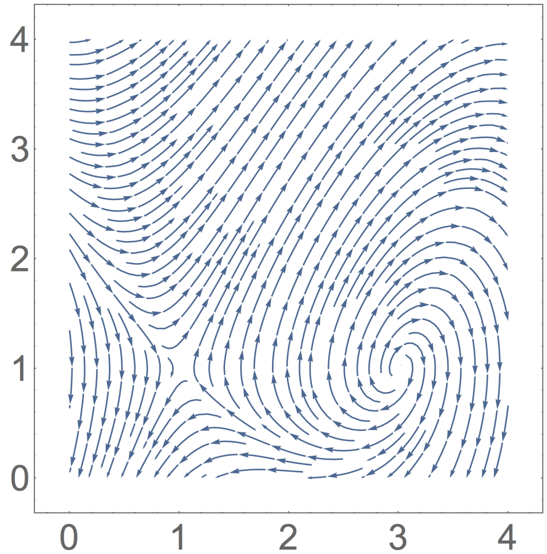
$$y''(x) = y'(x) + 2y(x)$$

   a. Introduce new variable(s) to write this second-order differential equation as a linear system of first-order differential equations.

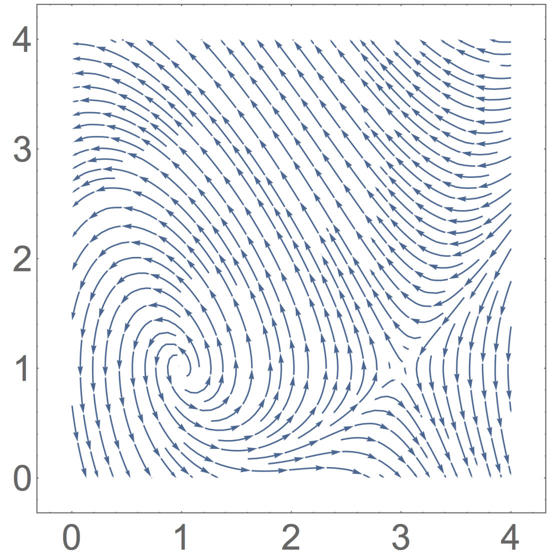
   b. Solve the system from part a.

   c. Use your solution from part b. to produce the general solution to the second-order differential equation above.

Larger versions of the answers to multiple choice question (x):



A



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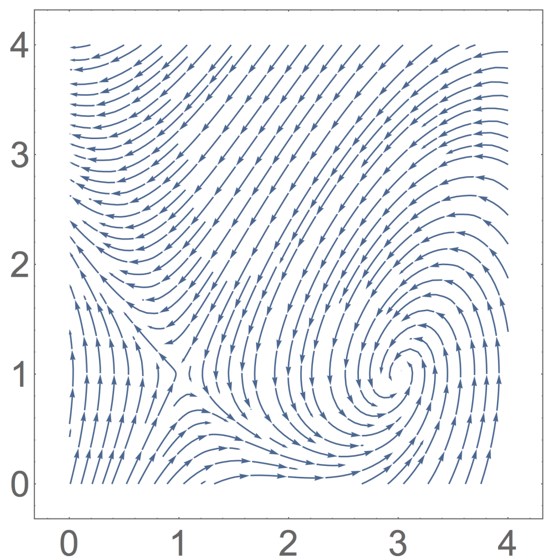

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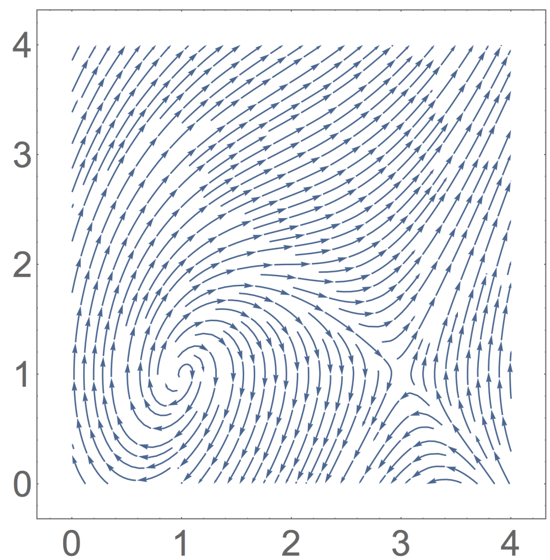
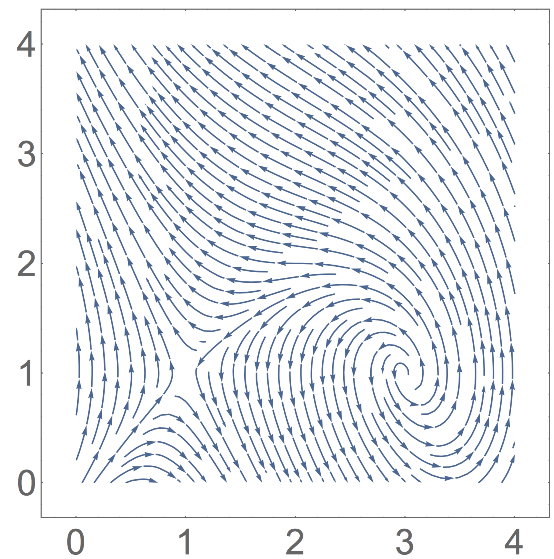


D



E