# MA137 – Calculus 1 with Life Science Applications
# **Discrete-Time Models**
## Sequences and Difference Equations
## (Sections 2.1 and 2.2)

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# What are sequences?

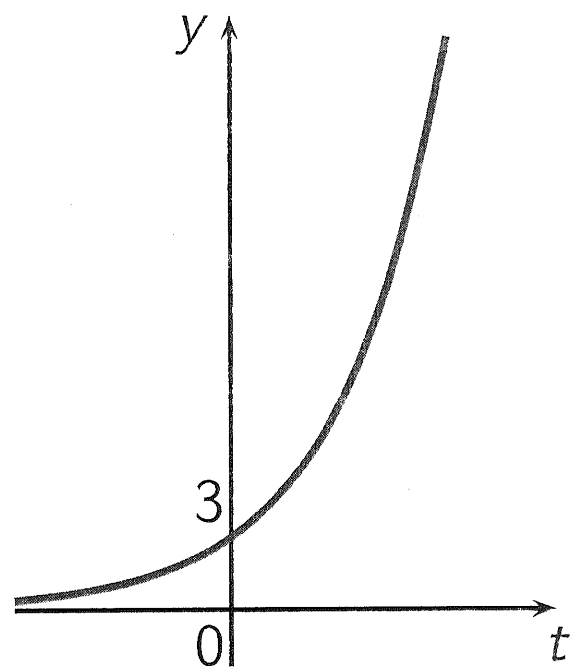
So far we have studied real valued functions whose domain consists of the real numbers, say:

$$f : \mathbb{R} \longrightarrow \mathbb{R}.$$

**For example**, consider the function

$$f(t) = 3 \cdot 2^t.$$

The graph of $f$ looks like:




**More generally**, we have considered functions of the form

$$P(t) = P_0(1 + r)^t,$$

where $r$ is a positive real number ($r \equiv$ growth rate).

Sometimes it makes sense to change the domain of the function to the nonnegative integers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

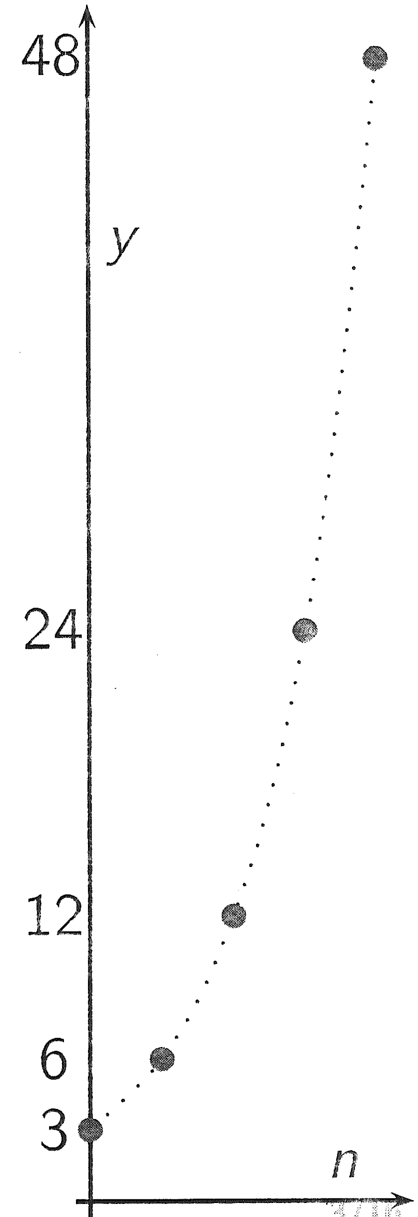
$$f : \mathbb{N} \longrightarrow \mathbb{R}, \qquad n \mapsto f(n).$$

**For example**, $f(n) = 3 \cdot 2^n$ with $n \in \mathbb{N}$.

A table is a useful tool to illustrate this function

| $n$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ |
|---|---|---|---|---|---|---|
| $3 \cdot 2^n$ | 3 | 6 | 12 | 24 | 48 | $\cdots$ |

The graph is useful too!




Because the domain consists of nonnegative integers, the graph consists of isolated points with coordinates

$$(0, f(0)) \quad (1, f(1)) \quad (2, f(2)) \quad (3, f(3)) \quad (4, f(4)) \quad \cdots$$

**Note:** we should not have connected the isolated points with the dotted curve. Please disregard it!!

# Definition and Notation

**Definition (Sequence/Notation)**

We can write the function

$$f : \mathbb{N} \longrightarrow \mathbb{R}, \qquad n \mapsto f(n)$$

as a list of numbers $f_0, f_1, f_2, f_3, \ldots,$ where $f_n = f(n)$.

We refer to this list as a **sequence**.

We write $\{f_n \mid n \in \mathbb{N}\}$ (or $\{f_n\}$ for short) to denote the entire sequence.

We list the values of the sequence $\{f_n\}$ in order of increasing $n$

$$f_0, \ f_1, \ f_2, \ f_3, \ \ldots$$

**Remark:** Instead of '$f$' we often use the letters '$a$' or '$b$' or '$c$' ... to denote sequences.

For example: $a_n = \dfrac{n}{n+1}$ $\qquad b_n = \dfrac{(-1)^n}{(n+1)^2}$ $\qquad c_n = 3 \cdot 2^n$

# Example 1:

Find a general formula for the general term $a_n$ for each of the following sequences starting with $a_0$:

(a) $0, 1, 4, 9, 16, 25, 36, 49, \ldots$

(b) $1, -1, 1, -1, 1, -1, \ldots$

(c) $1, \; -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \ldots.$

Repeat this problem starting this time with $a_1$.

(a) Consider $0, 1, 4, 9, 16, 25, 36, 49, \ldots$

these are all squares of numbers.

We want them to be labeled as

$a_0 = 0, \quad a_1 = 1, \quad a_2 = 4, \quad a_3 = 9, \quad a_4 = 16, \ldots$

thus $\boxed{a_n = n^2}$ is the nth term of the sequence

(b) We want: $a_0 = 1, \; a_1 = -1, \; a_2 = 1, \; a_3 = -1, \ldots$

So we have $\boxed{a_n = (-1)^n}$ for all $n \in \mathbb{N}$

(c) We want: $a_0 = 1, \; a_1 = -\frac{1}{2}, \; a_2 = \frac{1}{4}, \; a_3 = -\frac{1}{8}$

$a_4 = \frac{1}{16}$, etc ... Notice that all denominators

are powers of 2; there is an alternating sign: $\boxed{a_n = \left(-\frac{1}{2}\right)^n}$

Repeat :

(a) this time we want : $a_1 = 0$ , $a_2 = 1$ , $a_3 = 4$ , $a_4 = 9$ , $a_5 = 16$ , ..... Thus we need to shift the integers :

$$a_n = (n-1)^2 \qquad \text{for } n = 1, 2, 3, 4, \ldots$$

(b) We want : $a_1 = 1$ , $a_2 = -1$ , $a_3 = 1$ , $a_4 = -1$ , ... again we shift the integers :

$$a_n = (-1)^{n-1} \quad \text{or} \quad a_n = (-1)^{n+1} \qquad n = 1, 2, 3, 4, \ldots$$

(c) We want : $a_1 = 1$ , $a_2 = -\frac{1}{2}$ , $a_3 = \frac{1}{4}$ , $a_4 = -\frac{1}{8}$ , ....

$$a_n = \left(-\frac{1}{2}\right)^{n-1} \qquad n = 1, 2, 3, 4, \ldots$$

# Example 2:

Consider the sequence given by

$$a_n = 2 + \frac{(-1)^n}{n} \qquad n > 1.$$

List the first six terms of the sequence and plot them on the Cartesian plane.

$$a_n = 2 + \frac{(-1)^n}{n} \qquad n > 1$$

notice that the expression does <u>not</u> make sense for $n = 0$.

$$a_1 = 2 + \frac{(-1)^1}{1} = 2 - 1 = \underline{1}$$

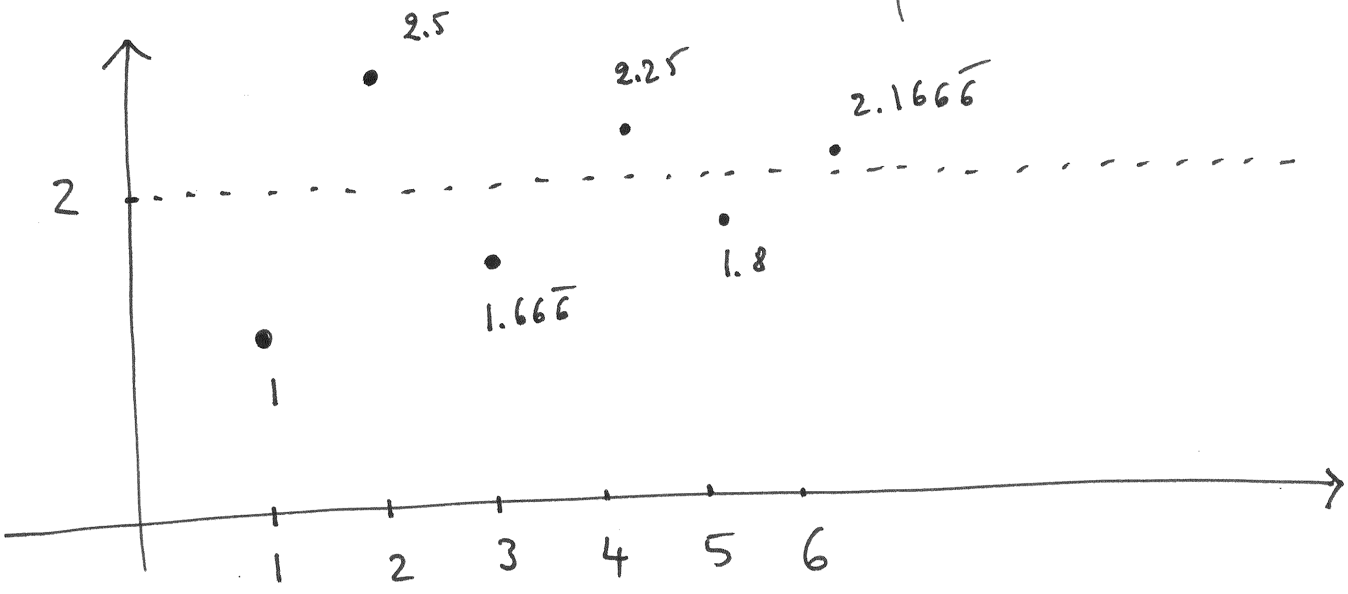
$$a_2 = 2 + \frac{(-1)^2}{2} = \underline{2.5}$$

$$a_3 = 2 + \frac{(-1)^3}{3} = 2 - \tfrac{1}{3} = \underline{1.66\bar{6}}$$

$$a_4 = 2 + \frac{(-1)^4}{4} = \underline{2.25}$$

$$a_5 = 2 + \frac{(-1)^5}{5} = \underline{1.8}$$

$$a_6 = 2 + \frac{(-1)^6}{6} = 2.166\bar{6}$$



# Recursions (or Recursive Sequences)

The exponential growth model we considered earlier

$$P_n = 3 \cdot 2^n$$

is an example of a sequence. Explicitly, we have

$$P_0 = 3, \quad P_1 = 6, \quad P_2 = 12, \quad P_3 = 24, \quad P_4 = 48, \quad \cdots$$

It is not difficult to observe that this sequence of numbers describes quantities that double after each unit of time.
More explicitly, we can write

$$P_1 = 2P_0, \qquad P_2 = 2P_1, \quad P_3 = 2P_2, \quad P_4 = 2P_3, \quad \cdots$$

We can summarize the above facts into a single expression. I.e.,

$$P_{n+1} = 2P_n$$

this expression gives a rule that is applied repeatedly to go from one time step (the $n$th) to the next one (the $(n+1)$st).
Such an expression is called a **recursion**.

# Example 3:

**(a)** List the first five terms of the recursively define sequence

$$a_0 = 1 \qquad a_{n+1} = (n+1)a_n.$$

Do you see something familiar?

**(b)** List the first five terms of the recursively define sequence

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = 1 + \frac{1}{a_n}.$$

Do you see something familiar?

**Caution:** While it is easy to compute terms in a recursive relation, there are 2 issues:

- In order to find $a_{100}$, we have to compute the previous 99 terms.
- We may not get a feeling for what will eventually happen.

$a_0 = 1 \qquad a_{n+1} = (n+1)\,a_n \qquad n = 0, 1, 2, 3, \ldots$

when $n=0$ $\quad a_1 = 1 \cdot a_0 = 1$

when $n=1$ $\quad a_2 = (1+1)\,a_1 = 2 \cdot 1 = 2!$

when $n=2$ $\quad a_3 = (2+1)\,a_2 = 3 \cdot a_2 = 3 \cdot 2 \cdot 1 = 3!$

when $n=3$ $\quad a_4 = (3+1)\,a_3 = 4 \cdot a_3 = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$

when $n=4$ $\quad a_5 = (4+1)\,a_4 = 5 \cdot a_4 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

In general the explicit form for the sequence is: $\boxed{a_n = n!}$ for $n = 0, 1, 2, \ldots$

$a_1 = 1 \qquad a_{n+1} = 1 + \frac{1}{a_n}$ for $n = 1, 2, 3, 4, 5, \ldots$

when $n = 1$ $\quad a_2 = 1 + \frac{1}{a_1} = 1 + \frac{1}{1} = \underline{\underline{2}}$

when $n = 2$ $\quad a_3 = 1 + \frac{1}{a_2} = 1 + \frac{1}{2} = \underline{\underline{3/2}} \cong 1.5$

when $n = 3$ $\quad a_4 = 1 + \frac{1}{a_3} = 1 + \frac{1}{3/2} = 1 + \frac{2}{3} = \underline{\underline{\frac{5}{3}}} \cong 1.\overline{666}$

when $n = 4$ $\quad a_5 = 1 + \frac{1}{a_4} = 1 + \frac{1}{5/3} = 1 + 3/5 = \underline{\underline{\frac{8}{5}}} \cong 1.6$

when $n = 5$ $\quad a_6 = 1 + \frac{1}{a_5} = 1 + \frac{1}{8/5} = 1 + \frac{5}{8} = \underline{\underline{\frac{13}{8}}} \cong 1.625$

this sequence is given by the quotient of 2 consecutive Fibonacci's numbers

when $n \to \infty$ this ratio tends to $1.618 \simeq \frac{1+\sqrt{5}}{2}$

**GOLDEN RATIO**

# Example 4: (Online Homework HW06, # 8)

**(a)** Find a recursive definition for the sequence $9, 11, 13, 15, 17, \ldots$ Assume the first term in the sequence is indexed by $n = 1$.

**(b)** Find a closed formula for the sequence $9, 11, 13, 15, 17, \ldots$ Assume the first term in the sequence is indexed by $n = 1$.

9, 11, 13, 15, 17, .....

every number is obtained from the previous one by adding <u>two</u>:

$a_1 = 9$, $a_2 = 11 = 9+2 = 9+2(1)$, $a_3 = 13 = 9+4 = 9+2(2)$, $a_4 = 15 = 9+6 = 9+2(3)$, $a_5 = 17 = 9+8 = 9+2(4)$

<u>Recursive</u>: $\boxed{a_1 = 9 \qquad a_{n+1} = a_n + 2}$ $\quad n = 1, 2, 3, \ldots$

<u>Explicit</u>: $\boxed{a_n = 9 + 2(n-1)}$ $\quad n = 1, 2, 3, 4, \ldots$

# Recap

We gave two descriptions of sequences: explicit and recursive.

- An **explicit description** is of the form $a_n = f(n)$, $n = 0, 1, 2, \ldots$ where $f(n)$ is a function of $n$.
- A **recursive description** is of the form $a_{n+1} = g(a_n)$, $n = 0, 1, 2, \ldots$ where $g(a_n)$ is a function of $a_n$.

**Remark 1:**
In the above situation the value of $a_{n+1}$ depends only on the value one time step back, namely, $a_n$. In this case the recursion is called a **first-order recursion** .

**Remark 2:**
The sequence defined by

$$a_0 = 1, \quad a_1 = 1, \quad a_{n+2} = a_n + a_{n+1} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

is an example of a **second-order recursion**.

# Recursive Sequences in the Life Sciences

Recursive sequences (or **difference equations**) are often used in biology to model, for example, cell division and insect populations.

In this biological context we usually replace $n$ by $t$, to denote time.

If we think of $t$ as the current time, then $t+1$ is one unit of time into the future. We also use $N_t$ to denote the population size.

Thus a first-order difference equation modeling population size has the form

$$N_{t+1} = f(N_t) \qquad t = 0, 1, 2, 3, \ldots$$

In this context we call $f$ an **updating function** because $f$ 'updates' the population from $N_t$ to $N_{t+1}$.

# Malthusian (or Exponential) Growth Model

One of our earlier examples can be rewritten as

$$N_{t+1} = 2N_t \quad N_0 = 3 \qquad \text{or} \qquad N_t = 3 \cdot 2^t.$$

This example is a special case of the so called **Malthusian Growth Model**, named after Thomas Malthus (1766-1834):

$$N_{t+1} = (1 + r)N_t$$

which says that the next generation is proportional to the population of the current generation.

It is typical to set $R = 1 + r$ so that the recursion becomes

$$N_{t+1} = RN_t.$$

This recursion has the following explicit form

$$N_t = N_0 R^t.$$

Hence the name of Exponential Growth Model.

# Example 5: (Online Homework HW06, # 11)

(a) A population of herbivores satisfies the growth equation $y_{n+1} = 1.05y_n$, where $n$ is in years. If the initial population is $y_0 = 6,000$, then determine the explicit expression of the population.

(b) A competing group of herbivores satisfies the growth equation $z_{n+1} = 1.06y_n$ If the initial population is $z_0 = 3,200$, then determine how long it takes for this population to double.

(c) Find when the two populations are equal.

(a) $y_n = 6{,}000\,(1.05)^n$

(b) $z_n = 3{,}200\,(1.06)^n$

we want to know $n$ such that

$$\underline{3{,}200\,(1.06)^n} = z_n = \underline{2 \cdot 3{,}200}$$

<u>i.e.</u> we want $(1.06)^n = 2$

take log (or $\ln$) of both sides

$$\log (1.06)^n = \log(2) \implies n = \frac{\log 2}{\log(1.06)}$$

$$\cong \underline{\underline{11.895}}$$

(c) We want to find $n$ such that the two populations are equal:

$$6{,}000\,(1.05)^n = 3{,}200\,(1.06)^n$$

Rewrite as:

$$\frac{6{,}000}{3{,}200} = \frac{(1.06)^n}{(1.05)^n} \quad \text{OR} \quad \frac{15}{8} = \left(\frac{1.06}{1.05}\right)^n$$

Take log (or ln) of both sides

$$\log\left(\frac{15}{8}\right) = \log\left[\left(\frac{1.06}{1.05}\right)^n\right]$$

$$\Rightarrow \quad n \log\left(\frac{1.06}{1.05}\right) = \log(15/8)$$

$$\therefore \quad n = \frac{\log(15/8)}{\log\left(\frac{1.06}{1.05}\right)} \cong \underline{\underline{66.3177}}$$

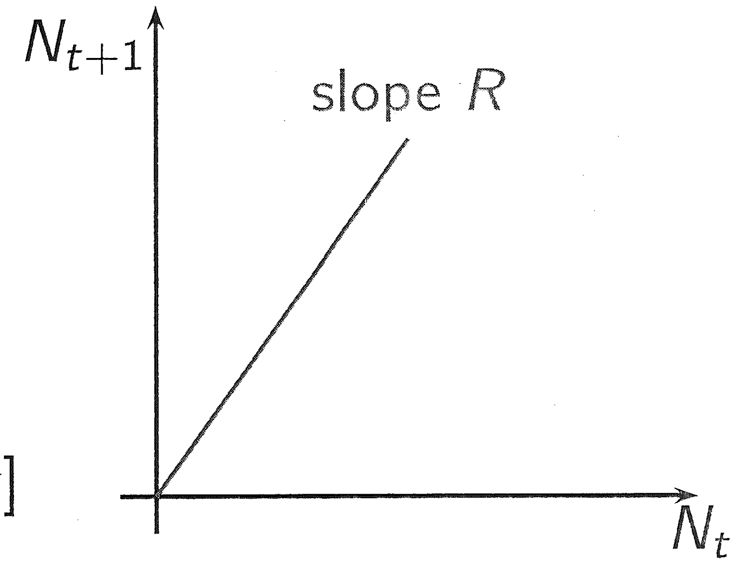
# Visualizing Recursions

We can visualize recursions by plotting $N_t$ on the horizontal axis and $N_{t+1}$ on the vertical axis. Since $N_t \geq 0$ for biological reasons, we restrict the graph to the first quadrant.

The exponential growth recursion

$$N_{t+1} = RN_t$$

is then a straight line through the origin with slope $R$.
[i.e., $N_{t+1} = f(N_t)$, where $f(x) = Rx$]




For any current population size $N_t$, the graph allows us to find the population size in the next time step, namely, $N_{t+1}$.

Unless we label the points according to the corresponding $t$-value, we would not be able to tell at what time a point $(N_t, N_{t+1})$ was realized. We say that **time is implicit in this graph**.

The hallmark of exponential growth is that the ratio of successive population sizes, $N_t/N_{t+1}$, is constant. More precisely, it follows from $N_{t+1} = RN_t$ that

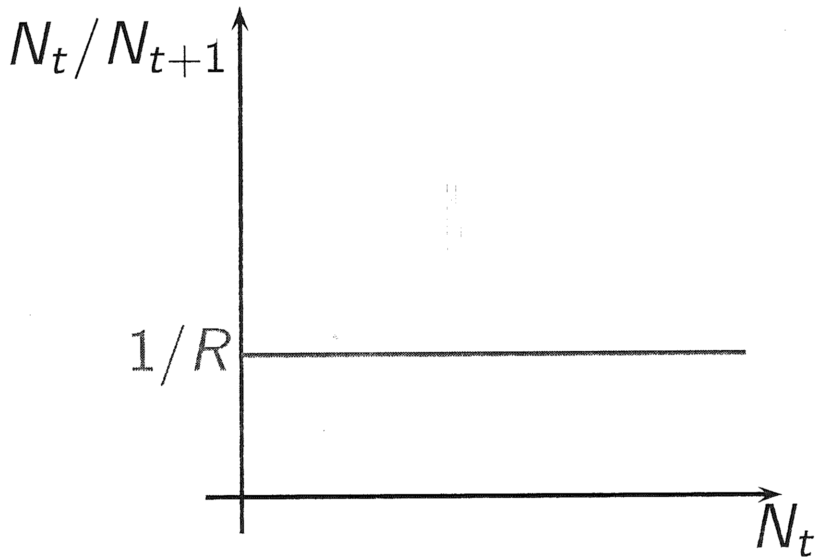
$$\frac{N_t}{N_{t+1}} = \frac{1}{R}$$

If the population consists of annual plants, we can interpret the ratio $N_t/N_{t+1}$ as the **parent-offspring ratio**.

If this ratio is constant, parents produce the same number of offspring, regardless of the current population density. Such growth is called **density independent**.

When $R > 1$, the parent-offspring ratio, is less than 1, implying that the number of offspring exceeds the number of parents. This model yields then an ever-increasing population size. It eventually becomes **biologically unrealistic**, since any population will sooner or later experience food or habitat limitations that will limit its growth.

Below is the graph of the parent-offspring ratio $N_t/N_{t+1}$ as a function of $N_t$ when $N_t > 0$.



# MA 137 – Calculus 1 with Life Science Applications

# **Discrete-Time Models**

## Sequences and Difference Equations: **Limits**

## (Section 2.2)

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# Long-Term Behavior

When studying populations over time, we are often interested in their long-term behavior.

Specifically, if $N_t$ is the population size at time $t$, $t = 0, 1, 2, \ldots$, we want to know how $N_t$ behaves as $t$ increases, or, more precisely, as $t$ tends to infinity.

Using our general setup and notation, we want to know the behavior of $a_n$ as $n$ tends to infinity and use the shorthand notation

$$\lim_{n \longrightarrow \infty} a_n$$

which we read as 'the limit of $a_n$ as $n$ tends to infinity.'

# Definition and Notation

**Definition (Informal)**

We say that the limit as $n$ tends to infinity of a sequence $a_n$ is a number $L$, written as $\lim_{n \longrightarrow \infty} a_n = L$, if we can make the terms $a_n$ as close to $L$ as we like by taking $n$ sufficiently large.

**Definition (Formal)**

The sequence $\{a_n\}$ has a limit $L$, written as $\lim_{n \longrightarrow \infty} a_n = L$, if, for any given any number $d > 0$, there is an integer $N$ so that

$$|a_n - L| < d$$

whenever $n > N$.

If the limit exists, the sequence **converges** (or is **convergent**).

Otherwise we say that the sequence **diverges** (or is **divergent**).

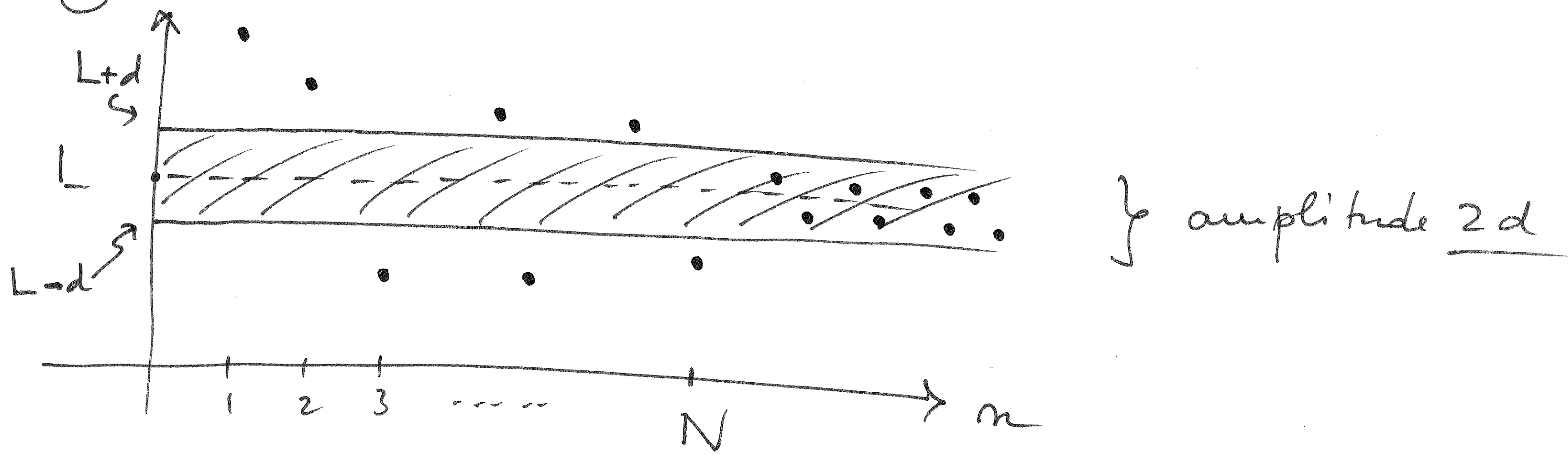
The informal definition of limit says that we can make the terms $a_n$ as close to the limit $L$ as we like.

The formal definition says that for any given number $d > 0$ there exists an integer $N$ so that $|a_n - L| < d$ whenever $n > N$.

If we rework it out we have

$$|a_n - L| < d \iff -d < a_n - L < d \iff \boxed{L - d < a_n < L + d}$$

geometrically, this means that if we plot the graph of the sequence in the Cartesian plane we have the following situation:




any number d defines a strip in the plane about the line L of amplitude 2d.

The points $(n, a_n)$ are perhaps not in that strip for $n \leq N$ ... however for $n > N$ all the points $(n, a_n)$ are in the strip.

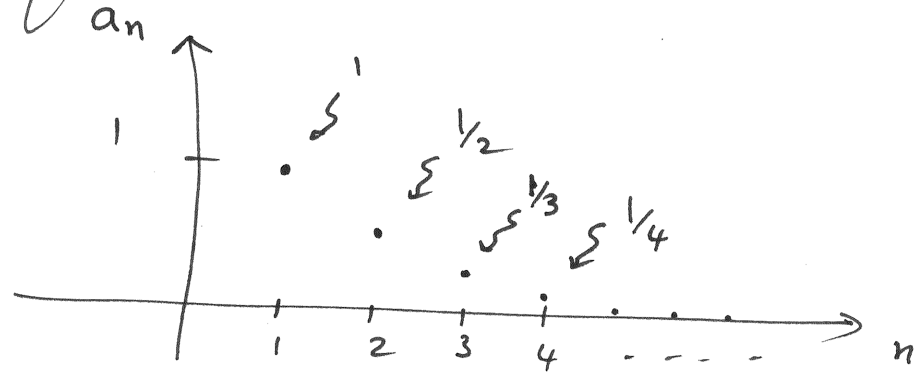
If we make "d" smaller, i.e. the strip is smaller, we can choose N larger.

# Example 1:

Let $a_n = \dfrac{1}{n}$ for $n = 1, 2, 3, \ldots$.

Show that $\displaystyle\lim_{n \longrightarrow \infty} \frac{1}{n} = 0$

Intuitively, the $\lim_{n \to \infty} \frac{1}{n}$ is equal to 0 because if we plot the points corresponding to this sequence in the cartesian plane we have




those points get closer and closer to the n-axis.

Formally, for any $d > 0$ we need to find $N$ such that $|a_n - L| < d$ whenever $n > N$.

But: $\left|\frac{1}{n} - 0\right| < d \iff \frac{1}{n} < d$ (as $n > 0$) $\iff \frac{1}{d} < n$. So choose $\boxed{N = \frac{1}{d}}$.

# Example 2:

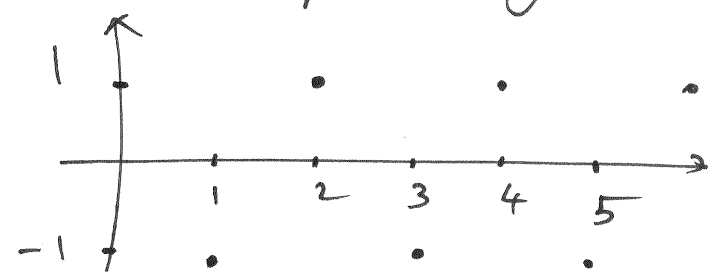
Let $a_n = (-1)^n$ for $n = 0, 1, 2, \ldots$.

Show that $\lim_{n \longrightarrow \infty} (-1)^n$ does not exist.

What about the limit of the sequence $b_n = \cos(\pi n)$ ?

$$\lim_{n \to \infty} (-1)^n = \text{does not exist}$$

If we plot the points corresponding to this sequence we get




This means that for consecutive values of the index, say $n$ and $n+1$ the difference $a_n - a_{n+1}$ is in absolute value always 2 ... even if $n$ goes to infinity. The do not get closer to a common value.

# Limit Laws

The operations of arithmetic, namely, addition, subtraction, multiplication, and division, all behave reasonably with respect to the idea of getting closer to as long as nothing illegal happens.

This is summarized by the following laws:

If $\lim\limits_{n\longrightarrow\infty} a_n$ and $\lim\limits_{n\longrightarrow\infty} b_n$ exist and $c$ is a constant, then

1. $\lim\limits_{n\longrightarrow\infty}(a_n + b_n) = (\lim\limits_{n\longrightarrow\infty} a_n) + (\lim\limits_{n\longrightarrow\infty} b_n)$
2. $\lim\limits_{n\longrightarrow\infty}(c\, a_n) = c(\lim\limits_{n\longrightarrow\infty} a_n)$
3. $\lim\limits_{n\longrightarrow\infty}(a_n b_n) = (\lim\limits_{n\longrightarrow\infty} a_n)(\lim\limits_{n\longrightarrow\infty} b_n)$
4. $\lim\limits_{n\longrightarrow\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\longrightarrow\infty} a_n}{\lim\limits_{n\longrightarrow\infty} b_n}$, provided $\lim\limits_{n\longrightarrow\infty} b_n \neq 0$

# Example 3:

Find $\lim_{n \longrightarrow \infty} \frac{n(1-3n^2)}{n^3+1}$.

Find $\lim_{n \longrightarrow \infty} \frac{n}{n^2+1}$.

(a) $\lim_{n\to\infty} \frac{n(1-3n^2)}{n^3+1}$ = using the limit laws

$$= \frac{\left(\lim_{n\to\infty} n\right)\left(\lim_{n\to\infty} 1-3n^2\right)}{\lim_{n\to\infty} (n^3+1)} = \text{etc....}$$

$$= \frac{\infty(-\infty)}{\infty} = \text{which is not defined.}$$

<u>However</u>, notice that $\lim_{n\to\infty} \frac{1}{n} = 0$

$$\Rightarrow \boxed{\lim_{n\to\infty} \frac{1}{n^p} = 0} \quad \text{for any } p>1$$

Thus we can rewrite our original limit as

$$\lim_{n\to\infty} \frac{n(1-3n^2)}{n^3+1} = \lim_{n\to\infty} \frac{n-3n^3}{n^3+1} =$$

$$= \lim_{n\to\infty} \frac{(n-3n^3)\cdot\frac{1}{n^3}}{(n^3+1)\cdot\frac{1}{n^3}} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^2}-3\right)}{\left(1+\frac{1}{n^3}\right)}$$

use now the properties of limits:

$$= \frac{\lim_{n\to\infty}\left(\frac{1}{n^2}-3\right)}{\lim_{n\to\infty}\left(1+\frac{1}{n^3}\right)} = \frac{\left[\lim_{n\to\infty}\left(\frac{1}{n^2}\right)\right]-\left[\lim_{n\to\infty} 3\right]}{\left[\lim_{n\to\infty} 1\right]+\left[\lim_{n\to\infty}\frac{1}{n^3}\right]}$$

$$= \frac{0-3}{1+0} = \frac{-3}{1} = \boxed{-3}$$

(b) $$\lim_{n\to\infty} \frac{n}{n^2+1} = \frac{\lim_{n\to\infty} n}{\lim_{n\to\infty}(n^2+1)} = \frac{+\infty}{+\infty}$$

However we can rewrite this limit as:

$$\lim_{n\to\infty} \frac{(n)\frac{1}{n^2}}{(n^2+1)\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}}$$

$$= \frac{\lim_{n\to\infty} \frac{1}{n}}{1+\lim_{n\to\infty}\frac{1}{n^2}} = \frac{0}{1+0} = \frac{0}{1} = \boxed{0}$$

Can you see a <u>general</u> rule?

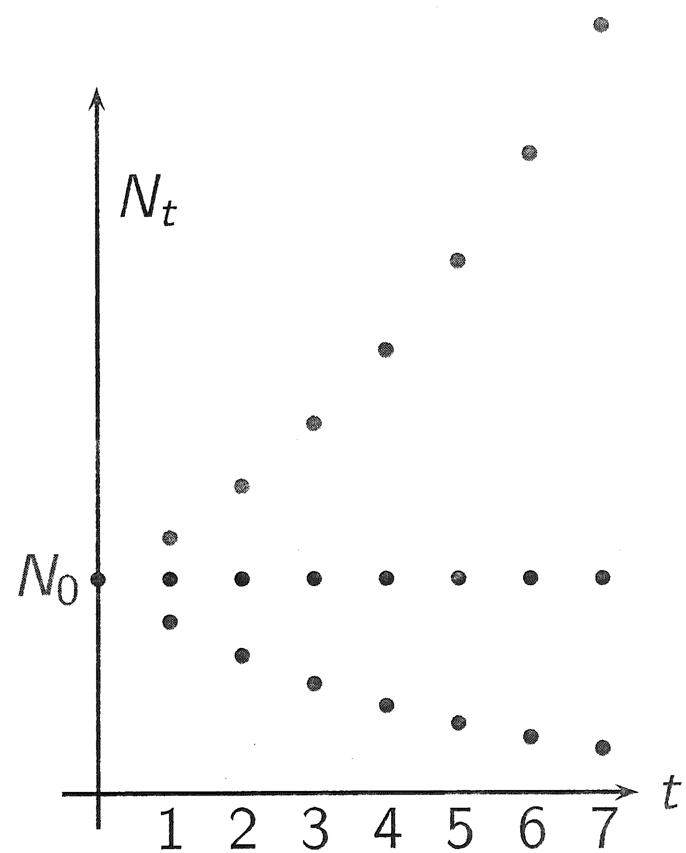
# Example 4:

For $R > 0$, we know that exponential growth is given by

$$N_t = N_0 R^n \qquad n = 0, 1, 2, \ldots$$

The figure below indicates that

$$\lim_{n \longrightarrow \infty} N_t = \begin{cases} 0 & \text{if } 0 < R < 1 \\ N_0 & \text{if } R = 1 \\ \infty & \text{if } R > 1 \end{cases}$$



# Example 5:

Find $\displaystyle \lim_{n \longrightarrow \infty} \frac{3 \cdot 4^n + 1}{4^n}$

$$\lim_{n\to\infty} \frac{3\cdot 4^n+1}{4^n} = \frac{\lim_{n\to\infty}(3\cdot 4^n+1)}{\lim_{n\to\infty} 4^n} = \frac{+\infty}{+\infty}$$

however we can rewrite the above limit as:

$$\lim_{n\to\infty}\left[\frac{3\cdot 4^n}{4^n} + \frac{1}{4^n}\right] = \lim_{n\to\infty}\left[3 + \left(\frac{1}{4}\right)^n\right]$$

$$= \left[\lim_{n\to\infty} 3\right] + \left[\lim_{n\to\infty}\left[\frac{1}{4}\right]^n\right] = 3 + 0$$

($\lim_{n\to\infty}\left[\frac{1}{4}\right]^n \to 0$ as $R = \frac{1}{4}$)

$$= \boxed{3}$$

# Squeeze (Sandwich) Theorem for Sequences

Sometimes the limit of a sequence can be difficult to calculate and we need to employ some other techniques. One of those techniques is to use the Squeeze (Sandwich) Theorem for Sequences.

## Squeeze (Sandwich) Theorem for Sequences

Consider three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ and suppose there exists an integer $N$ such that

$$a_n \leq b_n \leq c_n \qquad \text{for all} \quad n > N.$$

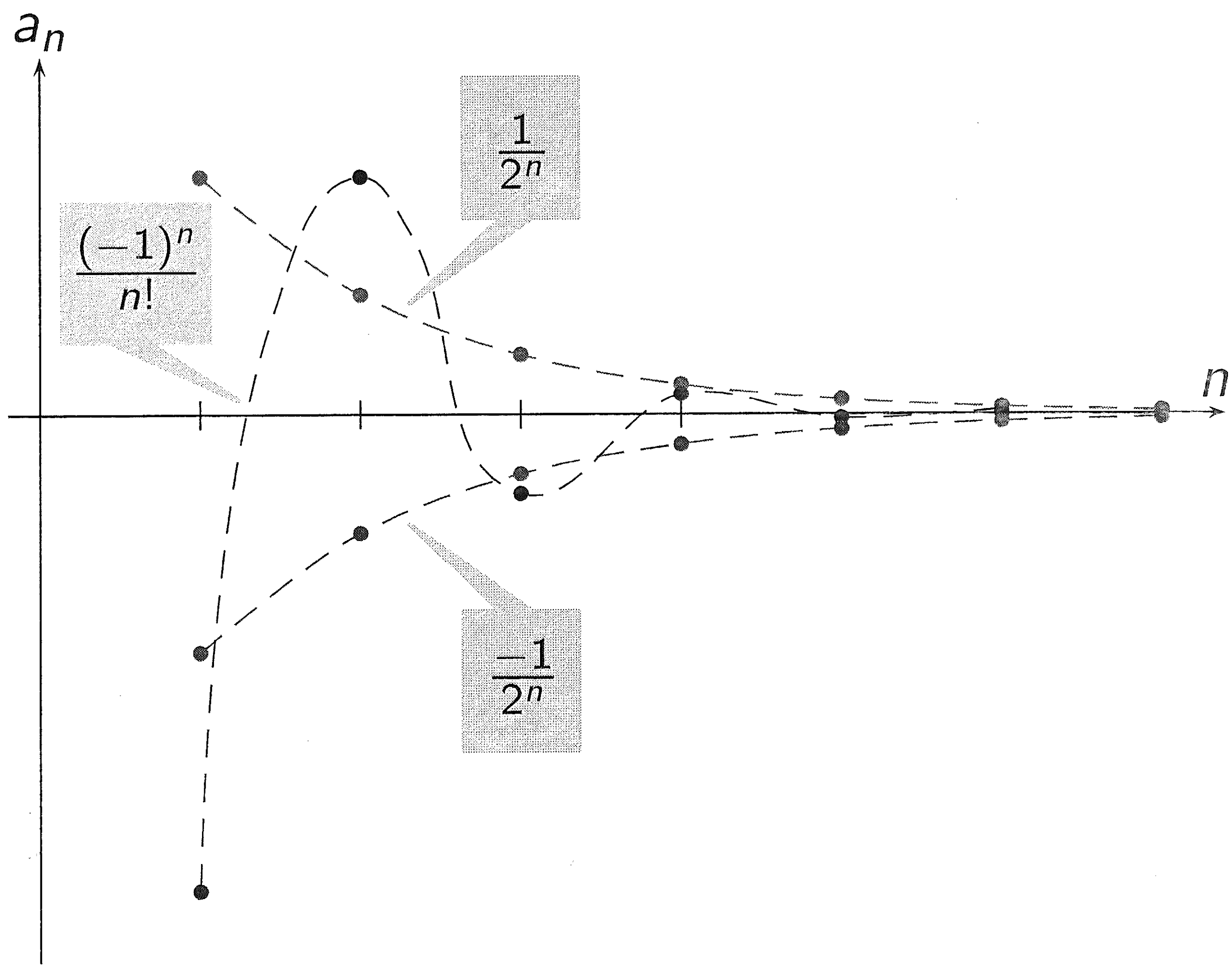
If $\lim_{n \longrightarrow \infty} a_n = L = \lim_{n \longrightarrow \infty} c_n$ then $\lim_{n \longrightarrow \infty} b_n = L$.

The values in the following table and the graph on the left

| $n$ | $1/n!$ | $1/2^n$ |
|---|---|---|
| 1 | 1 | 0.5 |
| 2 | 0.5 | 0.25 |
| 3 | $0.1\overline{6}$ | 0.125 |
| 4 | $0.041\overline{6}$ | 0.0625 |
| 5 | $0.008\overline{3}$ | 0.03125 |
| 6 | $0.0013\overline{8}$ | 0.015625 |
| 7 | 0.000198 | 0.0078125 |
| ⋮ | ⋮ | ⋮ |

suggest that for $n \geq 4$ we have

$$\frac{-1}{2^n} \leq \frac{(-1)^n}{n!} \leq \frac{1}{2^n} \qquad n \geq 4.$$




So by the Squeeze Theorem it follows that

$$\lim_{n \longrightarrow \infty} (-1)^n \frac{1}{n!} = 0.$$

# Example 6:

Find $$\lim_{n \longrightarrow \infty} \frac{2n + (-1)^n}{n}$$

$$b_n = \frac{2n + (-1)^n}{n} = 2 + \frac{(-1)^n}{n}$$

Observe that $-1 \le (-1)^n \le 1$ for every $n$

Thus

$$\boxed{a_n = 2 - \frac{1}{n}} \le \underbrace{2 + \frac{(-1)^n}{n}}_{b_n} \le \boxed{2 + \frac{1}{n} = c_n}$$

and $\lim_{n \to \infty} \left[2 - \frac{1}{n}\right] = 2 = \lim_{n \to \infty} \left[2 + \frac{1}{n}\right]$

so that $\boxed{\lim_{n \to \infty} \frac{2n + (-1)^n}{n} = 2}$

# Example 7:

Find $\displaystyle \lim_{n \longrightarrow \infty} \frac{5^n}{n!}$

Observe that

$$0 \leq \frac{5^n}{n!} = \frac{\overbrace{5 \cdot 5 \cdot 5 \cdot 5 \cdots 5 \cdot 5}^{n \text{ times}}}{n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1}$$

we can regroup those terms as

$$\underbrace{\left[\frac{5}{n} \cdot \frac{5}{n-1} \cdot \frac{5}{n-2} \cdots \frac{5}{6}\right]} \cdot \frac{5}{5} \cdot \frac{5}{4} \cdot \frac{5}{3} \cdot \frac{5}{2} \cdot 5$$

$$\leq \left(\frac{5}{6}\right)^{n-5} \cdot \frac{625}{24}$$

In other words: $0 \leq \frac{5^n}{n!} \leq \left(\frac{5}{6}\right)^{n-5} \cdot \frac{625}{24}$

But $\lim_{n \to 0} 0 = 0 = \lim_{n \to \infty} \left(5/6\right)^{n-5} \cdot \frac{625}{24}$ as $5/6 < 1$

So $\boxed{\lim_{n \to \infty} \frac{5^n}{n!} = 0}$

# MA 137 – Calculus 1 with Life Science Applications
# **Discrete-Time Models**
## Sequences and Difference Equations: **Limits**
## (Section 2.2)

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# Limits of Recursive Sequences

We now discuss how to find the limit when $a_n$ is defined by a recursive sequence of the first order

$$a_{n+1} = f(a_n)$$

Finding an explicit expression for $a_n$ is often not a feasible strategy, because solving recursions can be very difficult or even impossible.

How, then, can we say anything about the limiting behavior of a recursively defined sequence?

The following procedure will allow us to identify **candidates** for limits.

# Fixed Points (or Equilibria)

**Definition**

A **fixed point** (or **equilibrium**) of a recursive sequence

$$a_{n+1} = f(a_n)$$

is a number $\widehat{a}$ that is left unchanged by the (updating function) $g$, that is,

$$\widehat{a} = f(\widehat{a})$$

**Remark:**

A fixed point is only a candidate for a limit; a sequence does not have to converge to a given fixed point (unless $a_0$ is already equal to the fixed point).

# Example 1:

Let $a_{n+1} = 1 + \dfrac{1}{a_n}$. Find the fixed points of this recursion, and investigate the limiting behavior of $a_n$ when $a_1 = 1$.

Consider the recursive sequence $a_{n+1} = 1 + \frac{1}{a_n}$

(Notice that $a_{n+1} = f(a_n)$ where $f(x) = 1 + \frac{1}{x}$ )

To find the fixed points we need to solve for a in:

$$a = 1 + \frac{1}{a}$$

Multiply both sides by a : $a^2 = a\left(1 + \frac{1}{a}\right)$

$$\Longleftrightarrow \quad a^2 = a + 1 \quad \Longleftrightarrow \quad a^2 - a - 1 = 0$$

and use now the quadratic formula:

$$a_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} \begin{cases} \frac{1+\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \end{cases}$$

Thus there are two fixed points:

$$\hat{a}_1 = \frac{1+\sqrt{5}}{2} \cong 1.618 \qquad \hat{a}_2 = \frac{1-\sqrt{5}}{2} \cong -0.618$$

GOLDEN RATIO

Let's investigate $\lim_{n \to \infty} a_n$.

We already worked out a few terms of this sequence in an earlier lecture:

$a_{n+1} = 1 + \frac{1}{a_n}$

$a_1 = 1$

$a_2 = 1 + \frac{1}{a_1} = 1 + 1 = 2$

$a_3 = 1 + \frac{1}{a_2} = 1 + \frac{1}{2} = 3/2 = 1.5$

$a_4 = 1 + \frac{1}{a_3} = 1 + \frac{1}{3/2} = 1 + \frac{2}{3} = \frac{5}{3} = 1.6\overline{7}$

$a_5 = 1 + \frac{1}{a_4} = 1 + \frac{1}{5/3} = 1 + \frac{3}{5} = 8/5 = 1.6$

$a_6 = 1 + 1/a_5 = 1 + \frac{1}{8/5} = 1 + \frac{5}{8} = \frac{13}{8} = 1.625$

We realize that the n-th term of the sequence $a_n$ is the quotient of two consecutive Fibonacci's numbers (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...)

From the first few terms of the sequence we have worked out $a_1, a_2, \ldots, a_6$ it seems obvious that $\lim_{n \to \infty} a_n = \hat{a}_1 = \frac{1+\sqrt{5}}{2} \cong 1.618$

Aside, it takes quite some work and some mathematical skill to prove that there exists an explicit form of the Fibonacci's numbers

Namely:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for $n = 1, 2, 3, \ldots\ldots$

# Example 2:

Let $a_{n+1} = \sqrt{3a_n}$. Find the fixed points of this recursion, and investigate the limiting behavior of $a_n$ when $a_0 = 1$.

$a_{n+1} = \sqrt{3a_n}$

(notice than $a_{n+1} = f(a_n)$ when $f(x) = \sqrt{3x}$ )

To find the fixed points we have to solve

$a = \sqrt{3a} \iff a^2 = (\sqrt{3a})^2$ [i.e. we squared both sides]

$\iff a^2 = 3a \iff a^2 - 3a = 0$

$\iff a(a-3) = 0 \iff$ $\boxed{\hat{a}_1 = 0 \qquad \hat{a}_2 = 3}$

fixed points

We want to investigate $\lim_{n \to \infty} a_n$ with $a_0 = 1$

Then

$a_0 = 1$ ; $a_1 = \sqrt{3a_0} = \sqrt{3} \cong 1.732$ ;

$a_2 = \sqrt{3a_1} \cong 2.279$ ; $a_3 = \sqrt{3a_2} \cong 2.615$

$a_4 = \sqrt{3a_3} \cong 2.8$ ; $a_5 = \sqrt{3a_4} \cong 2.898$

$$a_6 = \sqrt{3 a_5} \cong 2.949 \quad ; \quad \text{etc.} \ldots$$

Hence all these calculations seem to suggest that

$$\boxed{\lim_{n \to \infty} a_n = 3}$$

that is the limit is the fixed point $\hat{a}_2 = 3$.

# Example 3:

Let $a_{n+1} = \dfrac{3}{a_n}$. Find the fixed points of this recursion, and investigate the limiting behavior of $a_n$ when $a_0$ is not equal to a fixed point.

$$a_{n+1} = \frac{3}{a_n}$$

[that is $a_{n+1} = f(a_n)$ with $f(x) = \frac{3}{x}$ ]

Fixed points: we need to solve the equation

$$a = \frac{3}{a} \iff a^2 = 3 \iff a = \pm\sqrt{3}$$

Thus there are two fixed points: $\boxed{\hat{a}_1 = \sqrt{3}\ ;\ \hat{a}_2 = -\sqrt{3}}$

(1) Suppose that $a_0 = \sqrt{3} \implies a_1 = \frac{3}{a_0} = \frac{3}{\sqrt{3}} = \sqrt{3}$

$a_2 = \frac{3}{a_1} = \frac{3}{\sqrt{3}} = \sqrt{3} \implies$ hence $a_n = \sqrt{3}$ for all $n$.

(2) Similarly if we start with $a_0 = -\sqrt{3}$ we get that $a_1 = \frac{3}{a_0} = \frac{3}{-\sqrt{3}} = -\sqrt{3}$ ; $a_2 = \frac{3}{a_1} = \frac{3}{-\sqrt{3}} = -\sqrt{3}$

i.e. $a_n = -\sqrt{3}$ for all $n$.

(3) However, let's start for example with $a_0 = 2$. We have $a_1 = \frac{3}{a_0} = \frac{3}{2} = 1.5$

$$a_2 = \frac{3}{a_1} = \frac{3}{3/2} = 2 \quad ; \quad a_3 = \frac{3}{a_2} = 3/2 \quad ; \quad \ldots$$

Hence we conclude that even if we started close to the fixed point $\boxed{\hat{a}_1 = \sqrt{3}}$, i.e. we picked $a_0 = 2$. We got

$$a_0 = a_2 = a_4 = a_6 = a_8 = \cdots = 2$$

$$a_1 = a_3 = a_5 = a_7 = a_9 = \cdots = 3/2$$

Hence $\lim_{n \to \infty} a_n =$ does not exist

# Comments

The previous examples illustrate that fixed points are only candidates for limits and that, depending on the initial condition, the sequence $\{a_n\}$ may or may not converge to a given fixed point.

If we know, however, that a sequence $\{a_n\}$ does converge, then the limit of the sequence must be one of the fixed points.

For this reason we say that a fixed point (or equilibrium) is **stable** if sequences that begin close to the fixed point approach that fixed point. It is called **unstable** if sequences that start close to the equilibrium move away from it.

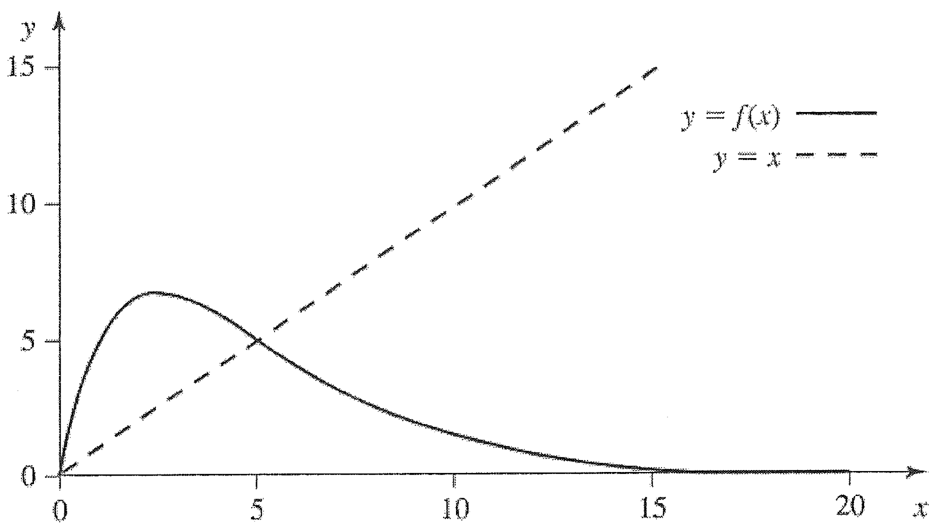
We will return to the relationship between fixed points and limits in Section 5.6, where we will learn methods that allow us to determine whether a sequence converges to a particular fixed point.

# A Graphical Way to Find Fixed Points

There is a graphical method for finding fixed points, which we mention briefly below.

Given a recursion of the form $a_{n+1} = f(a_n)$, then we know that a fixed point $\widehat{a}$ satisfies $\widehat{a} = f(\widehat{a})$.

This suggests that if we graph $y = f(x)$ and $y = x$ in the same coordinate system, then fixed points are located where the two graphs intersect, as shown in the picture below



# Example 4:

**(a)** Consider the sequence recursively defined by the relation

$$a_{n+1} = 2a_n(1 - a_n) \qquad a_0 = 0$$

and assume that $\lim_{n \to \infty} a_n$ exists.
Find all fixed points of $\{a_n\}$, and use a table or other reasoning to guess which fixed point is the limiting value for the given initial condition.

**(b)** Same as in **(a)** but with $a_0 = 0.1$.

Notice that $a_{n+1} = 2a_n(1-a_n)$ is of the form

$a_{n+1} = f(a_n)$ where $f(x) = 2x(1-x)$

this is a parabola with downward concavity

To find the fixed points we need to solve

$$a = 2a(1-a) \iff a = 0 \text{ or }$$

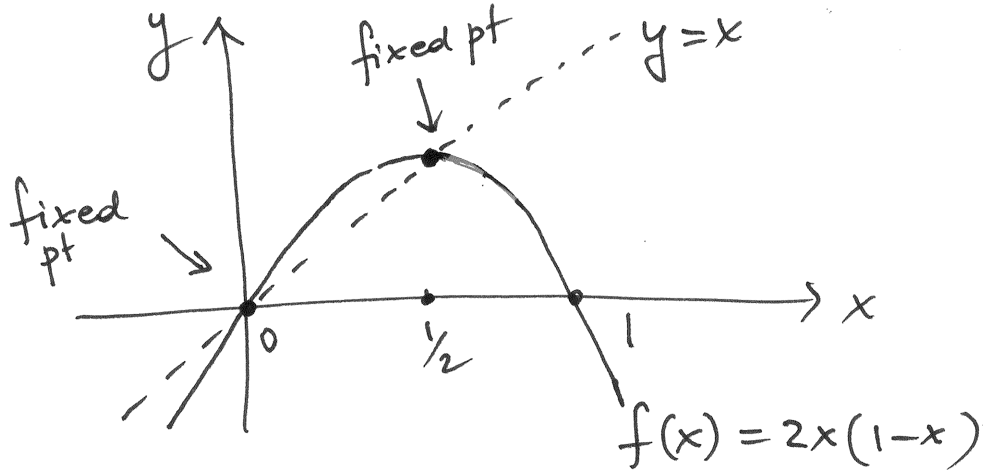
$$1 = 2(1-a) \iff \frac{1}{2} = 1-a \iff a = 1 - \frac{1}{2} = \frac{1}{2}$$

Thus the fixed points are:

$\boxed{\hat{a}_1 = 0}$ or $\boxed{\hat{a}_2 = \frac{1}{2}}$

Notice the the fixed points are geometrically given by the intersection points between

$$y = f(x) = 2x(1-x) \quad \text{and} \quad y = x$$




**About** : $\lim_{n \to \infty} a_n$

(1) if $a_0 = 0$ ; $a_1 = 2a_0(1-a_0) = 0$ ;

$a_2 = 2a_1(1-a_1) = 0$ etc....

so $$\boxed{\lim_{n \to \infty} a_n = 0}$$

(2) Let's consider the case $a_0 = 0.1$

That is we start from a point that is <u>very</u> close to the equilibrium/fixed point $0$.

$$a_0 = 0.1$$

$$a_1 = 2a_0(1-a_0) = 2\cdot(0.1)\cdot(0.9) = 0.18$$

$$a_2 = 2a_1(1-a_1) = 2(0.18)(0.82) = 0.2952$$

$$a_3 = 2a_2(1-a_2) = 2(0.2952)(0.7048) = 0.4161$$

$$a_4 = \dots = 0.486$$

Hence these values suggest $\boxed{\lim_{n\to\infty} a_n = 0.5}$

despite the fact that we started very close to $0$.

# MA137 – Calculus 1 with Life Science Applications
## Discrete-Time Models
## **More Population Models**
## (Section 2.3)

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September 19, 2016

## From: Simple Mathematical Models with Very Complicated Dynamics

*"First-order difference equations arise in many contexts in the biological, economic and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behaviour, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random fluctuations. There are consequently many fascinating problems, some concerned with delicate mathematical aspects of the fine structure of the trajectories, and some concerned with practical implications and applications."*

**Robert M. May**, Nature (1976)

# When are discrete time models appropriate?

- when studying seasonally breeding populations with non-overlapping generations where the population size at one generation depends on the population size of the previous generation. (Many insects and plants reproduce at specific time intervals or times of the year.)
- when studying populations censused at intervals. (These are the so-called metered models.)

The exponential (Malthusian) growth model described earlier fits into this category: $N_{t+1} = RN_t$.

We denote the population size at time $t$ by $N_t$, $t = 0, 1, 2, \ldots$ To model how the population size at generation $t+1$ is related to the population size at generation $t$, we write $N_{t+1} = f(N_t)$, where the function $f$ (updating function) describes the density dependence of the population dynamics.

A recursion of the form given before is called a first-order recursion because, to obtain the population size at time $t+1$, only the population size at the previous time step $t$ needs to be known.

A recursion is also called a difference equation or an iterated map.

The name **difference equation** comes from writing the dynamics in the form $N_{t+1} - N_t = g(N_t)$, which allows us to track population size changes from one time step to the next.

The name **iterated map** refers to the recursive definition.

When we study population models, we are frequently interested in asking questions about the long-term behavior of the population:

Will the population size reach a constant value?
Will it oscillate predictably?
Will it fluctuate widely without any recognizable patterns?

In the three examples that follow

- Beverton-Holt Recruitment Model,
- Discrete Logistic Equation,
- Ricker Logistic Equation,

we will see that discrete-time population models show very rich and complex behavior.

Earlier, we discussed the exponential growth model defined by the recursion $N_{t+1} = RN_t$ with $N_0 =$ population size at time 0.

When $R > 1$, the population size will grow indefinitely, if $N_0 > 0$.

Such growth, called **density-independent growth**, is biologically unrealistic. As the size of the population increases, individuals will start to compete with each other for resources, such as food or nesting sites, thereby reducing population growth.

We call population growth that depends on population density **density-dependent growth**.

# The Beverton-Holt Recruitment Model

To find a model that incorporates a reduction in growth when the population size gets large, we start with the ratio of successive population sizes in the exponential growth model and assume $N_0 > 0$:

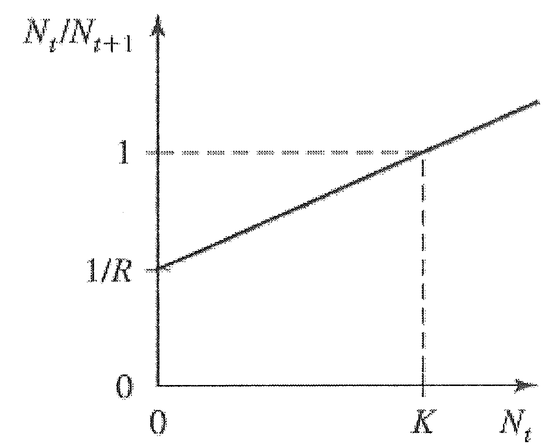
$$\frac{N_t}{N_{t+1}} = \frac{1}{R}.$$

The ratio $N_t/N_{t+1}$ is a constant. If we graphed this ratio as a function of the current population size $N_t$, we would obtain a horizontal line in a coordinate system in which $N_t$ is on the horizontal axis and the ratio $N_t/N_{t+1}$ is on the vertical axis.

Note that as long as the parent-offspring ratio $N_t/N_{t+1}$ is less than 1, the population size increases, since there are fewer parents than offspring. Once the ratio is equal to 1, the population size stays the same from one time step to the next. When the ratio is greater than 1, the population size decreases.

To model the reduction in growth when the population size gets larger, we drop the assumption that the parent-offspring ratio $N_t/N_{t+1}$ is constant and assume instead that the ratio is an increasing function of the population size $N_t$. That is, we replace the constant $1/R$ by a function that increases with $N_t$. The simplest such function is linear.

$$\frac{N_t}{N_{t+1}} = \frac{1}{R} + \frac{1 - \frac{1}{R}}{K} N_t$$




The population density where the parent-offspring ratio is equal to 1 is of particular importance, since it corresponds to the population size, which does not change from one generation to the next.
We call this population size the carrying capacity and denote it by $K$, where $K$ is a positive constant.

If we solve for $N_{t+1}$ we obtain

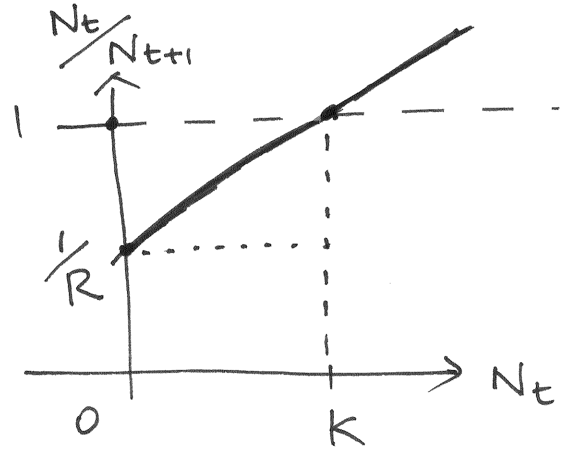
$$N_{t+1} = \frac{RN_t}{1 + \frac{R-1}{K} N_t}$$

This recursion is known as the Beverton-Holt recruitment curve.

We have two fixed points when $R > 1$: the fixed point $\widehat{N} = 0$, which we call trivial, since it corresponds to the absence of the population, and the fixed point $\widehat{N} = K$, which we call nontrivial, since it corresponds to a positive population size.

One can show that, when $K > 0$, $R > 1$, and $N_0 > 0$, we have that

$$\lim_{t \longrightarrow \infty} N_t = K.$$




the slope of the line is

$$\frac{\text{rise}}{\text{run}} = \frac{1 - 1/R}{K}$$

Hence the Beverton – Holt model is given by

$$\frac{N_t}{N_{t+1}} = \frac{1}{R} + \underbrace{\frac{1 - 1/R}{K}}_{\text{slope}} N_t$$

Solve for $N_{t+1}$ :

$$\frac{N_t}{1/R + \frac{1 - 1/R}{K} N_t} = N_{t+1} \qquad \underline{\text{OR}} \qquad N_{t+1} = \frac{N_t}{\frac{1}{R} + \frac{R-1}{RK} N_t}$$

(multiply top and bottom by R and get:)

$$\boxed{N_{t+1} = \frac{R N_t}{1 + \frac{R-1}{K} N_t}}$$

To find the fixed points of $N_{t+1} = \dfrac{R N_t}{1 + \frac{R-1}{K} N_t}$

We need to solve:

$$N = \frac{RN}{1 + \frac{R-1}{K} N}$$

one solution is clearly $\boxed{N = 0}$

If we simplify by $N$ on both sides we get

$$1 = \frac{R}{1 + \frac{R-1}{K} N} \iff 1 + \frac{R-1}{K} N = R$$

$$\iff \frac{R-1}{K} N = R - 1 \iff \text{(simplify } R-1\text{)}$$

$$\frac{N}{K} = 1 \iff \boxed{N = K}$$

So the fixed points are: $\hat{N}_1 = 0$ and $\hat{N}_2 = K$

# Possible Gen-Ed Project?

The Beverton-Holt stock recruitment model (1957) was used, originally, in fishery models. It is a special case (with $b = 1$) of the following more general model: the Hassell equation.

The Hassell equation (1975) takes into account intraspecific competition, more specifically scramble competition[1], and takes the form

$$N_{t+1} = \frac{R_0 N_t}{(1 + kN_t)^b}.$$

We have under-compensation for $0 < b < 1$;
we have exact compensation for $b = 1$;
we have over-compensation for $1 < b$.

[1] In ecology, scramble competition refers to a situation in which a resource is accessible to all competitors.

# The Discrete Logistic Equation

The most popular discrete-time single-species model is the discrete logistic equation, whose recursion is given by

$$N_{t+1} = N_t \left[ 1 + R \left( 1 - \frac{N_t}{K} \right) \right]$$

where $R$ and $K$ are positive constants. $R$ is called the **growth parameter** and $K$ is called the **carrying capacity**.

This model of population growth exhibits very complicated dynamics, described in an influential review paper by **Robert May** (1976).

We first rewrite the model in what is called the **canonical form**

$$x_{t+1} = r\, x_t (1 - x_t)$$

where $r = 1 + R$ and $x_t = \dfrac{R}{K(1+R)} N_t$.

# Advantages

The advantage of this canonical form is threefold:

(1) The recursion $x_{t+1} = r\,x_t(1 - x_t)$ is simpler;

(2) instead of two parameters, $R$ and $K$, there is just one, $r$;

(3) the quantity $x_t = \dfrac{R}{K(1+R)} N_t$ is dimensionless.

What does dimensionless mean? The original variable $N_t$ has units (or dimension) of number of individuals; the parameter $K$ has the same units. Dividing $N_t$ by $K$, we see that the units cancel and we say that the quantity $x_t$ is dimensionless. The parameter $R$ does not have a dimension, so multiplying $N_t/K$ by $R/(1+R)$ does not introduce any additional units. A dimensionless variable has the advantage that it has the same numerical value regardless of what the units of measurement are in the original variable.

# Reduction to the Canonical Form

$$\begin{aligned} N_{t+1} &= N_t\left[1+R\left(1-\frac{N_t}{K}\right)\right] \\ &= N_t\left[(1+R)-\frac{R}{K}N_t\right] \\ &= N_t(1+R)\left[1-\frac{R}{K(1+R)}N_t\right] \end{aligned}$$

$$\Longleftrightarrow$$

$$\frac{1}{1+R}N_{t+1} = N_t\left[1-\frac{R}{K(1+R)}N_t\right]$$

$$\Longleftrightarrow$$

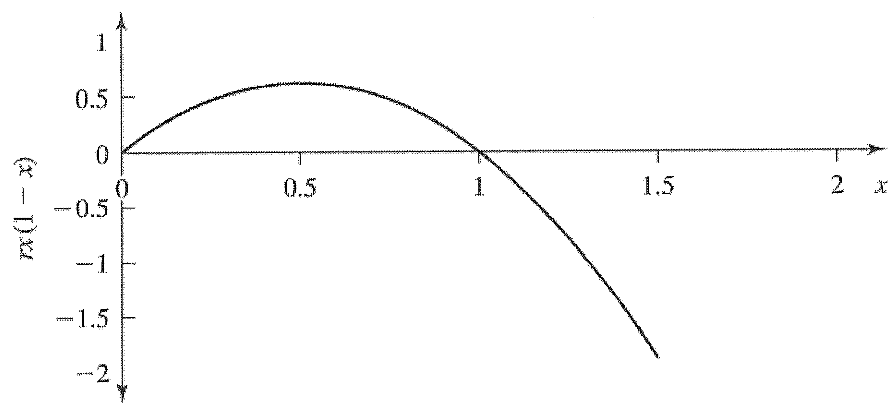
$$\frac{R}{K(1+R)}N_{t+1} = (1+R)\frac{R}{K(1+R)}N_t\left[1-\frac{R}{K(1+R)}N_t\right]$$

# $1 < r < 4$

Notice that we can write $x_{t+1} = r\,x_t(1 - x_t)$ as $x_{t+1} = f(x_t)$, where the function

$$f(x) = r\,x(1 - x)$$

is an upside-down parabola, since $r > 1$.




In order to make sure that $f(x_t) \in (0, 1)$ for all $t$, we also require that $r/4 < 1$, or $r < 4$. In fact, the maximum value of $f(x)$ occurs at $x = 1/2$, and $f(1/2) = r/4$.

Hence we need to impose the assumption that $1 < r < 4$.

# Fixed Points of $x_{t+1} = r\, x_t(1 - x_t)$

We first compute the fixed points of the discrete logistic equation written in standard form.

We need to solve $x = rx(1-x)$.

Solving immediately yields the solution $\widehat{x} = 0$. If $x \neq 0$, we divide both sides by $x$ and find that

$$1 = r(1-x), \qquad \text{or} \qquad \widehat{x} = 1 - \frac{1}{r}.$$

Provided that $r > 1$, both fixed points are in $[0, 1)$.

The fixed point $\widehat{x} = 0$ corresponds to the fixed point $\widehat{N} = 0$, which is why we call $\widehat{x} = 0$ a trivial equilibrium. When $\widehat{x} = 1 - 1/r$ we obtain that $\widehat{N} = K$ is the other fixed point.

$$N_{t+1} = N_t \left[ 1 + R\left(1 - \frac{N_t}{K}\right)\right]$$

$$\Longleftrightarrow \quad x_{t+1} = r\, x_t (1 - x_t) \qquad \text{" canonical form "}$$

where $r = 1 + R$ $\qquad x_t = \frac{R}{K(1+R)} N_t$

The fixed points in the canonical form are given by

$$x = r\,x(1-x) \Longleftrightarrow x = 0 \quad \text{or} \quad 1 = r(1-x)$$

$$\Longleftrightarrow x = 0 \quad \text{or} \quad \frac{1}{r} = 1 - x \Longleftrightarrow$$

$$\Longleftrightarrow \boxed{\hat{x}_1 = 0} \quad \text{or} \quad \boxed{\hat{x}_2 = 1 - \frac{1}{r}}$$

In the original model

$$0 = \hat{x}_1 = \frac{R}{K(1+R)} \hat{N}_1 \Longleftrightarrow \boxed{\hat{N}_1 = 0}$$

$$1 - \frac{1}{r} = \hat{x}_2 = \frac{R}{K(1+R)} \hat{N}_2 \Longleftrightarrow 1 - \frac{1}{r} = \frac{r-1}{K(r)} \hat{N}_2 \Longleftrightarrow \boxed{\hat{N}_2 = K}$$

# Long-term Behavior of $x_{t+1} = r\,x_t(1 - x_t)$

The long-term behavior of the discrete logistic equation is very complicated. We simply list the different cases.

If $1 < r < 3$ and $x_0 \in (0,1)$, $x_t$ converges to the fixed point $1 - 1/r$.

Increasing $r$ to a value between 3 and 3.449..., we see that $x_t$ settles into a cycle of period 2. That is, for $t$ large enough, $x_t$ oscillates back and forth between a larger and a smaller value.

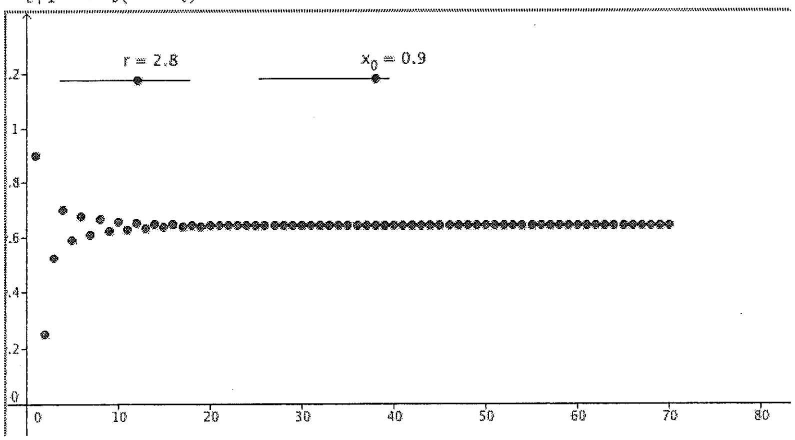
For $r$ between 3.449... and 3.544..., the period doubles: A cycle of period 4 appears for large enough times.

Increasing r continues to double the period: A cycle of period 8 is born when $r = 3.544...$, a cycle of period 16 when $r = 3.564...$, and a cycle of period 32 when $r = 3.567....$

This doubling of the period continues until $r$ reaches a value of about 3.57, when the population pattern becomes **chaotic**.

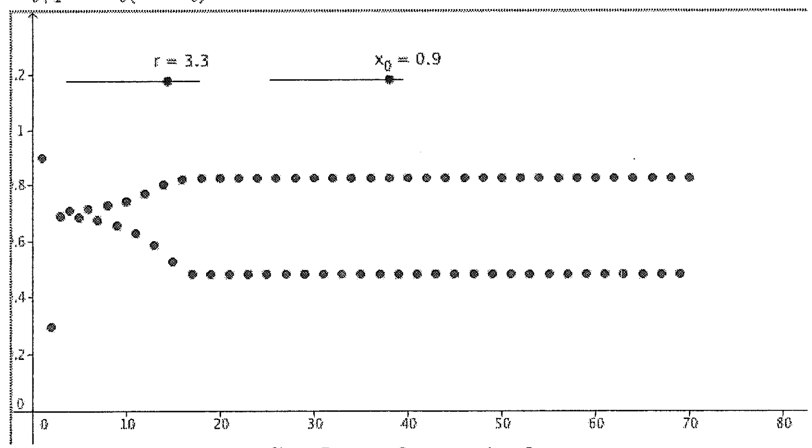
# Illustrations Using Applets built with GeoGebra

$x_0$ = initial population
$x_{t+1} = rx_t(1-x_t)$



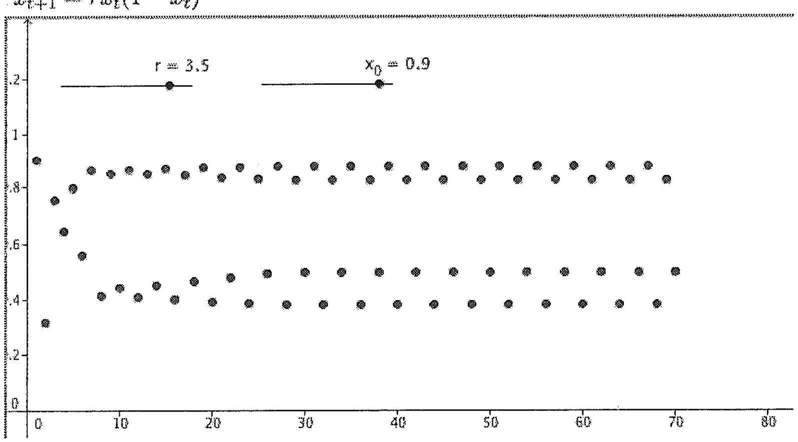

Convergence to the fixed point

$x_0$ = initial population
$x_{t+1} = rx_t(1-x_t)$



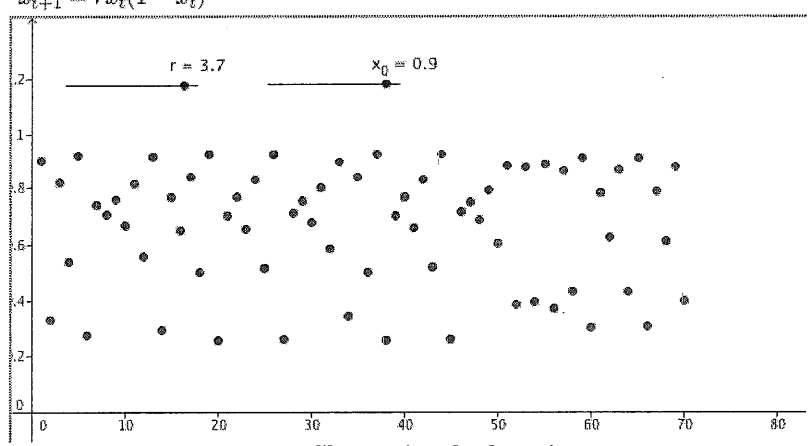

Cycle of period two

$x_0$ = initial population
$x_{t+1} = rx_t(1-x_t)$




Cycle of period four

$x_0$ = initial population
$x_{t+1} = rx_t(1-x_t)$

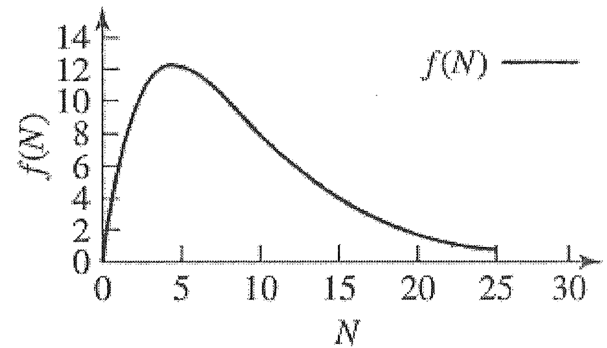



Chaotic behavior

# Ricker Logistic Equation

An iterated map that has the same (desirable) properties as the logistic map but does not admit negative population sizes (provided that $N_0$ is positive) is Rickers curve. The recursion, called the Ricker logistic equation, is given by

$$N_{t+1} = N_t \exp\left[R\left(1 - \frac{N_t}{K}\right)\right]$$




where $R$ and $K$ are positive parameters.

As in the discrete logistic model, $R$ is the growth parameter and $K$ is the carrying capacity. The fixed points are $\widehat{N} = 0$ and $\widehat{N} = K$.

The Ricker logistic equation shows the same complex dynamics as the discrete logistic map [convergence to the fixed point for small positive values of $R$, periodic behavior with the period doubling as $R$ increases, and chaotic behavior for larger values of $R$].

$$N_{t+1} = N_t \exp\left[R\left(1 - \frac{N_t}{K}\right)\right]$$

To find the fixed points we need to solve:

$$N = N \exp\left[R\left(1 - \frac{N}{K}\right)\right]$$

$$\iff N = 0 \quad \text{or} \quad 1 = \exp\left[R\left(1 - \frac{N}{K}\right)\right]$$

(Take ln of both sides)

$$\iff N = 0 \quad \text{or} \quad 0 = R\left(1 - N/K\right)$$

$$\iff \boxed{\hat{N}_1 = 0} \quad \text{or} \quad \boxed{\hat{N}_2 = K}$$

# Final Comments

- In Section 5.6 we will analyze in greater details and with more tools the stability of the equilibria in the previous models.
- On our class website there are three applets (created with the graphic package GeoGebra) that allow us to visualize the behavior of the previous three models by varying the various parameters. Please use them! These applets require the latest version of Java.
- What we described in Section 2 could be a great source for your Final project (which is due on December 4) both in terms of substantial mathematical component and adequate biological and/or medical content. Please start thinking about a possible project!