MA 137 – Calculus 1 with Life Science Applications
Formal Definition of the Derivative
(Section 4.1)

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Average Growth Rate

- Population growth in populations with discrete breeding seasons (as discussed in Chapter 2) can be described by the change in population size from generation to generation.
- By contrast, in populations that breed continuously, there is no natural time scale such as generations. Instead, we will look at how the population size changes over small time intervals.
- We denote the population size at time t by $N(t)$, where t is now varying continuously over the interval $[0, \infty)$. We investigate how the population size changes during the interval $[t_0, t_0 + h]$, where $h > 0$. The *absolute change* during this interval, denoted by ΔN, is

$$\Delta N = N(t_0 + h) - N(t_0).$$

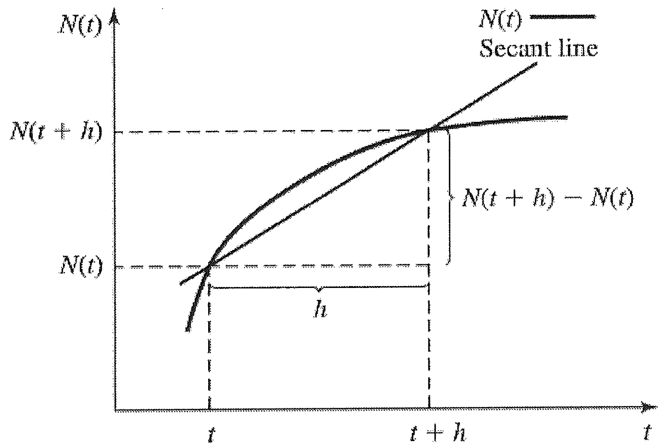
- To obtain the *relative change* during this interval, we divide ΔN by the length of the interval, denoted by Δt, which is h. We find that

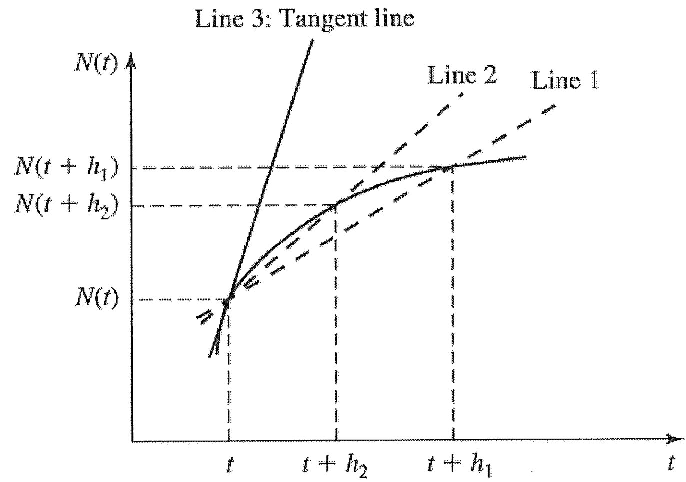
$$\frac{\Delta N}{\Delta t} = \frac{N(t_0 + h) - N(t_0)}{h}.$$

This ratio is called the **average growth rate**.

Geometric Interpretation

We see from the picture below [left] that $\Delta N/\Delta t$ is the slope of the **secant line** connecting the points $(t_0, N(t_0))$ and $(t_0 + h, N(t_0 + h))$.





Observe that the average growth rate $\Delta N/\Delta t$ depends on the length of the interval Δt.

This dependency is illustrated in the picture above [right], where we see that the slopes of the two secant lines (lines 1 and 2) are different.

But we also see that, as we choose smaller and smaller intervals, the secant lines converge to the **tangent line** at the point $(t_0, N(t_0))$ of the graph of $N(t)$ (line 3).

Instantaneous Growth Rate

The slope of the tangent line is called the **instantaneous growth rate** (at t_0) and is a convenient way to describe the growth of a continuously breeding population.

To obtain this quantity, we need to take a limit; that is, we need to shrink the length of the interval $[t_0, t_0 + h]$ to 0 by letting h tend to 0. We express this operation as

$$\lim_{\Delta t \to 0} \frac{\Delta N}{\Delta t} = \lim_{h \to 0} \frac{N(t_0 + h) - N(t_0)}{h}.$$

In the expression above, we take a limit of a quantity in which a continuously varying variable, namely, h, approaches some fixed value, namely, 0.

We denote the limiting value of $\Delta N / \Delta t$ as $\Delta t \to 0$ by $N'(t_0)$ (read "N prime of t_0") and call this quantity **the derivative of $N(t)$ at t_0**....provided that this limit exists!

The Derivative of a Function

We formalize the previous discussion for any function f.

The **average rate of change** of the function $y = f(x)$ between $x = x_0$ and $x = x_1$ is

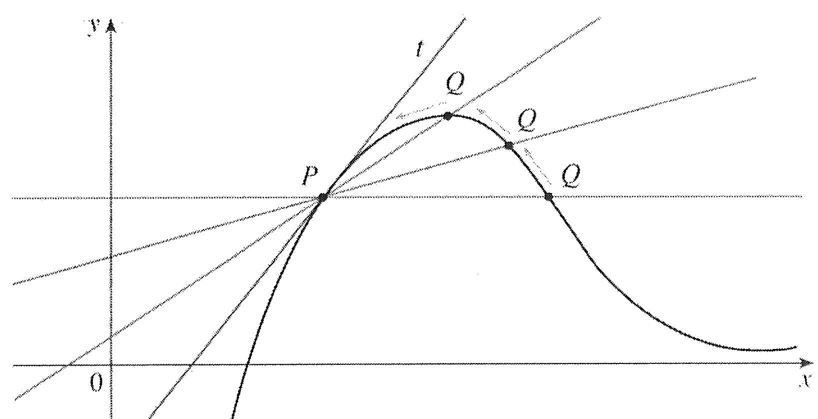
$$\frac{\text{change in } y}{\text{change in } x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

By setting $h = x_1 - x_0$, i.e., $x_1 = x_0 + h$, the above expression becomes

$$\frac{\Delta f}{\Delta x} = \frac{f(x_0 + h) - f(x_0)}{h}$$

Those quantities represent the slope of the secant line that passes through the points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$

[or $P(x_0, f(x_0))$ and $Q(x_0+h, f(x_0+h))$, respectively].



The instantaneous rate of change is defined as the result of computing the average rate of change over smaller and smaller intervals.

Definition

The **derivative of a function** f **at** x_0, denoted by $f'(x_0)$, is

$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that the limit exists.

In this case we say that the function f is **differentiable at** x_0.

Geometrically $f'(x_0)$ respresents the **slope of the tangent line**.

Note: To save on indices, we can also write $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ to denote the derivative of f at the point c.

- Now just drop the subscript 0 from the x_0 in the previous derivative formula, and you obtain the instantaneous rate of change of f with respect to x at a general point x. This is called the **derivative of f at x** and is denoted with $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}.$$

  It is a function of x...no longer a number!

- We say that f is **differentiable** on an open interval (a, b) if $f'(x)$ exists at every $x \in (a, b)$.

- **Notations:** There is more than one way to write the derivative of a function $y = f(x)$. The following expressions are equivalent:

$$y' = \frac{dy}{dx} = f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x).$$

  The notation $\frac{df}{dx}$ goes back to Leibniz and is called **Leibniz notation**.

  We can also write $\left.\frac{df}{dx}\right|_{x=x_0}$ to denote $f'(x_0)$.

Example 1: (Online Homework HW11, # 3)

Let $f(x)$ be the function $12x^2 - 2x + 11$. Then the difference quotient

$$\frac{f(1+h) - f(1)}{h}$$

can be simplified to $ah + b$ for $a =$ ________ and $b =$ ________.

Compute $\displaystyle\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$.

* $f(x) = 12x^2 - 2x + 11$

* $$\frac{f(1+h) - f(1)}{h} = \frac{\left[12(1+h)^2 - 2(1+h) + 11\right] - \left[12(1)^2 - 2(1) + 11\right]}{h}$$

$$= \frac{12(1+2h+h^2) - \cancel{2} - 2h + \cancel{11} - 12 + \cancel{2} - \cancel{11}}{h}$$

$$= \frac{\cancel{12} + 24h + 12h^2 - 2h - \cancel{12}}{h} = \frac{22h + 12h^2}{h}$$

$$= \underline{\underline{22 + 12h}}$$

* $$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \left[22 + 12h\right] = \underline{\underline{22}}$$

Example 2: (Online Homework HW11, # 4)

If $f(x) = ax^2 + bx + c$, find $f'(x)$, using the definition of derivative. (a, b, and c are constants.)

* $f(x) = ax^2 + bx + c$

* $$\frac{f(x+h) - f(x)}{h} = \frac{\{a[x+h]^2 + b[x+h] + c\} - \{ax^2 + bx + c\}}{h}$$

$$= \frac{\cancel{ax^2} + 2axh + ah^2 \cancel{+bx} + bh \cancel{+c} \cancel{-ax^2} \cancel{-bx} \cancel{-c}}{h}$$

$$= \frac{2axh + bh + ah^2}{h} = \frac{\cancel{h}(2ax + b + ah)}{\cancel{h}}$$

$$= \boxed{2ax + b + ah}$$

* $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2ax + b + ah) = \boxed{2ax + b}$$

($ah \to 0$)

Thus: $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$ (the constant c goes to zero)

In particular, if $\underline{a = 0}$; $f(x) = bx + c$ is such that $f'(x) = b$. Hence the derivative of a linear function is its *slope*.

Equation of the Tangent Line at a Point

If the derivative of a function f exists at $x = x_0$, then $f'(x_0)$ is the slope of the tangent line at the point $P(x_0, f(x_0))$.
The equation of the tangent line to the graph of f at P is given by

$$y - f(x_0) = f'(x_0)(x - x_0).$$

The importance of computing the equation of the tangent line to the graph of a function f at a point $P(x_0, f(x_0))$ lies in the fact that if we look at a portion of the graph of f near the point P, it becomes indistinguishable from the tangent line at P.

In other words, the values of the function are close to those of the linear function whose graph is the tangent line.

For this reason, the linear function whose graph is the tangent line to $y = f(x)$ at the point $P(x_0, f(x_0))$ is called the **linear approximation** of f near $x = x_0$.

Example 3: (Online Homework HW11, # 8)

If $f(x) = 4x + \frac{4}{x}$, find $f'(2)$, using the definition of derivative.

Use this to find the equation of the tangent line to the graph of $y = f(x)$ at the point $(2, f(2))$.

$$f(x) = 4x + \frac{4}{x}$$

In order to find the tangent line at $x=2$ we need $P(2, f(2)) = (2, 10)$ as $f(2) = 4 \cdot 2 + \frac{4}{2} \overset{!}{=} 8 + 2 \overset{!}{=} 10$

and the slope of the tg. line

$$\boxed{f'(2)}$$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\left[4(2+h) + \frac{4}{2+h}\right] - 10}{h}$$

$$= \lim_{h \to 0} \frac{4(2+h)^2 + 4 - 10(2+h)}{h(2+h)}$$

$$= \lim_{h \to 0} \frac{\cancel{16} + 16h + 4h^2 \cancel{+4} \cancel{-20} - 10h}{h(2+h)}$$

$$= \lim_{h \to 0} \frac{6h + 4h^2}{h(2+h)} = \lim_{h \to 0} \frac{h(6+4h)}{h(2+h)}$$

$$= \lim_{h \to 0} \frac{6+4h}{2+h} = \frac{6}{2} = \boxed{3}$$

Hence $f'(2) = 3$. Therefore the equation of the tangent line at $P(2, 10)$ is

$$\boxed{y - 10 = 3(x-2)}$$

or $\boxed{y = 3x + 4}$

Example 4:

If $f(x) = \sqrt{x}$, find $f'(x)$, using the definition of derivative.

* $f(x) = \sqrt{x}$

* $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} =$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}} \longleftarrow \text{derivative}$$

Example 5: (Online Homework HW11, # 11)

Assume that $f(x)$ is everywhere continuous and it is given to you that

$$\lim_{x\to 7} \frac{f(x)+9}{x-7} = 10.$$

It follows that $y =$ ______________ is the equation of the tangent line to $y = f(x)$ at the point (___, ___).

Recall that we can also write $f'(c)$ as:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Thus if we consider

$$10 = \lim_{x \to 7} \frac{f(x) + 9}{x - 7} = \lim_{x \to 7} \frac{f(x) - (-9)}{x - 7}$$

we have that $\boxed{c = 7}$ $f(c) = f(7) = -9$

and $f'(c) = f'(7) = 10$. Thus the equation of the tangent line at $P(7, -9)$ is

$$\boxed{y - (-9) = 10(x - 7)} \quad \text{or} \quad \boxed{y = 10x - 79}$$

Example 6: (Online Homework HW11, # 4)

The limit below represents a derivative $f'(a)$.

$$\lim_{h \to 0} \frac{(-4+h)^3 + 64}{h}.$$

Find $f(x)$ and a.

$$f'(a) = \lim_{h \to 0} \frac{(-4+h)^3 + 64}{h}$$

$$= \lim_{h \to 0} \frac{(-4+h)^3 - (-64)}{h}$$

$$= \lim_{h \to 0} \frac{(-4+h)^3 - (-4)^3}{h}$$

$$\therefore \quad \boxed{f(x) = x^3} \qquad \boxed{a = -4}$$

Differentiability and Continuity

A function f is differentiable at a point if the derivative at that point exists. That is, if the tangent line at that point is well defined.

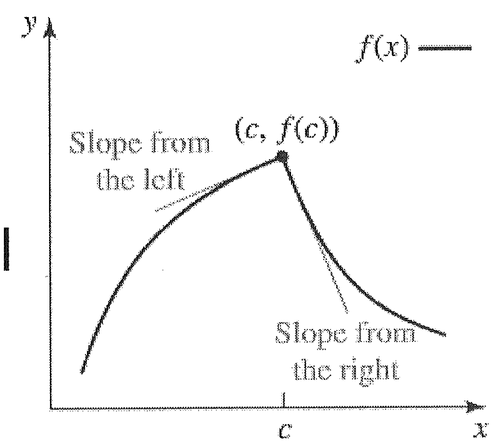
There are two ways that a tangent line might not exist.
It depends on how limits fail to exist:

(a) left-hand and right-hand limit do not agree;
(b) one of these limits is infinite.

Continuity alone is not enough for a function to be differentiable:

(a) The function $f(x) = |x|$ is continuous at all values of x, but it is not differentiable at $x = 0$. It has a **sharp corner** at $x = 0$

(b) The function $f(x) = x^{1/3}$ is continuous for all x, but it is not differentiable at $x = 0$. There is a **vertical tangent line** at $x = 0$.



Differentiability Implies Continuity

However, if a function is differentiable, it is also continuous.

Theorem

If f is differentiable at $x = x_0$, then f is also continuous at $x = x_0$.

Proof: To show that f is continuous at $x = x_0$, we must show that

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad \text{or} \qquad \lim_{x \to x_0} [f(x) - f(x_0)] = 0.$$

However

$$\begin{aligned}
\lim_{x \to x_0} [f(x) - f(x_0)] &= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right] \\
&= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right] \cdot \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \cdot \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \cdot 0 \\
&= 0.
\end{aligned}$$

Example 7: (Online Homework HW11, # 14)

Find a and b so that the function

$$f(x) = \begin{cases} x^2 - 2x + 3 & \text{if } x \leq 2 \\ ax^2 + 6x + b & \text{if } x > 2 \end{cases}$$

is both continuous and differentiable.

$f(x)$ is made of two pieces of parabolas.
These are continuous and differentiable for
$x < 2$ and $x > 2$. The problem is at $x = 2$.

We need to make sure that f is continuous
at $x = 2$ and that the derivative exists

at $x = 2$.

(1.) for the continuity we need: $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

Thus: $\lim_{x \to 2^-} (x^2 - 2x + 3) = \lim_{x \to 2^+} (ax^2 + 6x + b)$

↑ MUST

i.e. $2^2 - 2(2) + 3 = a(2)^2 + 6(2) + b$

$\iff$ $\boxed{4a + b = -9}$

(2) The derivative of f is

$$f'(x) = \begin{cases} 2x - 2 & \text{if } x < 2 \\ 2ax + 6 & \text{if } x > 2 \end{cases}$$

We need to make sure that it exists for $\underline{x = 2}$

$$\lim_{x \to 2^-} f'(x) = \lim_{x \to 2^+} f'(x) \text{ ; hence}$$

$$2(2) - 2 \underset{\uparrow \text{ MUST}}{=} 2a(2) + 6$$

$$\Longleftrightarrow \quad 2 = 4a + 6 \quad \Longleftrightarrow \quad 4a = -4 \quad \Longleftrightarrow \quad \boxed{a = -1}$$

<u>Hence</u>: $\begin{cases} a = -1 \\ 4a + b = -9 \end{cases} \Longrightarrow \boxed{b = -5}$

MA 137 – Calculus 1 with Life Science Applications
The Power Rule, the Basic Rules of Differentiation, and the Derivatives of Polynomials
(Section 4.2)

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Basic Rules

Since polynomials and rational functions are built up by the basic operations of addition, subtraction, multiplication, and division operating on power functions of the form $y = x^n, n = 0, 1, 2, \ldots,$ we need differentiation rules for such operations.

Theorem

Suppose c is a constant, n is a positive integer, and f(x) and g(x) are differentiable functions. Then the following relationships hold:

0. $$\frac{d}{dx}[c] = 0$$

1. $$\frac{d}{dx}[cf(x)] = c\,\frac{d}{dx}f(x)$$

2. $$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

3. $$\frac{d}{dx}[x^n] = nx^{n-1}$$

Example 1: (Neuhauser, Problem # 6, p. 149)

Differentiate $f(x) = -1 + 3x^2 - 2x^4$ with respect to x.

* $f(x) = -1 + 3x^2 - 2x^4$

* $$\frac{d}{dx} f(x) = \frac{d}{dx}(-1) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(-2x^4)$$

by property 2. —

$$= 0 + 3 \cdot \frac{d}{dx}(x^2) - 2 \cdot \frac{d}{dx}(x^4)$$

by property 1. —

$$= 3(2x) - 2 \cdot (4x^3)$$

by property 3. (power rule)

$$= \boxed{6x - 8x^3}$$

Example 2: (Neuhauser, Problem # 32, p. 150)

Differentiate

$$f(N) = \frac{bN^2 + N}{K + b}$$

with respect to N. Assume that b and K are positive constants.

$$f(N) = \frac{bN^2 + N}{K + b}$$

$$= \left(\frac{b}{K+b}\right) N^2 + \left(\frac{1}{K+b}\right) \cdot N$$

$$f'(N) = \frac{d}{dN} f(N) = \left(\frac{b}{K+b}\right) \cdot 2N + \left(\frac{1}{K+b}\right) \cdot 1$$

$$= \boxed{\frac{2bN + 1}{K + b}}$$

Example 3: (Neuhauser, Problem # 38, p. 150)

Differentiate

$$g(N) = rN\left(1 - \frac{N}{K}\right)$$

with respect to N. Assume that K and r are positive constants.

$$g(N) = rN\left(1 - \frac{N}{K}\right)$$ can be rewritten as

$$= rN - \frac{r}{K}N^2$$

Hence

$$\underline{\underline{g'(N)}} = \frac{dg}{dN} = r \cdot 1 - \frac{r}{K} \cdot 2N$$

$$= \boxed{r - \frac{2r}{K}N}$$

Example 4: (Neuhauser, Problem # 56, p. 150)

Find the tangent line to

$$f(x) = cx^3 - 2cx$$

at $x = -1$. Assume that c is a positive constant.

$f(x) = c x^3 - 2 c x$

We need to find the tangent line at the point where $x = -1$.

* Hence : $P(-1, f(-1)) = \boxed{(-1, c)}$

as $f(-1) = c(-1)^3 - 2c(-1) = -c + 2c = c$

* Now, for the derivative at $x = -1$:

$f'(x) = 3cx^2 - 2c$

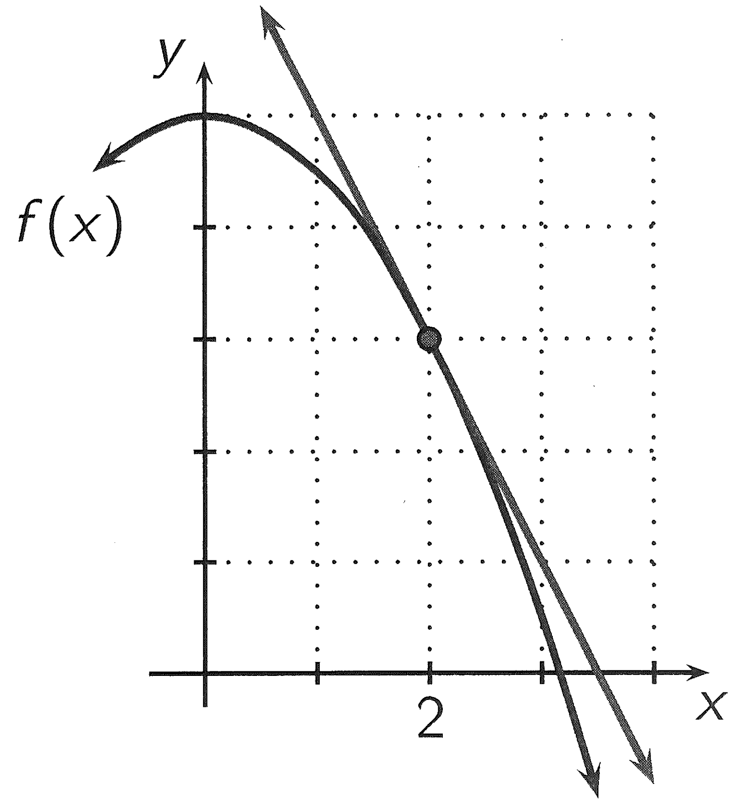
$f'(-1) = 3c(-1)^2 - 2c = 3c - 2c = \underline{\underline{c}}$

* Eq of tg. line : $\boxed{y - c = c(x+1)}$ or $\boxed{y = cx + 2c}$

Example 5:

A segment of the tangent line to the graph of $f(x)$ at x is shown in the picture. Using information from the graph we can estimate that

$$f(2) = \_\_\_\_\_\_ \qquad f'(2) = \_\_\_\_\_\_$$



hence the equation to the tangent line to the graph of

$$g(x) = 5x + f(x)$$

at $x = 2$ can be written in the form $y = mx + b$ where

$$m = \_\_\_\_\_\_ \qquad b = \_\_\_\_\_\_.$$

* From the graph: $f(2) = 3$ $\quad f'(2) = -\frac{4}{2} = -2$

* $g(x) = 5x + f(x)$

At $x=2$; $g(2) = 5 \cdot 2 + f(2) = 10 + 3 = 13$

About the derivative of g: $g'(x) = 5 + f'(x)$

so that $g'(2) = 5 + f'(2) = 5 - 2 = 3$

Therefore the equation of the tg. line to the graph of g at $(2, 13)$ is:

$$\boxed{y - 13 = 3(x-2)} \quad \text{OR} \quad \boxed{y = 3x + 7}$$

Example 6: (Online Homework HW12, # 11)

Lizards are cold-blooded animals whose temperatures roughly match the surrounding environment. Suppose the body temperature, $T(t)$, of a lizard is measured for a period of 18 hours from midnight until 6 PM. The body temperature (in $^\circ C$) of the lizard over this period of time (in hours) is found to be well approximated by the polynomial

$$T(t) = -0.009t^3 + 0.29t^2 - 1.7t + 15.5.$$

(a) Find the general expression for the rate of change of body temperature per hour, $T'(t)$.

(b) Use this information to find what the rate of change of body temperature is at: midnight; 4 AM; 8 AM; noon; 4 PM.

(c) Which of these times gives the fastest increase in the body temperature and which shows the most rapid cooling of the lizard?

$T(t) = -0.009t^3 + 0.29t^2 - 1.7t + 15.5$

temperature of the lizard

(a) $T'(t) = -0.027t^2 + 0.58t - 1.7$

(b) $T'(0) = -1.7$

$T'(4) = 0.188$

$T'(8) = 1.212$

$T'(12) = 1.372$

$T'(16) = 0.668$

(c) Fastest increase at noon : $T'(12) = 1.372$

Most rapid cooling at midnight $T'(0) = -1.7$

Proofs:

0. Define $f(x) = c$ and use the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$

1. We use the definition of the derivative and one of the Limit Laws:

$$[cf(x)]' = \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} = c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = cf'(x).$$

2. We use the definition of the derivative, rewrite the numerator and then use one of the Limit Laws:

$$\begin{aligned}
[f+g]'(x) &\stackrel{\text{def}}{=} \lim_{h\to 0} \frac{[f+g](x+h) - [f+g](x)}{h} \\
&\stackrel{\text{def}}{=} \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h} \\
&\stackrel{\text{rule}}{=} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).
\end{aligned}$$

Special product formulas: The powers of certain binomials occur so frequently that we should memorize the following formulas. We can verify them by performing the multiplications.

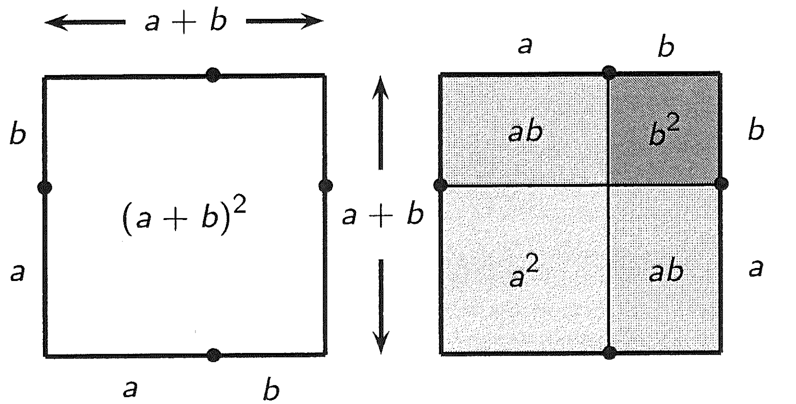
If A and B are any real numbers or algebraic expressions, then:

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
4. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Visualizing a formula:

Many of the special product formulas can be seen as geometrical facts about length, area, and volume. The ancient Greeks always interpreted algebraic formulas in terms of geometric figures.

For example, the figure below

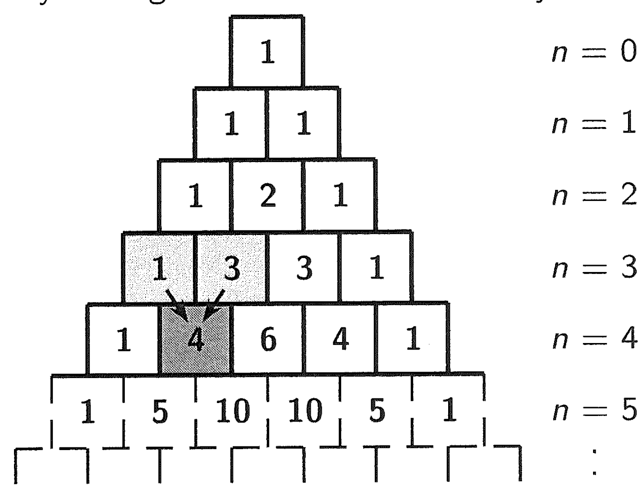


shows how the formula for the square of a binomial (formula **1**) can be interpreted as a fact about areas of squares and rectangles.

Pascal's triangle:

The coefficients (without sign) of the expansion of a binomial of the form $(a \pm b)^n$ can be read off the n-th row of the following 'triangle' named **Pascal's triangle** (after Blaise Pascal, a 17th century French mathematician and philosopher).

To build the triangle, start with '1' at the top, then continue placing numbers below it in a triangular way. Each number is simply obtained by adding the two numbers directly above it.



3. We use the definition of the derivative and the Binomial Theorem. The Binomial Theorem tells us

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

Let's now use the definition of the derivative with $f(x) = x^n$:

$$\begin{aligned}
f'(x) &\overset{\text{def}}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\{x^n + nx^{n-1}h + [n(n-1)]/2\, x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\} - x^n}{h} \\
&= \lim_{h\to 0} \frac{nx^{n-1}h + [n(n-1)]/2\, x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0} \frac{\{nx^{n-1} + [n(n-1)]/2\, x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\}h}{h} \\
&= \lim_{h\to 0} \{nx^{n-1} + [n(n-1)]/2\, x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\} \\
&= nx^{n-1}
\end{aligned}$$

3'. Define $f(x) = x^n$. We know from the alternate limit form of the definition of the derivative that the derivative $f'(x)$ is given by,

$$f'(x) = \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{x_1 \to x} \frac{x_1^n - x^n}{x_1 - x}.$$

Now we have the following formula,

$$x_1^n - x^n = (x_1 - x)(x_1^{n-1} + x x_1^{n-2} + x^2 x_1^{n-3} + \cdots + x^{n-3} x_1^2 + x^{n-2} x_1 + x^{n-1})$$

which we can verify by simply multiplying the two factors together.

Let's now use the alternative definition of the derivative with $f(x) = x^n$:

$$\begin{aligned}
f'(x) &\stackrel{\text{def}}{=} \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{x_1 \to x} \frac{x_1^n - x^n}{x_1 - x} \\
&= \lim_{x_1 \to x} \frac{(x_1 - x)(x_1^{n-1} + x x_1^{n-2} + x^2 x_1^{n-3} + \cdots + x^{n-3} x_1^2 + x^{n-2} x_1 + x^{n-1})}{x_1 - x} \\
&= \lim_{x_1 \to x} (x_1^{n-1} + x x_1^{n-2} + x^2 x_1^{n-3} + \cdots + x^{n-3} x_1^2 + x^{n-2} x_1 + x^{n-1}) \\
&= n x^{n-1} \qquad \text{[as there are } n \text{ equal terms in the expression]}
\end{aligned}$$

MA 137 – Calculus 1 with Life Science Applications
The Product and Quotient Rule
and the Derivatives of Rational and Power Functions
(Section 4.3)

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Basic Rules (cont'd)

Theorem

Suppose $f(x)$ and $g(x)$ are differentiable functions. Then the following relationships hold:

4. $$\frac{d}{dx}[f(x)\cdot g(x)] = \frac{d}{dx}[f(x)]\cdot g(x) + f(x)\cdot\frac{d}{dx}[g(x)]$$

(in prime notation) $(fg)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

5. $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]\cdot g(x) - f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

(in prime notation) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}$

The Power Rule for Negative Exponents

The quotient rule allows us to extend the power rule to the case where the exponent is a negative integer:

Theorem

If $f(x) = x^{-n}$, where n is a positive integer, then $f'(x) = -nx^{-n-1}$.

Proof: We write $f(x) = \dfrac{1}{x^n}$ and use the quotient rule

$$f'(x) = \frac{0 \cdot x^n - 1 \cdot nx^{n-1}}{[x^n]^2} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{(n-1)-2n} = -nx^{-n-1}.$$

There is a general form of the power rule in which the exponent can be any real number. In Section 4.4, we give the proof for the case when the exponent is rational; we prove the general case in Section 4.7.

Theorem (General Form)

If $f(x) = x^r$, where r is any real number, then $f'(x) = r\,x^{r-1}$.

Example 1: (Neuhauser, Example # 1, p. 153)

Differentiate $f(x) = (3x + 1)(2x^2 - 5)$.

* $f(x) = (3x+1)(2x^2-5)$

Let's use the product rule

$$f'(x) = 3\cdot(2x^2-5) + (3x+1)(4x)$$

If we want to expand we get

$$f'(x) = 6x^2 - 15 + 12x^2 + 4x$$

$$= 18x^2 + 4x - 15$$

* We could also have multiply the factors in $f(x)$:

$$f(x) = 6x^3 - 15x + 2x^2 - 5$$

$$= 6x^3 + 2x^2 - 15x - 5$$

so that

$$f'(x) = 18x^2 + 4x - 15$$

Example 2: (Online Homework HW12, # 17)

Differentiate $Y(u) = (u^{-2} + u^{-3})(u^5 + u^2)$.

* We use the product and power rule for integer exponents.

$$Y(u) = (u^{-2} + u^{-3})(u^5 + u^2)$$

$$Y'(u) = \left[-2u^{-3} - 3u^{-4}\right](u^5 + u^2) + (u^{-2} + u^{-3})(5u^4 + 2u)$$

We could simplify it now....

* I personally would have simplified first $Y(u)$ as follows:

$$Y(u) = u^{-2}u^5 + u^{-2}u^2 + u^{-3}u^5 + u^{-3}u^2$$

$$= u^3 + 1 + u^2 + u^{-1}$$

Hence $\Upsilon(u) = u^3 + u^2 + 1 + u^{-1}$

and $\Upsilon'(u) = 3u^2 + 2u - u^{-2}$

$$= 3u^2 + 2u - \frac{1}{u^2}$$

$$= \boxed{\frac{3u^4 + 2u^2 - 1}{u^2}}$$

Example 3: (Neuhauser, Problem # 39, p. 158)

Assume that $f(x)$ is differentiable.

Find an expression for the derivative of

$$y = -5x^3 f(x) - 2x$$

at $x = 1$, assuming that $f(1) = 2$ and $f'(1) = -1$.

$$y = -5x^3 f(x) - 2x$$

$$f(1) = 2$$

$$f'(1) = -1$$

$$y' = -15x^2 f(x) - 5x^3 f'(x) - 2$$

Hence when $x = 1$

$$y'(1) = -15(1)^2 f(1) - 5(1)^3 f'(1) - 2$$

$$= -15(2) - 5(-1) - 2$$

$$= -30 + 5 - 2 = \boxed{-27}$$

Example 4: (Online Homework HW12, # 19)

Differentiate $$f(x) = \frac{ax+b}{cx+d} \quad ,$$

where a, b, c, and d are constants and $ad - bc \neq 0$.

$$f(x) = \frac{ax+b}{cx+d}$$

We use the quotient rule

$$f'(x) = \frac{a(cx+d) - (ax+b)c}{(cx+d)^2}$$

$$= \frac{\cancel{acx} + ad - \cancel{acx} - bc}{(cx+d)^2}$$

$$= \frac{ad-bc}{(cx+d)^2} \neq 0 \quad \left(\text{unless } x = -\frac{d}{c}\right)$$

Example 5: (Online Homework HW12, # 22)

Find an equation of the tangent line to the given curve at the specified point:

$$y = \frac{\sqrt{x}}{x+3} \qquad P(4, 2/7).$$

$$y = \frac{\sqrt{x}}{x+3} \qquad P\left(4, \frac{2}{7}\right)$$

notice that indeed $y(4) = \frac{\sqrt{4}}{4+3} = \frac{2}{7}$

We need $y'(4)$ to write the equation of the tangent line at P.

We use the quotient rule; also recall that

$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ by the generalized power rule.

$$y' = \frac{\frac{1}{2\sqrt{x}}(x+3) - \sqrt{x}\,(1)}{(x+3)^2} = \frac{\dfrac{x+3 - (2\sqrt{x})(\sqrt{x})}{2\sqrt{x}}}{(x+3)^2}$$

$$= \frac{x+3-2x}{2\sqrt{x}\,(x+3)^2} = \boxed{\frac{3-x}{2\sqrt{x}\,(x+3)^2}}$$

Since $y' = \dfrac{3-x}{2\sqrt{x}\,(x+3)^2}$ we have that

$$y'(4) = \frac{3-4}{2\sqrt{4}\,(4+3)^2} = \frac{-1}{4\cdot 49}$$

So:

$$y - \frac{2}{7} = -\frac{1}{4\cdot 49}(x-4)$$

$$y = -\frac{1}{196}x + \frac{1}{49} + \frac{2}{7}$$

$$\boxed{y = -\frac{1}{196}x + \frac{15}{49}}$$

Example 6: (Neuhauser, Example # 6, p. 155)

Differentiate the Monod growth function

$$f(R) = \frac{aR}{k + R}$$

where a and k are positive constants.

$$f(R) = \frac{aR}{k+R}$$ where a, k are positive constants

$$f'(R) = \frac{d}{dR} f = \frac{(a \cdot 1)(k+R) - aR(1)}{(k+R)^2}$$

$$= \frac{ak + \cancel{aR} - \cancel{aR}}{(k+R)^2} = \boxed{\frac{ak}{(k+R)^2}}$$

Since $a, k > 0$ then notice that

$\boxed{f'(R) > 0}$ always except for $R = -k$

Example 7: (Neuhauser, Problem # 84, p. 159)

Assume that $f(x)$ is differentiable.

Find an expression for the derivative of

$$y = \frac{f(x)}{x^2 + 1}$$

at $x = 2$, assuming that $f(2) = -1$ and $f'(2) = 1$.

$$y = \frac{f(x)}{x^2+1} \qquad f(2) = -1 \qquad f'(2) = 1$$

Want $y'(2)$.

$$y' = \frac{f'(x)(x^2+1) - f(x)\cdot 2x}{(x^2+1)^2}$$

Hence:

$$y'(2) = \frac{f'(2)(2^2+1) - f(2)\cdot 2\cdot 2}{(2^2+1)^2}$$

$$= \frac{1\cdot 5 - (-1)\cdot 4}{25} = \boxed{\frac{9}{25}}$$

Proofs:

4. We use the definition of the derivative, rewrite the numerator in a 'tricky' way and use the limit laws and the continuity of the functions.

$$
\begin{aligned}
(fg)'(x) &\overset{\text{def}}{=} \lim_{h\to 0} \frac{(fg)(x+h)-(fg)(x)}{h} \\
&\overset{\text{def}}{=} \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&\overset{\text{trick}}{=} \lim_{h\to 0} \frac{f(x+h)g(x+h) \boxed{-f(x)g(x+h)+f(x)g(x+h)} - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h}\, g(x+h) + f(x)\, \frac{g(x+h)-g(x)}{h}\right] \\
&\overset{\text{rule}}{=} \left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right]\left[\lim_{h\to 0} g(x+h)\right] + f(x)\left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right] \\
&\overset{\text{cont.}}{=} f'(x)\, g(x) + f(x)\, g'(x).
\end{aligned}
$$

5. We use the definition of the derivative, rewrite the numerator in a 'tricky' way and use the limit laws and the continuity of the functions.

$$(f/g)'(x) =$$

$$\stackrel{\text{def}}{=} \lim_{h\to 0} \frac{(f/g)(x+h)-(f/g)(x)}{h}$$

$$\stackrel{\text{def}}{=} \lim_{h\to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x)\,g(x+h)}$$

$$\stackrel{\text{trick}}{=} \lim_{h\to 0} \frac{f(x+h)g(x) \boxed{-f(x)g(x)+f(x)g(x)} - f(x)g(x+h)}{h\,g(x)\,g(x+h)}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h\,g(x+h)} - f(x)\,\frac{g(x+h)-g(x)}{h\,g(x)g(x+h)} \right]$$

$$\stackrel{\text{rule}}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0} \frac{1}{g(x+h)} - \lim_{h\to 0} \frac{f(x)}{g(x)g(x+h)} \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$\stackrel{\text{cont.}}{=} f'(x)\frac{1}{g(x)} - \frac{f(x)}{[g(x)]^2}g'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}.$$

MA 137 – Calculus 1 with Life Science Applications
A First Look at Differential Equations
(Section 4.1.2)

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Differential Equations ($\equiv$ DEs)

A **differential equation** is an equation that contains an unknown function and one (or more) of its derivatives.

For example

- $\dfrac{dy}{dx} + 6y = 7;$
- $\dfrac{dy}{dt} + 0.2\,t\,y = 6t;$
- $\dfrac{dP}{dt} = \sqrt{P\,t};$
- $xy' + y = y^2.$

If a differential equation contains only the first derivative, it is called a **first-order differential equation**: $\dfrac{dy}{dx} = h(x, y)$.

Example 1

Consider the differential equation $$(t+1)\frac{dy}{dt} - y + 6 = 0.$$

Which of the following functions

$$y_1(t) = t + 7 \qquad y_2(t) = 3t + 21 \qquad y_3(t) = 3t + 9$$

are solutions for all t?

$$(t+1)\frac{dy}{dt} - y + 6 = 0$$

(1.) Consider $y_1 = t+7$; $\frac{dy_1}{dt} = 1$ and now plug into the equation

$$(t+1)\cdot(1) - (t+7) + 6 = t + 1 - t - 7 + 6 = 0 \checkmark$$

(2.) Consider $y_2 = 3t+21$; $\frac{dy_2}{dt} = 3$ and now plug into the equation

$$(t+1)\cdot(3) - [3t+21] + 6 =$$

$$= \cancel{3t} + 3 - \cancel{3t} - 21 + 6 = -12 \neq 0$$

(3.) Consider $y_3 = 3t+9$; $\frac{dy_3}{dt} = 3$ and now plug into the equation

$$(t+1)(3) - (3t+9) + 6 =$$

$$= 3t + 3 - 3t - 9 + 6 = 0 \checkmark$$

Hence y_1 and y_3 are both solutions of the differential equation

$$(t+1)\frac{dy}{dt} - y + 6 = 0$$

Example 2

Verify that the function

$$y = 1 - \frac{1}{x^3 + C} \qquad C = \text{any constant}$$

is a (family of) solution(s) of the differential equation

$$y' - 3(y-1)^2x^2 = 0.$$

Verify that $y = 1 - \dfrac{1}{x^3+C}$ is a family of solutions of the DE:

$$y' - 3(y-1)^2 x^2 = 0$$

We need y'. Using the quotient rule

$$y' = 0 - \frac{0\cdot(x^3+C) - 1\cdot(3x^2)}{(x^3+C)^2} = \frac{+3x^2}{(x^3+C)^2}$$

Hence, let us plug into the equation

$$\left[\frac{3x^2}{(x^3+C)^2}\right] - 3\left[\cancel{1} - \frac{1}{x^3+C} - \cancel{1}\right]^2 x^2 =$$

$$\frac{3x^2}{(x^3+C)^2} - \frac{3x^2}{(x^3+C)^2} = 0 \quad \checkmark$$

Yes!

DEs arise for example in biology (e.g. models of population growth), economics (e.g. models of economic growth), and many other areas.

exponential growth model: $$\frac{dN}{dt} = rN \qquad N(0) = N_0;$$

logistic growth model: $$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \qquad N(0) = N_0;$$

Newton's law of cooling: $$\frac{dT}{dt} = -k(T - T_e) \qquad T(0) = T_0;$$

von Bertalanffy models: $$\frac{dL}{dt} = k(L_\infty - L) \qquad L(0) = L_0;$$

$$\frac{dW}{dt} = \eta W^{2/3} - \kappa W \qquad W(0) = W_0;$$

Solow's economic growth model: $$\frac{dk}{dt} = sk^\alpha - \delta k \qquad k(0) = k_0.$$

The Exponential Growth Model

A biological population with plenty of food, space to grow, and no threat from predators, tends to grow at a rate that is proportional to the population – that is, in each unit of time, a certain percentage of the individuals produce new individuals.

If reproduction takes place more or less continuously, then this growth rate is represented by

$$\frac{dN}{dt} = rN,$$

where $N = N(t)$ is the population as a function of time t and r is the growth rate.

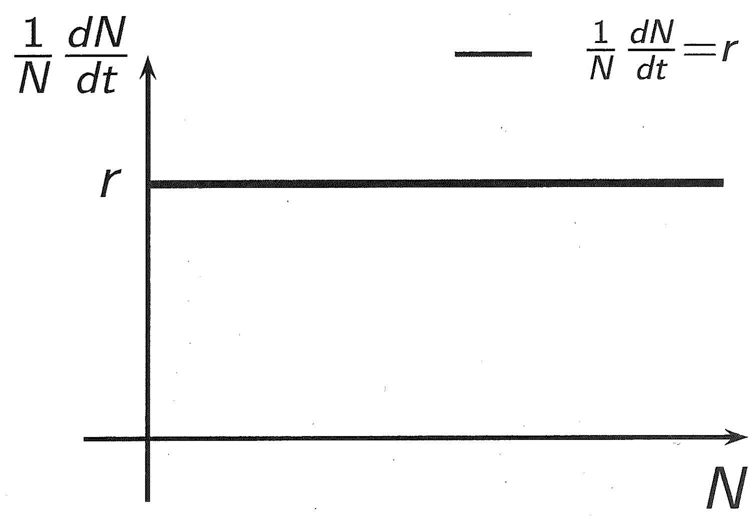
Assume also that N_0 is the population at time $t = 0$.

Note: $r =$ birth rate $-$ mortality rate.

Rewriting this differential equation as

$$\frac{1}{N}\frac{dN}{dt} = r$$

says that the per capita growth rate in the exponential model is a constant function of population size.



We will show (later) that the solution to this differential equation is

$$\boxed{N(t) = N_0 e^{rt}}.$$

The Logistic Growth Model ($\equiv$ Verhulst Model)

- In short, unconstrained natural growth is exponential growth.
- However, we may account for the growth rate declining to 0 by including a factor $1 - N/K$ in the model, where K is a positive constant.
- The factor $1 - N/K$ is close to 1 (that is, has no effect) when N is much smaller than K, and is close to 0 when N is close to K.
- The resulting model,

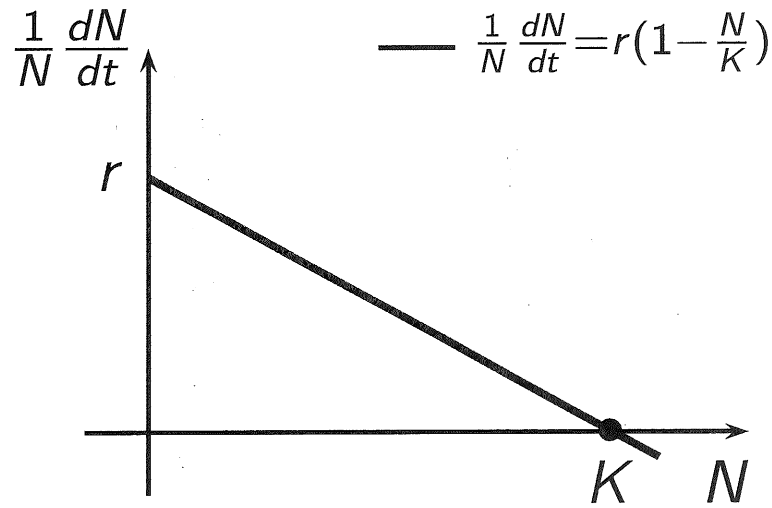
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \qquad \text{with} \qquad N(0) = N_0$$

is called the **logistic growth model** or the **Verhulst model**.

The word "logistic" has no particular meaning in this context, except that it is commonly accepted. The second name honors **Pierre François Verhulst** (1804–1849), a Belgian mathematician who studied this idea in the 19th century. Using data from the first five U.S. censuses, he made a prediction in 1840 of the U.S. population in 1940 – and was off by less than 1%.

Rewriting this differential equation as

$$\frac{1}{N}\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)$$



says that the per capita growth rate in the logistic equation is a linearly decreasing function of population size.

Note: r ($\equiv$growth rate) and K ($\equiv$carrying capacity) are positive constants.

We will show (later) that the solution to this differential equation is

$$\boxed{N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right)e^{-rt}}}$$

Observe that $\lim_{t\to\infty} N(t) = K$.

This justifies that the constant K is dubbed **carrying capacity**.

Compare the logistic growth DE

$$\frac{1}{N} \cdot \frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)$$

to the discrete logistic model :

$$N_{t+1} = N_t \left[1 + R\left(1 - \frac{N_t}{K}\right)\right] \quad \text{for } t = 0, 1, 2, \ldots$$

We can rewrite the latter as :

$$N_{t+1} = N_t + R N_t \left(1 - \frac{N_t}{K}\right)$$

$$\Longleftrightarrow$$

$$\frac{N_{t+1} - N_t}{1} = R N_t \left(1 - \frac{N_t}{K}\right)$$

$$\Longleftrightarrow$$

$$\frac{1}{N_t}\left(\frac{N_{t+1} - N_t}{1}\right) = R\left(1 - \frac{N_t}{K}\right)$$

!!

Newton's Law of Cooling

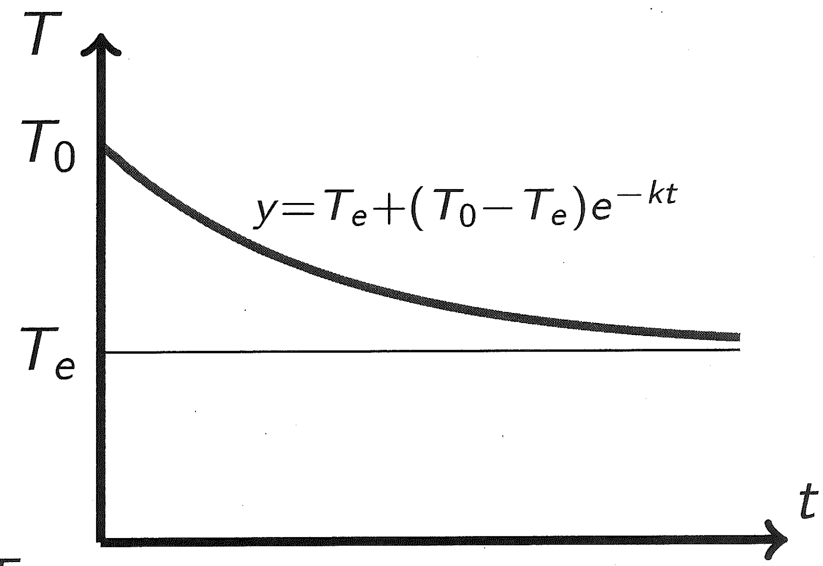
It states that the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium:

$$\frac{dT}{dt} = -k(T - T_e) \qquad T(0) = T_0,$$

where k is a positive constant.

We can show that the solution of this IVP is given by

$$\boxed{T(t) = T_e + (T_0 - T_e)e^{-kt}}.$$



Notice also that

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty}[T_e+(T_0-T_e)e^{-kt}] = T_e.$$

The Von Bertalanffy (Restricted) Growth Equation

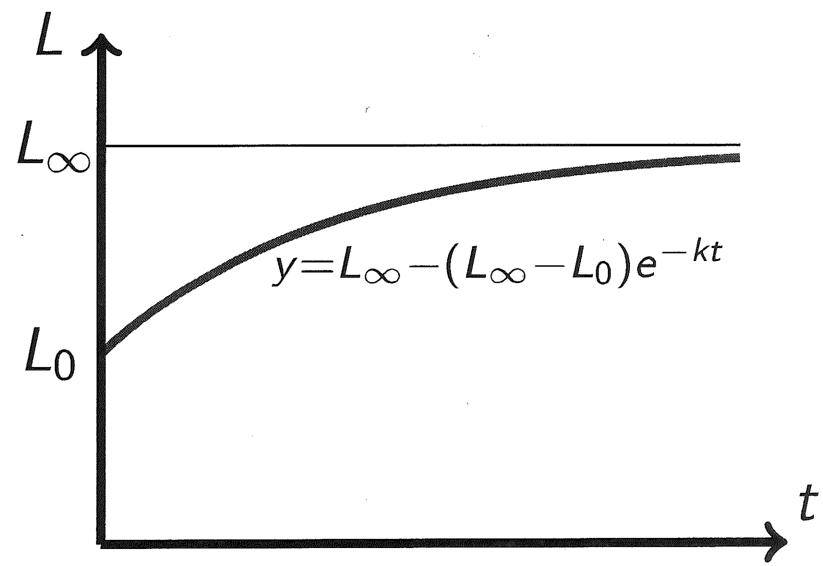
A commonly used DE for the growth, in length, of an individual fish is

$$\frac{dL}{dt} = k(L_\infty - L) \qquad L(0) = L_0,$$

where $L(t)$ is length at age t, L_∞ is the asymptotic length and k is a positive constant. The DE captures the idea that the rate of growth is proportional to the difference between asymptotic and current length.

We can show that the solution of this IVP is given by

$$\boxed{L(t) = L_\infty - (L_\infty - L_0)e^{-kt}}.$$



Notice also that

$$\lim_{t\to\infty} L(t) = \lim_{t\to\infty} [L_\infty - (L_\infty - L_0)e^{-kt}] = L_\infty.$$

Allometric Growth

In biology, **allometry** is the study of the relationship between sizes of parts of an organism (e.g., skull length and body length, or leaf area and stem diameter).

We denote by $L_1(t)$ and $L_2(t)$ the respective sizes of two organs of an individual of age t. We say that L_1 and L_2 are related through an allometric law if their specific growth rates are proportional—that is, if

$$\frac{1}{L_1} \cdot \frac{dL_1}{dt} = k \frac{1}{L_2} \cdot \frac{dL_2}{dt}$$

for some constant k. If k is equal to 1, then the growth is called isometric; otherwise it is called allometric.

We will show that he solution to this differential equation is

$$\boxed{L_1 = C\, L_2^k}$$

for some constant C.

Homeostasis

The nutrient content of a consumer can range from reflecting the nutrient content of its food to being constant. A model for homeostatic regulation is provided in Sterner and Elser (2002). It relates a consumers nutrient content (denoted by y) to its foods nutrient content (denoted by x) as

$$\frac{dy}{dx} = \frac{1}{\theta}\frac{y}{x}$$

where $\theta \geq 1$ is a constant.

We can show that $\boxed{y = C\,x^{1/\theta}}$ for some positive constant C.

Absence of homeostasis means that the consumer reflects the food's nutrient content. This occurs when $y = Cx$ and thus when $\theta = 1$.

Strict homeostasis means that the nutrient content of the consumer is independent of the nutrient content of the food; that is, $y = C$; this occurs in the limit as $\theta \longrightarrow \infty$.

Equilibria of an Autonomous DE

Many of the DEs that model biological situations are of the form

$$\frac{dy}{dx} = g(y)$$

where the right-hand side does not depend explicitly on x. (We will typically think of x as time.) The equations are called **autonomous differential equations**.

Constant solutions form a special class of solutions of autonomous differential equations. These solutions are called (point) **equilibria**.

Example For example

$$N_1(t) = 0 \qquad \text{and} \qquad N_2(t) = K$$

are constant solutions to the logistic equation
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

Finding Equilibria

If $\widehat{y}$ (read "y hat") satisfies

$$g(\widehat{y}) = 0$$

then $\widehat{y}$ is an equilibrium of the autonomous differential equation

$$\frac{dy}{dx} = g(y).$$

Basic Property

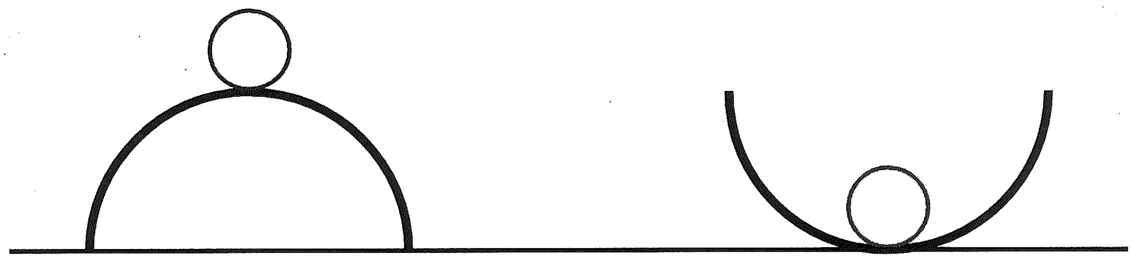
The basic property of equilibria is that if, initially (say, at $x = 0$), $y(0) = \widehat{y}$ and $\widehat{y}$ is an equilibrium, then $y(x) = \widehat{y}$ for all $x \geq 0$.

Stability of Equilibria

Of great interest is the stability of equilibria of a differential equation. This is best explained by the example of a ball on a hill vs a ball in a valley:

a ball rests on top of a hill

a ball rests at the bottom of a valley



In either case, the ball is in equilibrium because it does not move.

If we perturb the ball by a small amount (i.e., if we move it out of its equilibrium slightly) the ball on the left will roll down the hill and not return to the top, whereas the ball on the right will return to the bottom of the valley.

The ball on the **left** is **unstable** and the ball on the **right** is **stable**.

Stability for Equilibria of DE

Suppose that $\widehat{y}$ is an equilibrium of $\dfrac{dy}{dx} = g(y)$; that is, $g(\widehat{y}) = 0$.

We look at what happens to the solution when we start close to the equilibrium; that is, we consider the solution of the DE when we move away from the equilibrium by a small amount, called a *small perturbation*.

We say that $\widehat{y}$ is **locally stable** if the solution returns to the equilibrium $\widehat{y}$ after a small perturbation;

We say that $\widehat{y}$ is **unstable** if the solution does not return to the equilibrium $\widehat{y}$ after a small perturbation.

We will discuss stability of equilibria in great detail in MA 138.

MA 137 – Calculus 1 with Life Science Applications
The Chain Rule and Higher Derivatives
(Section 4.4)

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The Chain Rule

Theorem

If g is differentiable at x and f is differentiable at $y = g(x)$, then the composite function $(f \circ g)(x) = f[g(x)]$ is differentiable at x, and the derivative is given by

$$(f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$

- The proof of the theorem is on p. 164 of the Neuhauser's textbook.
- The function g is the inner function; the function f is the outer function.
- The expression $f'[g(x)] \cdot g'(x)$ thus means that we need to find the derivative of the outer function, evaluated at $g(x)$, and the derivative of the inner function, evaluated at x, and then multiply the two together.
- A special case of the chain rule is called the **power chain rule**:

If $\quad y = [f(x)]^n \quad$ then $\quad \dfrac{dy}{dx} = n[f(x)]^{n-1} \cdot f'(x)$

The Chain Rule in Leibniz Notation

The derivative of $f \circ g$ can be written in Leibniz notation.

If we set $u = g(x)$, then

$$\begin{aligned} \frac{d}{dx}[(f \circ g)(x)] &= \frac{d}{dx} f[g(x)] \\ &\overset{u=g(x)}{=} \frac{d}{dx} f(u) \\ &= \frac{df}{du} \cdot \frac{du}{dx} \end{aligned}$$

This form of the chain rule emphasizes that, in order to differentiate $f \circ g$, we multiply the derivative of the outer function and the derivative of the inner function, the former evaluated at u, the latter at x.

Example 1: (Online Homework HW13, # 3)

Let $F(x) = f(f(x))$ and $G(x) = (F(x))^2$ and suppose that

$$f(5) = 3 \qquad f(7) = 5 \qquad f'(5) = 8 \qquad f'(7) = 13$$

Find $F'(7)$ and $G'(7)$.

$f(5) = 3 \qquad f(7) = 5 \qquad f'(5) = 8 \qquad f'(7) = 13$

(1) $\boxed{F(x) = f(f(x))}$

By the chain rule $\quad F'(x) = f'(f(x)) \cdot f'(x)$

Thus $F'(7) = f'(f(7)) \cdot f'(7) = f'(5) \cdot f'(7)$

$= 8 \cdot 13 = \underline{\underline{104}}$

(2) $\boxed{G(x) = (F(x))^2}$

By the (power) chain rule $\quad G'(x) = 2\, F(x) \cdot F'(x)$

Thus $\quad G'(7) = 2\, F(7) \cdot F'(7) = 2 \cdot 3 \cdot 104 = \underline{\underline{624}}$

$F(7) = f(f(7)) = f(5) = 3$ →

Example 2: (Online Homework HW13, # 6)

Let $f(x) = \dfrac{9}{(2x^2 - 3x + 6)^4}$. Find $f'(x)$.

$$f(x) = \frac{9}{(2x^2-3x+6)^4} = 9(2x^2-3x+6)^{-4}$$

Hence, by the (power) chain rule

$$f'(x) = 9\cdot(-4)(2x^2-3x+6)^{-4-1}\cdot(4x-3)$$

$$= \boxed{\frac{-36(4x-3)}{(2x^2-3x+6)^5}}$$

The Quotient Rule Using the Chain Rule

We can prove quotient rule using the product and (power) chain rules. Treat the quotient f/g as a product of f and the reciprocal of g. I.e.,

$$\frac{f(x)}{g(x)} = f(x) \cdot g(x)^{-1}.$$

Next, apply the product rule

$$\left(\frac{f(x)}{g(x)}\right)' = [f(x) \cdot g(x)^{-1}]' = f'(x) \cdot g(x)^{-1} + f(x) \cdot [g(x)^{-1}]'$$

and apply the (power) chain rule to find $[g(x)^{-1}]'$. We obtain

$$= f'(x) \cdot [g(x)^{-1}] + f(x) \cdot [(-1)g(x)^{-2} \cdot g'(x)].$$

Finish by writing the expression with a common denominator of $[g(x)]^2$

$$= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}.$$

Example 3: (Neuhauser, Example # 5, p. 161)

Find the derivative of $h(x) = \left(\dfrac{x}{x+1}\right)^2$.

$$h(x) = \left(\frac{x}{x+1}\right)^2$$

By the (power) chain rule

$$h'(x) = 2\left(\frac{x}{x+1}\right)^{2-1} \cdot \frac{d}{dx}\left(\frac{x}{x+1}\right)$$

$$= 2 \cdot \frac{x}{x+1} \cdot \frac{1\cdot(x+1) - x(1)}{(x+1)^2}$$

$$\boxed{= 2 \cdot \frac{x}{(x+1)^3}}$$

Example 4: (Neuhauser, Problem # 32, p. 172)

Differentiate $g(N) = \dfrac{N}{(k + bN)^3}$ with respect to N.

Assume that b and k are positive constants.

$$g(N) = \frac{N}{(k + bN)^3}$$

Use the quotient rule and the chain rule:

$$g'(N) = \frac{1 \cdot (k+bN)^3 - N \cdot 3\,(k+bN)^{3-1} \cdot (b)}{\left[(k+bN)^3\right]^2}$$

$$= \frac{(k+bN)^3 - 3bN\,(k+b)^2}{(k+bN)^6}$$

$$= \frac{(k+bN)^2\left[\,k+bN - 3bN\,\right]}{(k+bN)^6}$$

$$= \boxed{\frac{k - 2bN}{(k+bN)^4}}$$

Example 5: (Neuhauser, Problem # 39, p. 172)

Find the derivative of

$$\frac{[f(x)]^2}{g(2x) + 2x}$$

assuming that f and g are both differentiable functions.

$$y = \frac{[f(x)]^2}{g(2x) + 2x}$$

We find y' using the quotient rule and the chain rule:

$$y' = \frac{\{[f(x)]^2\}' \cdot (g(2x) + 2x) - [f(x)]^2 \cdot \{g(2x) + 2x\}'}{(g(2x) + 2x)^2}$$

$$= \frac{2f(x) \cdot f'(x) \cdot (g(2x) + 2x) - [f(x)]^2 \cdot (g'(2x) \cdot 2 + 2)}{(g(2x) + 2x)^2}$$

$$= \boxed{\frac{2f(x) \cdot [f'(x)(g(2x) + 2x) - f(x)(g'(2x) + 1)]}{[g(2x) + 2x]^2}}$$

Higher Derivatives

- The derivative of a function f is itself a function. We refer to this derivative as the **first derivative**, denoted f'. If the first derivative exists, we say that the function is once differentiable.
- Given that the first derivative is a function, we can define its derivative (where it exists). This derivative is called the **second derivative** and is denoted f''. If the second derivative exists, we say that the original function is twice differentiable.
- This second derivative is again a function; hence, we can define its derivative (where it exists). The result is the **third derivative**, denoted f'''. If the third derivative exists, we say that the original function is three times differentiable.
- We can continue in this manner; from the fourth derivative on, we denote the derivatives by $f^{(4)}, f^{(5)}$, and so on.
  If the nth derivative exists, we say that the original function is n times differentiable.

- Polynomials are functions that can be differentiated as many times as desired. The reason is that the first derivative of a polynomial of degree n is a polynomial of degree $n-1$. Since the derivative is a polynomial as well, we can find its derivative, and so on. Eventually, the derivative will be equal to 0.
- We can write higher-order derivatives in Leibniz notation: The nth derivative of $f(x)$ is denoted by

$$\frac{d^n f}{dx^n}$$

Example 6: (Online Homework HW13, # 4)

Find the first and second derivatives of the following function

$$f(x) = (5 - 3x^2)^4$$

$$f(x) = (5 - 3x^2)^4$$

Then :

$$f'(x) = 4(5-3x^2)^3 \cdot (-6x)$$

$$= -24x(5-3x^2)^3$$

$$f''(x) = -24(5-3x^2)^3 - 24x \cdot \left[3(5-3x^2)^2 \cdot (-6x)\right]$$

$$= -24(5-3x^2)^3 + 18 \cdot 24 x^2 (5-3x^2)^2$$

$$= 24(5-3x^2)^2 \cdot \left[-(5-3x^2) + 18x^2\right]$$

$$= \boxed{24(5-3x^2)^2 \cdot (21x^2 - 5)}$$

Example 7: (Online Homework HW13, # 16)

Find the first and second derivatives of the following function

$$y = \frac{1 - 4u}{1 + 3u}$$

$$y = \frac{1-4u}{1+3u}$$

$$y' = \frac{-4(1+3u) - (1-4u)(3)}{(1+3u)^2}$$

$$= \frac{-4 - \cancel{12u} - 3 + \cancel{12u}}{(1+3u)^2} = \boxed{\frac{-7}{(1+3u)^2}}$$

$$= -7(1+3u)^{-2}$$

$$y'' = -7(-2)(1+3u)^{-2-1} \cdot (3)$$

$$= \boxed{\frac{42}{(1+3u)^3}}$$

Velocity and Acceleration

The velocity of an object that moves on a straight line is the derivative of the objects position. The derivative of the velocity is the acceleration.

If $s(t)$ denotes the position of an object moving on a straight line, $v(t)$ its velocity, and $a(t)$ its acceleration, then the three quantities are related as follows:

$$v(t) = \frac{ds}{dt} \qquad \text{and} \qquad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Example 8: (Neuhauser, Problem # 87, p. 173)

Neglecting air resistance, the height h (in meters) of an object thrown vertically from the ground with initial velocity v_0 is given by

$$h(t) = v_0 t - \frac{1}{2} g t^2$$

where $g = 9.81 \mathrm{m/s}^2$ is the earth's gravitational constant and t is the time (in seconds) elapsed since the object was released.

(a) Find the velocity and the acceleration of the object.

(b) Find the time when the velocity is equal to 0. In which direction is the object traveling right before this time? in which direction right after this time?

$$h(t) = v_0 t - \tfrac{1}{2} g t^2$$

(a) $$v(t) = h'(t) = \frac{dh}{dt} = v_0 - \frac{1}{2} g \cdot 2t$$

$$= \boxed{v_0 - gt}$$

$$a(t) = v'(t) = h''(t) = \frac{d^2h}{dt^2} = \boxed{-g}$$

(b) $$v(t) = 0 \iff v_0 - gt = 0 \iff$$

$$\boxed{t = \frac{v_0}{g}}$$

Before $\frac{v_0}{g}$ we have that $v(t)$ is positive so the object goes up; after $\frac{v_0}{g}$ the velocity is negative so the object goes down.

MA 137 – Calculus 1 with Life Science Applications
Implicit Differentiation
(Section 4.4)

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Implicit Differentiation

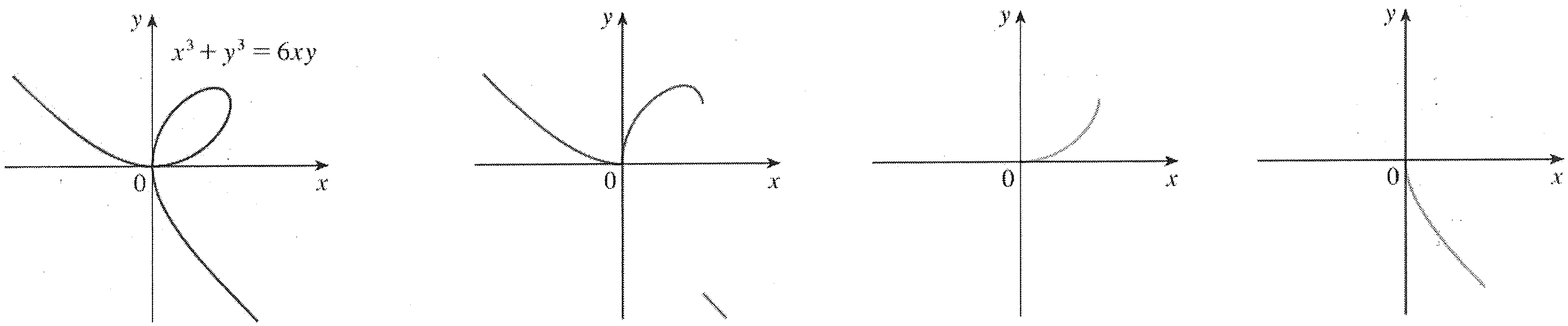
So far, we have considered only functions of the form $y = f(x)$, which define y explicitly as a function of x.

It is also possible to define y implicitly as a function of x, as in the following equation:

$$x^3 + y^3 = 6xy \tag{1}$$

Here, y is still given as a function of x (i.e., y is the dependent variable), but there is no obvious way to solve for y.

Below are the graphs of three such functions related to equation (1), dubbed the folium of Descartes.



When we say that f is implicitly defined by the equation given in (1), we mean that the equation

$$x^3 + [f(x)]^3 = 6\,x\,f(x)$$

is true for all values of x in the domain of f.

Fortunately, there is a very useful technique, based on the chain rule, that will allow us to find dy/dx for implicitly defined functions.

This technique is called **implicit differentiation**.

We summarize the steps we take to find dy/dx when an equation defines y implicitly as a differentiable function of x:

1. Differentiate both sides of the equation with respect to x, keeping in mind that y is a function of x.

   [**Note:** differentiating terms involving y typically requires the chain rule.]
2. Solve the resulting equation for dy/dx.

Example 1:

(a) Find y' if y is implicitly defined by $x^3 + y^3 = 6xy$.

(b) Find an equation for the tangent line to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

(a) Consider $x^3 + y^3 = 6xy$. Take $\frac{d}{dx}$ of both sides:

$$\frac{d}{dx}\left[x^3 + y^3\right] = \frac{d}{dx}(6xy)$$

$$\Leftrightarrow \quad \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 6\frac{d}{dx}(xy)$$

$$\Leftrightarrow \quad 3x^2 + \underbrace{\frac{d}{dy}(y^3)\cdot\frac{dy}{dx}}_{\text{chain rule}} = \underbrace{6\cdot y + 6x\frac{dy}{dx}}_{\text{product rule}}$$

$$\Leftrightarrow \quad 3x^2 + 3y^2\cdot\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$\Leftrightarrow \quad 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

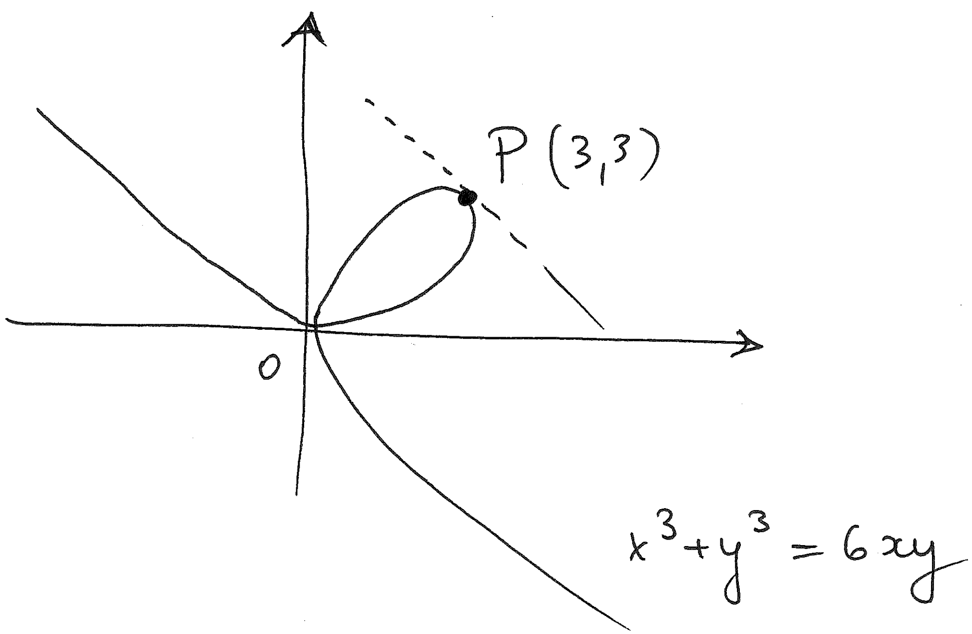
$$\Leftrightarrow \quad \boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

(b)

$$\left.\frac{dy}{dx}\right|_{P(3,3)} = \frac{2(3) - 3^2}{3^2 - 2\cdot 3} = \frac{6-9}{9-6} = \frac{-3}{3} = \boxed{-1}$$

Hence the equation of the tangent line at $P(3,3)$ is

$$y - 3 = (-1)(x-3) \qquad \boxed{y = -x + 6}$$



Example 2: (Online Homework HW14, # 2)

Given $xy + 2x + 3x^2 = -4$:

(a) Find y' by implicit differentiation.

(b) Solve the equation for y and differentiate to get y' in terms of x.
(The answers should be consistent!)

(a) Given $xy + 2x + 3x^2 = -4$. Take $\frac{d}{dx}$ of both sides

$$\frac{d}{dx}\left[xy + 2x + 3x^2\right] = \frac{d}{dx}(-4)$$

$$\frac{d}{dx}(xy) + 2\frac{d}{dx}(x) + 3\frac{d}{dx}(x^2) = 0$$

$$1 \cdot y + x\frac{dy}{dx} + 2 + 3(2x) = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-2 - 6x - y}{x}}$$

(b) $xy = -2x - 3x^2 - 4 \Rightarrow y = \frac{-2x - 3x^2 - 4}{x}$

So explicitely $\boxed{y = -2 - 3x - \frac{4}{x}}$

$$y' = -3 - 4(-1)x^{-2} = \boxed{-3 + \frac{4}{x^2}}$$

Example 3: (Neuhauser, Problem # 54, p. 172)

Find dy/dx by implicit differentiation if

$$\frac{x}{xy+1} = 2xy.$$

$$\frac{x}{(xy+1)} = 2xy \iff x = 2xy(xy+1)$$

$$\iff x = 2x^2y^2 + 2xy .$$

Take the derivative of both sides w.r.t. x :

$$\frac{d}{dx}[x] = \frac{d}{dx}\left[2x^2y^2 + 2xy\right]$$

$$1 = 4xy^2 + 2x^2\left(2y\frac{dy}{dx}\right) + 2\cdot 1\cdot y + 2x\frac{dy}{dx}$$

$$1 - 4xy^2 - 2y = 4x^2y\frac{dy}{dx} + 2x\frac{dy}{dx}$$

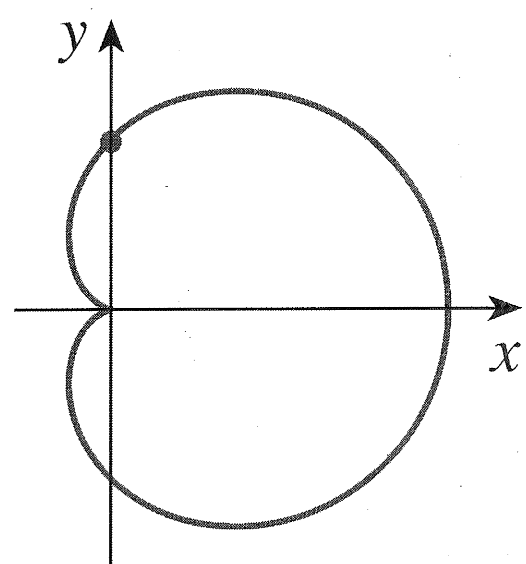
hence $$\boxed{\frac{dy}{dx} = \frac{1-4xy^2-2y}{4x^2y+2x}}$$

Example 4: (Online Homework HW14, # 6)

Use implicit differentiation to find an equation of the tangent line to the curve (called **cardioid**)

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

at the point $(0, 1/2)$.



$$x^2+y^2=(2x^2+2y^2-x)^2 \qquad \text{cardioid}$$

We need the tangent line at $P(0,1/2)$.

We need the slope: so $\left.\frac{dy}{dt}\right|_P = ?$

We use implicit differentiation:

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}\left[(2x^2+2y^2-x)^2\right]$$

$$\Longleftrightarrow$$

$$2x+\frac{d}{dx}(y^2)=\underbrace{2(2x^2+2y^2-x)^{2-1}}_{\text{chain rule}}\cdot\frac{d}{dx}(2x^2+2y^2-x)$$

$$\Longleftrightarrow$$

$$2x + \underbrace{2y \cdot \frac{dy}{dx}}_{\text{chain rule}} = 2(2x^2+2y^2-x) \cdot \left[4x + 4y\frac{dy}{dx} - 1\right]$$

factor

$$2y\frac{dy}{dx} - 8y(2x^2+2y^2-x)\frac{dy}{dx} = 2(2x^2+2y^2-x)(4x-1) - 2x$$

$$\Longleftrightarrow$$

$$\frac{dy}{dx}\left(2y - 8y(2x^2+2y^2-x)\right) = 2(2x^2+2y^2-x)(4x-1) - 2x$$

simplify 2 on both sides

$$\therefore \boxed{\frac{dy}{dx} = \frac{(2x^2+2y^2-x)(4x-1) - x}{y - 4y(2x^2+2y^2-x)}}$$

Evaluate the derivative when $x = 0$ and $y = 1/2$

$$\left.\frac{dy}{dx}\right|_{P} = \frac{\left(2\left(\tfrac{1}{2}\right)^2\right)(-1) - 0}{\tfrac{1}{2} - 4\left(\tfrac{1}{2}\right)\left(2\left(\tfrac{1}{2}\right)^2\right)} = \frac{-1/2}{1/2 - 1}$$

$$= \frac{-1/2}{-1/2} = \boxed{1}$$

Thus the equation of the tg. line is

$$\boxed{y - \tfrac{1}{2} = 1\cdot(x - 0)} \quad \text{or} \quad \boxed{y = x + \tfrac{1}{2}}$$

Power Rule for Rational Exponents

We now provide a proof of the generalized form of the power rule when the exponent r is a rational number: $\dfrac{d}{dx}(x^r) = r\,x^{r-1}$.
We write $r = p/q$, where p and q are integers and are in lowest terms. (If q is even, we require x and y to be positive.) Then

$$y = x^r \qquad \Longleftrightarrow \qquad y = x^{p/q} \qquad \Longleftrightarrow \qquad y^q = x^p.$$

Differentiating both sides of $y^q = x^p$ with respect to x, we find that

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

Hence

$$\begin{aligned}\frac{dy}{dx} &= \frac{p}{q}\,\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\,\frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q}\,\frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q}\,x^{p-1-(p-p/q)}\\ &= \frac{p}{q}\,x^{p/q-1} = r\,x^{r-1}\end{aligned}$$

MA 137 – Calculus 1 with Life Science Applications
Related Rates
(Section 4.4)

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Related Rates

An important application of implicit differentiation is related-rates problems.

In a related-rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured).

For instance, suppose that y is a function of x and both y and x depend on time. If we know how x changes with time (i.e., if we know dx/dt), then we might want to know how y changes with time (i.e, dy/dt).

It is almost always better to use Leibniz's notation $\frac{dy}{dt}$, if we are differentiating, for instance, the function y with respect to time t. The y' notation is more ambiguous when working with rates and should therefore be avoided.

Neuhauser, p. 167 — ($\approx$ Online Homework HW14, # 9)

Consider a parcel of air rising quickly in the atmosphere. The parcel expands without exchanging heat with the surrounding air.

Laws of physics tell us that the volume (V) and the temperature[1] (T) of the parcel of air are related via the formula

$$TV^{\gamma-1} = C$$

where γ is approximately 1.4 for sufficiently dry air and C is a constant.

To determine how the temperature of the air parcel changes as it rises, we implicitly differentiate $TV^{\gamma-1} = C$ with respect to time t:

$$\frac{dT}{dt}V^{\gamma-1} + T(\gamma-1)V^{\gamma-2}\frac{dV}{dt} = 0 \qquad \text{or} \qquad \frac{dT}{dt} = -(\gamma-1)\frac{T}{V}\frac{dV}{dt}.$$

Since rising air expands with time, we express this relationship as $dV/dt > 0$. We conclude then that the temperature decreases (i.e., $dT/dt < 0$), since both T and V are positive and $\gamma \approx 1.4$.

[1]The temperature is measured in kelvin, a scale chosen so that the temperature is always positive. The Kelvin scale is the absolute temperature scale.

Related-rates Problems Guideline

1. Read the problem and identify the variables.

   Time is often an understood variable. If the problem involves geometry, draw a picture and label it. Label anything that does not change with a constant. Label anything that does change with a variable.

2. Write down which derivatives you are given.

   Use the units to help you determine which derivatives are given. The word "per" often indicates that you have a derivative.

3. Write down the derivative you are asked to find.

   "How fast..." or "How slowly..." indicates that the derivative is with respect to time.

4. Look at the quantities whose derivatives are given and the quantity whose derivative you are asked to find. Find a relationship between all of these quantities.

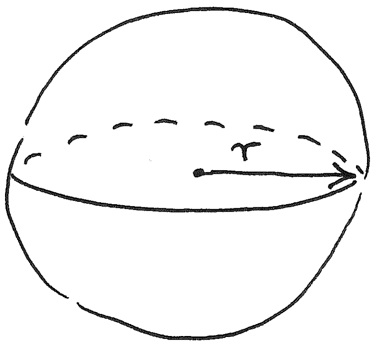
5. Use the chain rule to differentiate the relationship.

6. Substitute any particular information the problem gives you about values of quantities at a particular instant and solve the problem.

   To find all of the values to substitute, you may have to use the relationship you found in step **4**. That is, take a snapshot of the picture at that particular instant.

Example 1: (Online Homework HW14, # 7)

A spherical balloon is inflated so that its volume is increasing at the rate of 2.1 ft^3/min. How rapidly is the diameter of the balloon increasing when the diameter is 1.5 feet?



V = volume of the spherical balloon

r = radius of the balloon

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 2.1 \ ft^3/min$$

We have and seek information in terms of the diameter D of the spherical balloon: $D = 2r$

So $V = \frac{4}{3}\pi\left[\frac{D}{2}\right]^3 = \frac{4}{3}\pi\frac{D^3}{8} = \frac{\pi D^3}{6}$

Take derivative with respect to time:

$$\frac{dV}{dt} = \frac{\pi}{6} 3D^2 \cdot \frac{dD}{dt} \quad \therefore \quad \frac{dV}{dt} = \frac{\pi}{2}D^2 \cdot \frac{dD}{dt}$$

or

$$\frac{dD}{dt} = \frac{2}{\pi D^2} \cdot \frac{dV}{dt}$$

Substitute our data in

$$\frac{dD}{dt} = \frac{2}{\pi D^2} \cdot \frac{dV}{dt}$$

$\frac{dV}{dt} = 2.1 \ ft^3/min$ and $D = 1.5$ feet

$$\therefore \quad \frac{dD}{dt} \text{ at that time } = \frac{2}{\pi(1.5)^2} \cdot 2.1 \ ft/min$$

$$= 3.008 \ ft/min$$

Example 2: (Online Homework HW14, # 11)

Brain weight B as a function of body weight W in fish has been modeled by the power function $B = .007W^{2/3}$, where B and W are measured in grams.

A model for body weight as a function of body length L (measured in cm) is $W = .12L^{2.53}$.

If, over 10 million years, the average length of a certain species of fish evolved from 15cm to 20cm at a constant rate, how fast was the species' brain growing when the average length was 18cm?

[**Note:** 1 nanogram (ng) $= 10^{-9}$ g.]

$$B = 0.007\, W^{2/3}$$

B = brain weight in g.
W = body weight

Moreover $W = .12\, L^{2.53}$
where L is the body length

We can substitute and relate directly the brain weight and the body length:

$$B = 0.007\left(0.12\, L^{2.53}\right)^{2/3} = 0.001703 \cdot L^{1.6867}$$

Hence

$$\frac{dB}{dt} = 0.001703 \cdot 1.6867\, L^{0.6867} \cdot \frac{dL}{dt}$$

$$\therefore \quad \boxed{\frac{dB}{dt} = 0.00287245 \cdot L^{0.6867} \frac{dL}{dt}}$$

We need to plug in our data.

The value for L is 18 cm. What about $\frac{dL}{dt}$?

Since the average length of this species of fish has evolved at a constant rate from 15 cm to 20 cm over 10 million years:

$$\frac{dL}{dt} = \frac{20-15}{10 \text{ million}} = \frac{5}{10^7} = \underline{5 \cdot 10^{-7}}$$

Hence

$$\frac{dB}{dt} = 0.00287245\,(18)^{0.6867} \cdot 5 \cdot 10^{-7}$$

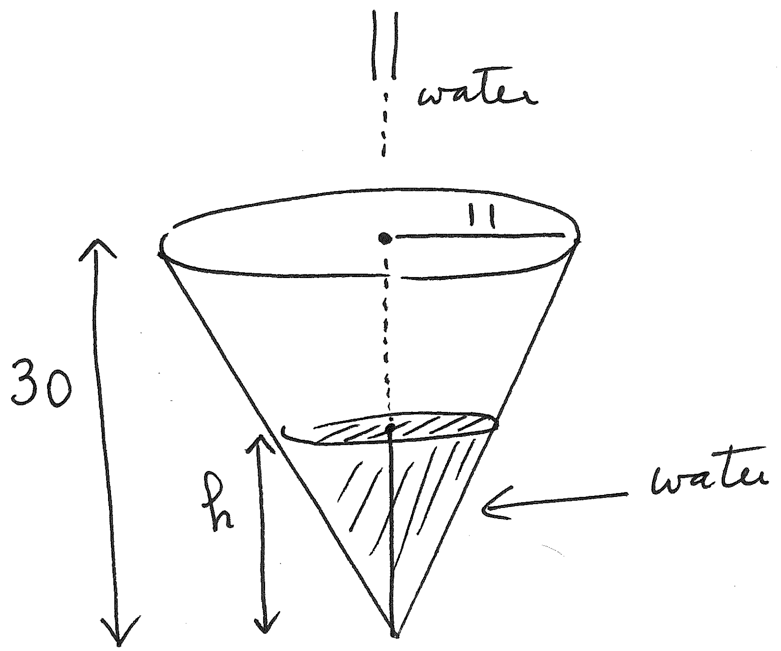
$$= 0.1045245 \cdot 10^{-7} \text{ g/year}$$

$$= 10.45245 \; 10^{-9} \text{ g/year} = 10.45245 \frac{\text{ng}}{\text{year}}$$

ng = nanograms

Example 3: (Online Homework HW14, # 13)

A conical water tank with vertex down has a radius of 11 feet at the top and is 30 feet high.
If water flows into the tank at a rate of 10 ft^3/min, how fast is the depth of the water increasing when the water is 18 feet deep?



water flow in at a rate of $10\ ft^3/min$: $\boxed{\dfrac{dV}{dt} = 10}$

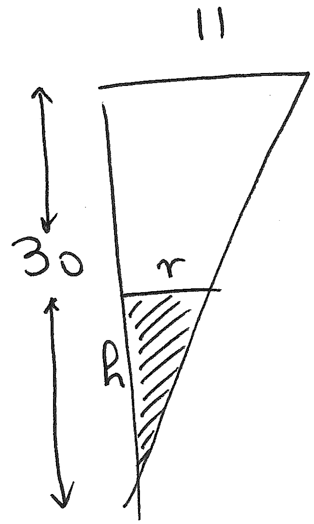
h = height of water

$\boxed{\dfrac{dh}{dt} = ?}$

Need to find a relation between V = volume of water and the height h.

$V = \text{cone} = \frac{1}{3}\pi \cdot r^2 \cdot h$ when r is the radius of the cone.

There is a relation between the h and r in this cone.



there are two similar triangles

$$\frac{30}{11} = \frac{h}{r}$$

ratio of corresponding sides is the same!

Thus $\boxed{r = \frac{11}{30} h}$

Hence $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left[\frac{11}{30} h\right]^2 \cdot h = \frac{121\pi}{2700} \cdot h^3$

From $V = \frac{121}{2700} \cdot \pi h^3$ we get $\frac{dV}{dt} = \frac{121}{2700}\pi \frac{d}{dt}\left[h^3\right]$

and by the chain rule

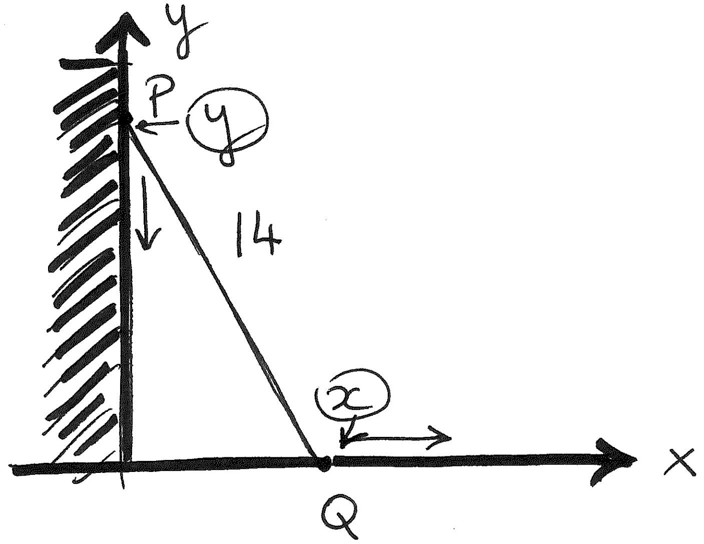
$$\frac{dV}{dt} = \frac{121}{\cancel{2700}_{900}}\pi\, \cancel{3} h^2 \frac{dh}{dt}$$

$$\therefore \boxed{\frac{dh}{dt} = \frac{900}{121\pi} \cdot \frac{1}{h^2}\frac{dV}{dt}}$$

and with our data: $\frac{dh}{dt} = \frac{900}{121\pi} \cdot \frac{1}{18^2} \cdot 10 = 0.07307 \frac{ft}{min}$

Example 4: (Online Homework HW14, # 17)

A 14 foot ladder is leaning against a wall. If the top slips down the wall at a rate of 3 ft/s, how fast will the foot be moving away from the wall when the top is 11 feet above the ground?



$P(0,y)$ denotes the top of the ladder

$Q(x,0)$ denotes the foot of the ladder

$$\frac{dy}{dt} = -3 \text{ ft/sec} \qquad \frac{dx}{dt} = ? \text{ ft/sec}$$

Relationship between the quantities:

$$x^2 + y^2 = 14^2$$

By Pythagoras' theorem

Use implicit differentiation

$$\frac{d}{dt}\left[x^2 + y^2\right] = \frac{d}{dt}(14^2)$$

$$\frac{d}{dx}(x^2)\frac{dx}{dt} + \frac{d}{dy}(y^2) \cdot \frac{dy}{dt} = 0$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\therefore \quad \boxed{\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}}$$

With our data: top is 11 feet above the ground so $y = 11$. At that time the foot will be:

$$x^2 + 11^2 = 14^2 \implies x = \sqrt{14^2 - 11^2} = \sqrt{75}$$

Hence $$\frac{dx}{dt} = -\frac{11}{\sqrt{75}} \cdot (-3) = \frac{33}{\sqrt{75}} \cong \boxed{3.81 \text{ ft/sec}}$$

MA 137 — Calculus 1 with Life Science Applications
Derivatives of Trigonometric Functions
(Section 4.5)

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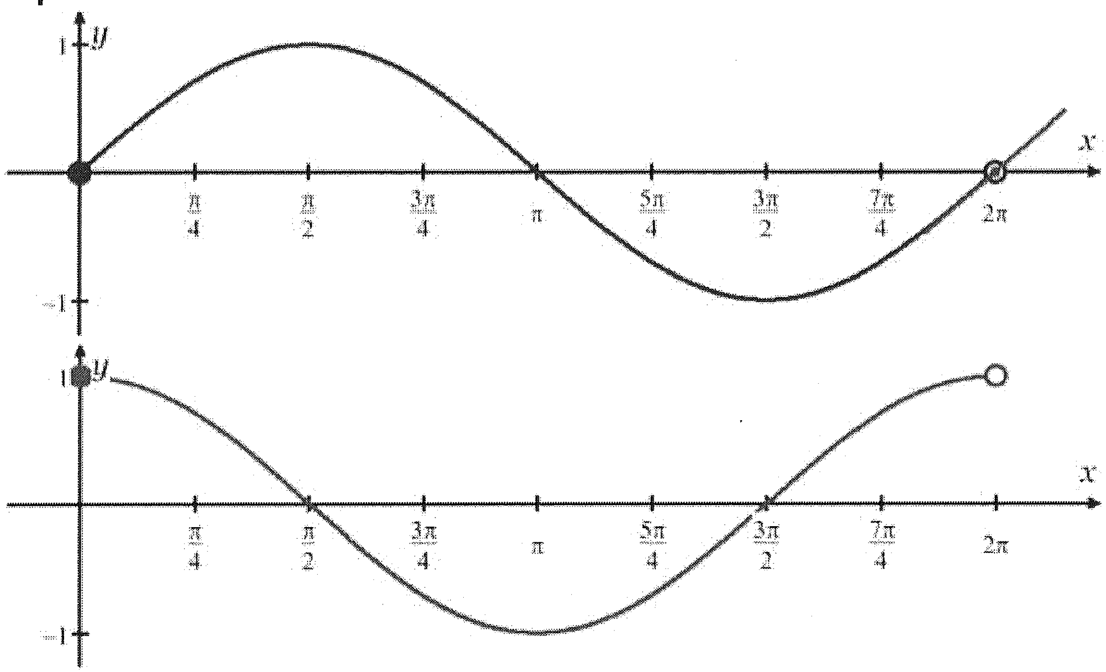
Cyclic phenomena are most easily modeled by sines and cosines:

- length of day;
- length of season;
- some population models (e.g. ideal predator-prey models).

We need to know how fast they change.
Let's compare $\sin x$ and $\cos x$:



The Derivative of Sine and Cosine

Theorem

The functions $\sin x$ *and* $\cos x$ *are differentiable for all* x*, and*

$$\frac{d}{dx}\sin x = \cos x \quad \textit{and} \quad \frac{d}{dx}\cos x = -\sin x$$

We need the trigonometric limits from Section 3.4 to compute the derivatives of the sine and cosine functions. Namely,

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$$

We also need the addition formulas for sine and cosine

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Note that all angles are measured in radians.

Proof for Cosine

We use the formal definition of derivatives:

$$\begin{aligned}
\frac{d}{dx}\cos x &\overset{\text{def}}{=} \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&\overset{\text{add. form.}}{=} \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\,(\cos h-1)-\sin x\sin h}{h} \\
&= \lim_{h\to 0}\left[\cos x\,\frac{\cos h-1}{h}-\sin x\,\frac{\sin h}{h}\right] \\
&\overset{\text{laws}}{=} \cos x\lim_{h\to 0}\frac{\cos h-1}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h} \\
&\overset{\text{fund. lim.}}{=} \cos x\cdot 0-\sin x\cdot 1 \\
&= -\sin x
\end{aligned}$$

Proof for Sine

We use the formal definition of derivatives:

$$
\begin{aligned}
\frac{d}{dx}\sin x &\overset{def}{=} \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&\overset{\text{add. form.}}{=} \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\,(\cos h-1)+\cos x\sin h}{h} \\
&= \lim_{h\to 0}\left[\sin x\,\frac{\cos h-1}{h}+\cos x\,\frac{\sin h}{h}\right] \\
&\overset{\text{laws}}{=} \sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h} \\
&\overset{\text{fund. lim.}}{=} \sin x\cdot 0+\cos x\cdot 1 \\
&= \cos x
\end{aligned}
$$

Derivatives of Remaining Trigonometric Functions

The derivatives of the other trigonometric functions can be found using the following identities and the quotient rule:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

For example: $\frac{d}{dx}(\tan x) = \cdots = \sec^2 x = 1 + \tan^2 x.$

Example 1: (Online Homework HW15, # 3)

Find the equation of the tangent line to the curve $y = 6x \cos x$ at the point $(\pi, -6\pi)$.

$y = 6x \cos x$

We want the equation of the tangent line at $P(\pi, -6\pi)$.

Notice $y(\pi) = 6 \cdot \pi \cdot \underbrace{\cos(\pi)}_{-1} = -6\pi$ ✓

We need the derivative evaluated at $x = \pi$:

$y' = 6 \cdot 1 \cdot \cos x + 6x \cdot (-\sin x)$

$= 6 \cos x - 6x \sin x$

$y'\big|_{x=\pi} = 6 \cdot \underbrace{\cos \pi}_{-1} - 6\pi \underbrace{\sin(\pi)}_{=0} = \boxed{-6}$

Hence $\boxed{y - (-6\pi) = -6(x - \pi)}$

OR $y = -6x + \cancel{6\pi} - \cancel{(6\pi)}$ $\therefore$ $\boxed{y = -6x}$

Example 2: (Online Homework HW15, # 4)

(a) Let $f(x) = \sin^3(x)$. Find $f'(x)$.

(b) Let $g(x) = \sin(x^3)$. Find $g'(x)$.

(a) $f(x) = \sin^3(x) = [\sin(x)]^3$

that's the meaning

So, using the power chain rule, we get

$f'(x) = 3[\sin(x)]^{3-1} \cdot \cos(x)$

$= \boxed{3\sin^2(x) \cdot \cos(x)}$ Ans

(b) $g(x) = \sin(x^3)$

Using the chain rule we get

$g'(x) = \cos(x^3) \cdot \underbrace{3x^2}$

$= \boxed{3x^2 \cos(x^3)}$

Example 3: (Online Homework HW15, # 7)

Find the derivative of the following function:

$$f(x) = \frac{\cos(2x)}{6 - \sin(2x)}$$

$$f(x) = \frac{\cos(2x)}{6 - \sin(2x)}$$

$$f'(x) = \frac{(\cos(2x))' \cdot (6 - \sin(2x)) - \cos(2x)\,(6 - \sin(2x))'}{[6 - \sin(2x)]^2}$$

$$= \frac{-\sin(2x) \cdot 2\,(6 - \sin(2x)) - \cos(2x) \cdot [-\cos(2x) \cdot 2]}{[6 - \sin(2x)]^2}$$

$$= \frac{-12\sin(2x) + \underbrace{2\sin^2(2x) + 2\cos^2(2x)}_{=2}}{[6 - \sin(2x)]^2}$$

$$= \boxed{\frac{2 - 12\sin(2x)}{[6 - \sin(2x)]^2}}$$

///

Example 4: (Online Homework HW15, # 8)

Find the derivative of the following function:

$$f(x) = \left(x^3 - \cos(6x^2)\right)^5$$

$$f(x) = \left[x^3 - \cos(6x^2)\right]^5$$

We need to use the chain rule to find $f'(x)$:

$$f'(x) = 5\left[x^3 - \cos(6x^2)\right]^{5-1} \cdot \left(x^3 - \cos(6x^2)\right)'$$

$$= 5\left[x^3 - \cos(6x^2)\right]^4 \cdot \left(3x^2 - (-\sin(6x^2) \cdot 12x)\right)$$

$$= \boxed{5\left[x^3 - \cos(6x^2)\right]^4 \cdot \left(3x^2 + 12x\sin(6x^2)\right)}$$

Example 5:

Human heart goes through cycles of contraction and relaxation (called systoles). During cycles, blood pressure goes up and down repeatedly; as heart contracts, pressure rises, and as heart relaxes (for a split second), pressure drops.
Consider approximate function for blood pressure of a patient

$$P(t) = 100 + 20\cos\left(\frac{\pi t}{35}\right) \quad \text{mmHg}$$

where t is measured in minutes . Find and interpret $P'(t)$.

$$P(t) = 100 + 20\cos\left(\frac{\pi t}{35}\right)$$

$$P'(t) = 0 + 20 \cdot \left[-\sin\left(\frac{\pi t}{35}\right) \cdot \frac{\pi}{35}\right]$$

$$= -\frac{\overset{4}{\cancel{20}} \cdot \pi}{\underset{7}{\cancel{35}}} \sin\left(\frac{\pi t}{35}\right)$$

$$= \boxed{-\frac{4\pi}{7} \sin\left(\frac{\pi t}{35}\right)} \quad \text{mm Hg/min}$$

$P'(t)$ is measured in mm Hg/min; it is the change in blood pressure due to normal cycle of the heart.

Example 6: (Online Homework HW15, # 9)

During the human female menstrual cycle, the gonadotropin, FSH or follicle stimulating hormone, is released from the pituitary in a sinusoidal manner with a period of approximately 28 days.
Guyton's text on Medical Physiology shows that if we define day 0 ($t = 0$) as the beginning of menstruation, then FSH, $F(t)$, cycles with a high concentration of about 4.4 (relative units) around day 9 and a low concentration of about 1.2 around day 23.

a. Consider a model of the concentration FSH (in relative units) given by

$$F(t) = A + B\cos(\omega(t - \varphi)),$$

where A, B, ω, and φ (with $0 \leq \varphi \leq 28$) are constants and t is in days. Use the data above to find the four parameters.

If ovulation occurs around day 14, then what is the approximate concentration of FSH at that time?

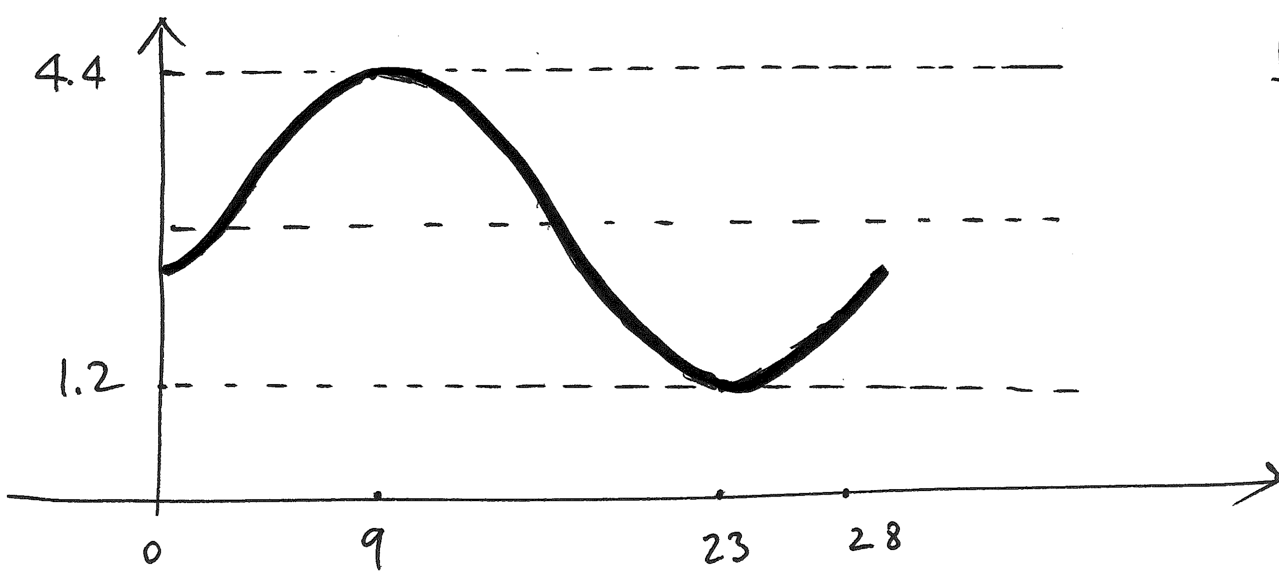
You should sketch a graph of the concentration of FSH over one period.

b. Find the derivative of $F(t)$. Give its value at the time of ovulation ($t = 14$).

We need to find the correct values in

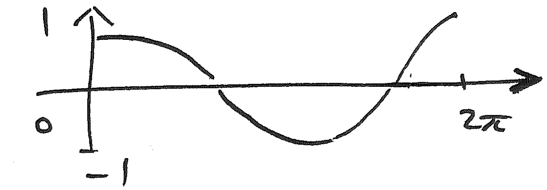
$$F(t) = A + B\cos(\omega(t - \varphi))$$

So that we have a complete cycle in 28 days; the high concentration is around day nine and has value 4.4; the low concentration is around day 23 and has value 1.2.



Here is how the graph should look like!

We need to shift, stretch and translate the graph of cos(t):



A = gives the vertical shift of the x-axis

B = gives half of the amplitude of the cycle.

A is the middle point between 4.4 and 1.2

$= \boxed{2.8} = \dfrac{4.4+1.2}{2}$

$B = \dfrac{4.4-1.2}{2}$ = half of the amplitude $= \boxed{1.6}$

the pick that occurs when $t=0$ for cos(t) now occurs when $t=9$. Thus $\boxed{\varphi = 9}$

gives the horizontal shift to the right

Finally we need to find the frequency "ω":

Notice that $F(t) = F(t+28) = F(t+56) = \ldots$

i.e. the values repeat every 28 days:

Thus: $F(t) = F(t+28)$ gives

$$A + B\cos(\omega(t-\varphi)) = A + B\cos(\omega(t+28-\varphi))$$

$$\Longleftrightarrow$$

$$\cos(\omega(t-\varphi)) = \cos(\omega(t-\varphi) + \omega \cdot 28)$$

Thus $\omega \cdot 28$ must be 2π

$$\therefore \quad \omega = \frac{2\pi}{28} \cong 0.2244$$

Hence $\boxed{F(t) = 2.8 + 1.6\cos(0.2244(t-9))}$

(b) $F'(t) = 1.6\left[-\sin(0.2244(t-9)) \cdot (0.2244)\right]$

$= \boxed{-0.3590\sin(0.2244(t-9))}$

$F'(14) = -0.3590 \cdot \sin(1.122)$

$\cong -0.32345$

$F(14) = 2.8 + 1.6\cos(1.122) \cong 3.49421$

MA 137 — Calculus 1 with Life Science Applications
Derivatives of Exponential Functions
(Section 4.6)

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The Derivative of the Natural Exponential Function

Theorem

The function e^x *is differentiable for all* x, *and* $\frac{d}{dx} e^x = e^x$.

In particular, if $g(x)$ *is a differentiable function, it follows from the chain rule that*

$$\frac{d}{dx} e^{g(x)} = e^{g(x)} \cdot g'(x).$$

We need to know the following limit to compute the derivative of the natural exponential function. Namely,

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Although we cannot rigorously prove this result here, the table below should convince you of its validity

h	-0.1	-0.01	-0.001	$\cdots$	0.001	0.01	0.1
$\frac{e^h - 1}{h}$	0.9516	0.9950	0.9995		1.0005	1.0050	1.0517

Proof

We use the formal definition of the derivative. In the final step, we will be able to write the term e^x in front of the limit because e^x does not depend on h.

$$
\begin{aligned}
\frac{d}{dx} e^x &\stackrel{\text{def}}{=} \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&\stackrel{\text{exp. prop.}}{=} \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h} \\
&\stackrel{\text{laws}}{=} e^x \lim_{h \to 0} \frac{e^h - 1}{h} \\
&\stackrel{\text{fund. lim.}}{=} e^x \cdot 1 \\
&= e^x
\end{aligned}
$$

The Derivative of ANY Exponential Function

Theorem

The function a^x *is differentiable for all* x, *and* $\frac{d}{dx}a^x = a^x \cdot \ln a.$
In particular, if $g(x)$ *is a differentiable function, it follows from the chain rule that*

$$\frac{d}{dx}a^{g(x)} = a^{g(x)} \cdot \ln a \cdot g'(x).$$

We can prove the above result using the definition of the derivative and the limit

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \ln a,$$

in the same manner that we did for the natural exponential function.

Alternatively, we can use the following identity

$$a^x = e^{\ln a^x} = e^{x \ln a}$$

and the chain rule. Namely,

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \ln a = a^x \cdot \ln a.$$

Example 1: (Nuehauser, Example # 1, p. 179)

Find the derivative of $f(x) = e^{-x^2/2}$.

$$f(x) = e^{-x^2/2}$$

We need to use the chain rule

$$f'(x) = e^{-x^2/2} \cdot \frac{d}{dx}\left(-\frac{x^2}{2}\right)$$

$$= e^{-x^2/2} \cdot \left(-\frac{1}{2} \cdot 2x\right) = \boxed{-x e^{-x^2/2}}$$

Example 2:

Find the derivative with respect to x of $g(x) = xe^{-x}$.

Evaluate $g'(x)$ at $x = 1$.

$$g(x) = x e^{-x}$$

$$g'(x) = 1 \cdot e^{-x} + x \cdot \frac{d}{dx}\left(e^{-x}\right)$$

↑ product rule

$$= e^{-x} + x\left[e^{-x}(-1)\right]$$

↑ chain rule

$$= e^{-x} - x e^{-x}$$

$$\boxed{= e^{-x}(1-x)}$$

$$g'(1) = e^{-1} \cdot (1-1) = \underline{\underline{0}}$$

Example 3: (Online Homework HW15, # 14)

The cutlassfish is a valuable resource in the marine fishing industry in China. A von Bertalanffy model is fit to data for one species of this fish giving the length of the fish, $L(t)$ (in mm), as a function of the age, a (in yr). An estimate of the length of this fish is

$$L(a) = 593 - 378e^{-0.166a}.$$

(a) Find the L-intercept.
Find an equation for the horizontal asymptote of $L(a)$.
Find the maximum possible length of this fish.

(b) Determine how long it takes for this fish to reach 90 percent of its maximum length.

(c) Differentiate $L(a)$ with respect to a.

(a) $L(a) = 593 - 378e^{-0.166a}$

To find the L-intercept we set $a = 0$

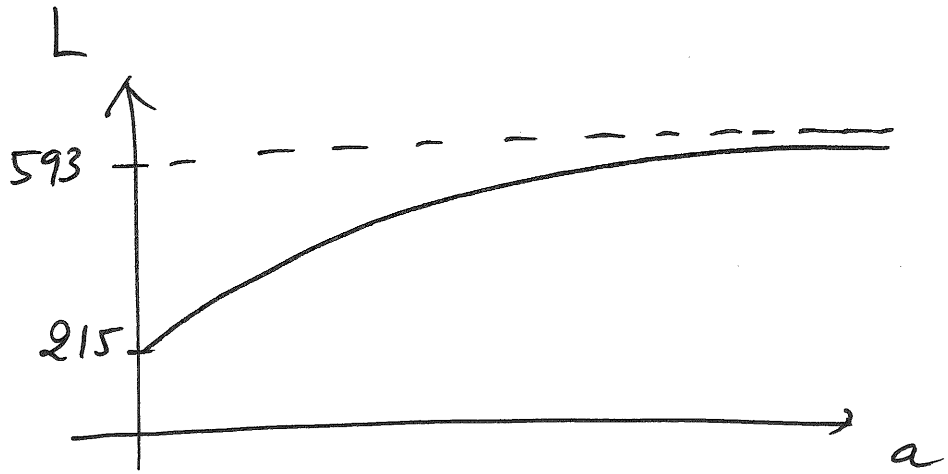
$L(0) = 593 - 378e^{-0.166a} = 593 - 378\underbrace{e^{0}}_{1}$

$= 593 - 378 = \underline{215}$

To get the equation of the horizontal asymptote we need to evaluate $\lim_{a \to \infty} \left(593 - 378e^{-0.166a}\right)$

$= 593 - 378 \underbrace{\lim_{a \to \infty} e^{-0.166a}}_{0} = 593 - 0 = \underline{\underline{593}}$

Hence the maximum possible length of the fish is (close to) 593



graph of $L(a)$

(b) We need to find the age such that

$$\underbrace{0.9 \cdot 593}_{90\% \text{ of max. length}} = L(a) = 593 - 378\, e^{-0.166a}$$

$$\Longleftrightarrow \quad 378\, e^{-0.166a} = 593 - 533.7$$

$$\Longleftrightarrow \quad e^{-0.166a} = \frac{59.3}{378} \cong 0.15688$$

$$\Longleftrightarrow \quad -0.166a = \ln(0.15688)$$

$$\Longleftrightarrow \quad a = \frac{\ln(0.15688)}{-0.166} \cong \underline{\underline{11.1583}}$$

(c) $L(a) = 593 - 378\, e^{-0.166a}$

$$\frac{dL}{da} = L'(a) = 0 - 378 \cdot \underbrace{e^{-0.166a} \cdot (-0.166)}_{\text{chain rule}}$$

$$= 378 \cdot (0.166)\, e^{-0.166a}$$

$$= \underline{\underline{62.748\, e^{-0.166a}}}$$

notice that the derivative is always positive !!

Example 4: (Neuhauser, Example # 5, p. 180)

Radioactive Decay: Show that the function $W(t) = W_0 e^{-rt}$ satisfies the differential equation

$$\frac{dW}{dt} = -rW(t) \qquad W(0) = W_0.$$

[W_0 is the amount of material at time $t = 0$ and r is called the radioactive decay rate.]

$W(t) = W_0 e^{-rt}$

notice that at $t=0$

$$W(0) = W_0 \underbrace{e^{-r \cdot 0}}_{=1} = W_0 \checkmark$$

Let's compute the derivative of $W(t) = W_0 e^{-rt}$

$$\frac{dW}{dt} = W_0 \underbrace{e^{-rt} \cdot (-r)}_{\text{chain rule.}}$$

Substitute in the D.E. $\frac{dW}{dt} = -rW$

$$\underbrace{W_0 e^{-rt}(-r)} \stackrel{?}{=} -r\left(W_0 e^{-rt}\right)$$

the two sides are identical! ✓

Example 5: (Neuhauser, Example # 6, p. 181)

Exponential Growth: Show that the function $N(t) = N_0 e^{rt}$ satisfies the differential equation

$$\frac{dN}{dt} = rN(t) \qquad N(0) = N_0.$$

[N_0 is the population size at time $t = 0$ and r is called the growth rate.]

$N(t) = N_0 e^{rt}$ notice that at $t=0$ we get

$$N(0) = N_0 \underbrace{e^{r \cdot 0}}_{1} = N_0 \quad \checkmark$$

Let's compute the derivative of $N(t) = N_0 e^{rt}$

$$\frac{dN}{dt} = \underbrace{N_0 e^{rt} \cdot (r)}_{\text{chain rule}}$$

Substitute in the D.E. $\frac{dN}{dt} = rN$.

$$N_0 e^{rt}(r) \stackrel{?}{=} r(N_0 e^{rt})$$

the two sides are identical! $\checkmark$

Example 6: (Neuhauser, Problem # 63, p. 182)

(a) Find the derivative of the logistic growth curve (Example 3, Section 3.3, p.112)

$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}}$$

(b) Show that $N(t)$ satisfies the differential equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \qquad N(0) = N_0$$

(c) Plot the per capita rate of growth $\frac{1}{N}\frac{dN}{dt}$ as a function of N, and note that it decreases with increasing population size.

(a) $$N(t) = \frac{K}{1+\left(\frac{K}{N_0}-1\right)e^{-rt}} = K\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right]^{-1}$$

Let's compute the derivative using this form ↑ instead of the quotient rule:

$$\frac{dN}{dt} = N' = K(-1)\cdot\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right]^{-2}\cdot\underbrace{\left(\frac{K}{N_0}-1\right)e^{-rt}(-r)}_{\text{chain rule}}$$

$$= \boxed{\frac{Kr\left(\frac{K}{N_0}-1\right)e^{-rt}}{\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right]^{2}}}$$

Let us substitute the derivative and the function into the D.E $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$

$$\frac{dN}{dt} \stackrel{?}{=} rN\left(1 - N/K\right)$$

$$\frac{Kr\left(K/N_0 - 1\right)e^{-rt}}{\left[1 + \left(K/N_0 - 1\right)e^{-rt}\right]^2} \stackrel{?}{=} r\,\frac{K}{1+\left(K/N_0 - 1\right)e^{-rt}} \cdot \left(1 - \frac{\cancel{K}}{\cancel{K}\left(1+\left(K/N_0 - 1\right)e^{-rt}\right)}\right)$$

$$\stackrel{?}{=} r\,\frac{K}{1+\left(\frac{K}{N_0}-1\right)e^{-rt}} \cdot \frac{\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right] - 1}{1+\left(\frac{K}{N_0}-1\right)e^{-rt}}$$

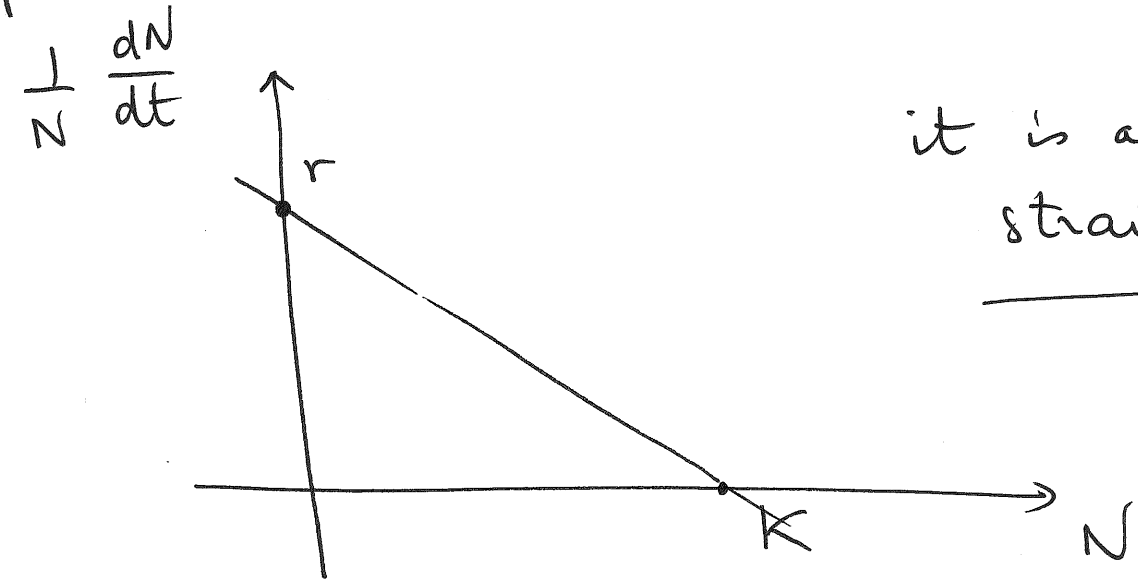
$$\stackrel{?}{=} \frac{rK \cdot \left(K/N_0 - 1\right)e^{-rt}}{\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right]^2}$$

Yes!!

(c) As we have discussed in a previous lecture

$$\frac{1}{N}\frac{dN}{dt} = r - \frac{r}{K}N$$

as a function of N has the following graph :



it is a decreasing straight line !

MA 137 — Calculus 1 with Life Science Applications
Derivatives of Logarithmic Functions and Logarithmic Differentiation
(Section 4.7)

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The Derivative of the Natural Logarithmic Function

Theorem

The function $\ln x$ *is differentiable for all* $x > 0$, *and* $\frac{d}{dx}\ln x = \frac{1}{x}$.

In particular, if $g(x)$ *is a differentiable function, it follows from the chain rule that*

$$\frac{d}{dx}\ln g(x) = \frac{1}{g(x)}\,g'(x).$$

We can use the derivative of e^x and the relationship between the exponential and the natural logarithmic functions to find the derivative of the function $\ln x$. Namely, we start by taking the derivative with respect to x of both sides of $e^{\ln x} = x$. We obtain

$$\frac{d}{dx}e^{\ln x} = \frac{d}{dx}x \iff e^{\ln x}\frac{d}{dx}\ln x = 1 \iff \frac{d}{dx}\ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Alternative Proof

We use the formal definition of the derivative and $e^x = \lim_{u\to\infty}\left(1+\frac{x}{u}\right)^u$

$$
\begin{aligned}
\frac{d}{dx}\ln x &\overset{\text{def}}{=} \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} \\
&\overset{\text{ln prop.}}{=} \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) \\
&= \lim_{h\to 0}\frac{1}{x}\,\frac{x}{h}\ln\left(1+\frac{1}{x/h}\right) && u = x/h \\
&\overset{\text{laws}}{=} \frac{1}{x}\lim_{u\to\infty}\ln\left(1+\frac{1}{u}\right)^u \\
&\overset{\text{cont.}}{=} \frac{1}{x}\ln\left[\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u\right] \\
&= \frac{1}{x}\ln e = \frac{1}{x}
\end{aligned}
$$

The Derivative of ANY Logarithmic Function

Theorem

The function $\log_a x$ *is differentiable for* $x > 0$, *and* $\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x}$.

In particular, if $g(x)$ *is a differentiable function, it follows from the chain rule that*

$$\frac{d}{dx}\log_a g(x) = \frac{1}{(\ln a)g(x)}g'(x).$$

From the base change formula for logarithms we have that

$$\log_a x = \frac{\ln x}{\ln a}$$

Thus it is enough to find the derivative of $\ln x$. Hence the formula.

Example 1: (Nuehauser, Problems # 28/34/52, p. 192)

Find $\dfrac{dy}{dx}$ when $y = \ln(1 - x^2)$.

Find $\dfrac{dy}{dx}$ when $y = \left[\ln(1 - x^2)\right]^3$.

Find $\dfrac{dy}{ds}$ when $y = \ln(\ln s)$.

(a) $y = \ln(1-x^2)$

$$\frac{dy}{dx} = \underbrace{\frac{1}{1-x^2}\cdot(-2x)}_{\text{chain rule}} = \boxed{\frac{-2x}{1-x^2}}$$

(b) $y = \left[\ln(1-x^2)\right]^3$

$$\frac{dy}{dx} = 3\left[\ln(1-x^2)\right]^{3-1}\cdot\left(\ln(1-x^2)\right)'$$

$$= 3\left[\ln(1-x^2)\right]^2\cdot\frac{1}{1-x^2}\cdot(-2x) = \boxed{\frac{-6x\left[\ln(1-x^2)\right]^2}{1-x^2}}$$

(c) $y = \ln(\ln s)$

$$\frac{dy}{ds} = \frac{1}{\ln s}\cdot(\ln s)' = \boxed{\frac{1}{\ln s}\cdot\frac{1}{s}}$$

Example 2: (Nuehauser, Problem # 56, p. 193)

Find $\dfrac{dy}{dx}$ when $y = \log(3x^2 - x + 2)$.

[**Note:** $\log = \log_{10}$]

$$y = \log(3x^2 - x + 2)$$

$$\frac{dy}{dx} = \frac{1}{\ln(10)\cdot(3x^2 - x + 2)} \cdot (6x - 1)$$

Recall the formula:

$$\frac{d}{dx}\left[\log_a g(x)\right] = \frac{1}{[\ln(a)]\cdot g(x)} \cdot g'(x)$$

Example 3: (Nuehauser, Problem # 62, p. 193)

Assume that $f(x)$ is differentiable with respect to x. Show that

$$\frac{d}{dx}\ln\left[\frac{f(x)}{x}\right] = \frac{f'(x)}{f(x)} - \frac{1}{x}$$

$$y = \ln\left[\frac{f(x)}{x}\right]$$

We want to compute $\frac{dy}{dx}$.

1st method direct computation using the chain rule and the quotient rule:

$$y' = \frac{1}{\frac{f(x)}{x}} \cdot \left[\frac{f(x)}{x}\right]' = \frac{x}{f(x)} \cdot \frac{f'(x)\cdot x - f(x)}{x^2}$$

$$= \frac{f'(x)\,x - f(x)}{x\,f(x)} = \frac{f'(x)}{f(x)} - \frac{1}{x}$$

2nd method

$$y = \ln\left[\frac{f(x)}{x}\right] = \ln[f(x)] - \ln x$$

Hence

$$y' = \frac{1}{f(x)} \cdot f'(x) - \frac{1}{x} = \boxed{\frac{f'(x)}{f(x)} - \frac{1}{x}}$$

Logarithmic Differentiation

In1695,Leibniz introduced logarithmic differentiation, following Johann Bernoulli's suggestion to find derivatives of functions of the form

$$y = [f(x)]^x.$$

Bernoulli generalized this method and published his results two years later.

> The **basic idea** is to take logarithms on both sides and then to use implicit differentiation.

Example 4: (Neuhauser, Example # 10, p. 190)

Find $\dfrac{dy}{dx}$ when $y = x^x$.

What about $\dfrac{d}{dx}\left[(2x)^{2x}\right]$?

$$y = x^x$$

to find y' we can take ln of both sides:

$$\ln y = \ln x^x \quad \underline{\text{OR}} \quad \ln y = x \cdot \ln x$$

Hence, now take $\frac{d}{dx}$ of both sides and use the chain rule:

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[x \cdot \ln x\right]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = y\left[\ln x + 1\right] = \underline{x^x\left[\ln x + 1\right]}$$

Alternative method :

$$y = x^x = e^{\ln x^x} = e^{\underline{\underline{x \ln x}}}$$

Hence $$y' = e^{x \ln x} \cdot (x \ln x)'$$

$$= e^{x \ln x} \cdot \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right)$$

$$= e^{\ln x^x} \cdot (\ln x + 1)$$

$$= x^x \cdot (\ln x + 1)$$

same answer as before !!!

About $y = (2x)^{2x}$

1st method: $\ln y = \ln\left[(2x)^{2x}\right] \iff \ln y = 2x \cdot \ln(2x)$

Take the derivative: $\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[2x \cdot \ln(2x)\right]$

$$\iff \frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \ln(2x) + 2x \cdot \left[\frac{1}{2x} \cdot 2\right]$$

$$\frac{dy}{dx} = y\left[2\ln(2x) + 2\right] = (2x)^{2x}\left[2\ln(2x) + 2\right]$$

2nd method: $y = (2x)^{2x} = e^{\ln\left[(2x)^{2x}\right]} = e^{2x \cdot \ln(2x)}$

Hence $y' = e^{2x \cdot \ln(2x)} \cdot \left[2x \cdot \ln(2x)\right]'$

$$\overset{!}{=} e^{\ln(2x)^{2x}} \cdot \left[2\ln(2x) + 2x \cdot \left(\frac{1}{2x} \cdot 2\right)\right]$$

$$= (2x)^{2x} \cdot \left[2\ln(2x) + 2\right]$$

Example 5: (Neuhauser, Problems # 66/73/74, p. 193)

Use logarithmic differentiation to find the first derivative of the functions

$$y = (\ln x)^{3x} \qquad y = x^{\cos x} \qquad y = (\cos x)^{x}$$

(a) $y = (\ln x)^{3x}$

Take $\ln y$ of both sides: $\ln y = \ln\left[(\ln x)^{3x}\right]$

So $\ln y = 3x \cdot \ln[\ln(x)]$. Take $\frac{d}{dx}$ of both sides:

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}\left[3x \cdot \ln[\ln(x)]\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 3 \cdot \ln(\ln(x)) + 3x \cdot \left(\frac{1}{\ln x} \cdot \frac{1}{x}\right)$$

$$\therefore \frac{dy}{dx} = y\left[3\ln(\ln(x)) + \frac{3}{\ln x}\right]$$

$$= (\ln x)^{3x} \cdot \left[3\ln(\ln(x)) + \frac{3}{\ln x}\right]$$

(b) Consider $y = x^{\cos x}$. Take: $\ln y = \ln\left[x^{\cos x}\right]$

$$\iff \quad \ln y = \cos x \cdot \ln x$$

Take $\frac{d}{dx}$ of both sides: $\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\cos x \cdot \ln x\right]$

$$\underbrace{\frac{1}{y}\frac{dy}{dx}}_{\text{chain rule}} = -\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = y \cdot \left[-\sin x \ln x + \frac{\cos x}{x}\right]$$

OR

$$\boxed{\frac{dy}{dx} = x^{\cos x} \cdot \left[-\sin x \ln x + \frac{\cos x}{x}\right]}$$

(c) $\quad y = (\cos x)^{x} \iff \ln y = x \cdot \ln(\cos x)$

$$\implies \frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln(\cos x) + x \cdot \frac{1}{\cos x}(-\sin x)$$

$$\implies \boxed{\frac{dy}{dx} = (\cos x)^{x} \cdot \left[\ln(\cos x) - x \tan x\right]}$$

Example 6: (Neuhauser, Problem # 75, p. 193)

Use logarithmic differentiation to find the first derivative of the function

$$y = \frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}}$$

$$y = \frac{e^{2x} \cdot (9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}}$$

If we try to do the derivative with the quotient rule, it will be complicated.

Let's take "ln" of both sides:

$$\Longleftrightarrow \ln y = \ln\left[\frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}}\right]$$

$$\Longleftrightarrow \ln y = \ln\left[e^{2x}(9x-2)^3\right] - \ln\left[(x^2+1)(3x^3-7)\right]^{1/4}$$

$$\Longleftrightarrow \ln y = \ln(e^{2x}) + \ln\left[(9x-2)^3\right] - \frac{1}{4}\left[\ln\left[(x^2+1)(3x^3-7)\right]\right]$$

$$\Longleftrightarrow \ln y = 2x + 3\ln(9x-2) - \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(3x^3-7)$$

Hence, after expanding, we have that

$$y = \frac{e^{2x} \cdot (9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}} \iff$$

$$\boxed{\ln y = 2x + 3\ln(9x-2) - \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(3x^3-7)}$$

Take the derivative w.r.t x of both sides:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 + 3\frac{1}{9x-2} \cdot (9) - \frac{1}{4}\frac{1}{x^2+1} \cdot (2x) - \frac{1}{4}\frac{1}{3x^3-7} \cdot (9x^2)$$

hence $$\frac{dy}{dx} = y\left[2 + \frac{27}{9x-2} - \frac{x}{2(x^2+1)} - \frac{9x^2}{4(3x^3-7)}\right]$$

or $$\frac{dy}{dx} = \frac{e^{2x}(9x-2)^3}{\sqrt[4]{(x^2+1)(3x^3-7)}} \cdot \left[2 + \frac{27}{9x-2} - \frac{x}{2(x^2+1)} - \frac{9x^2}{4(3x^3-7)}\right]$$

Power Rule (General Form)

Theorem

Let $f(x) = x^r$, where r is any real number. Then

$$\frac{d}{dx} x^r = r\, x^{r-1}$$

Proof: We set $y = x^r$ and use logarithmic differentiation to obtain

$$\begin{aligned} \frac{d}{dx}\left[\ln y\right] &= \frac{d}{dx}\left[\ln x^r\right] \\ \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}\left[r \ln x\right] \\ \frac{1}{y}\frac{dy}{dx} &= r\frac{1}{x} \end{aligned}$$

Solving for dy/dx yields

$$\frac{dy}{dx} = r\frac{1}{x}y = r\frac{1}{x}x^r = r\, x^{r-1}$$

MA 137 — Calculus 1 with Life Science Applications
Linear Approximations
(Section 4.8)

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Tangent Line Approximation

Assume that $y = f(x)$ is differentiable at $x = a$; then

$$L(x) = f(a) + f'(a)(x - a)$$

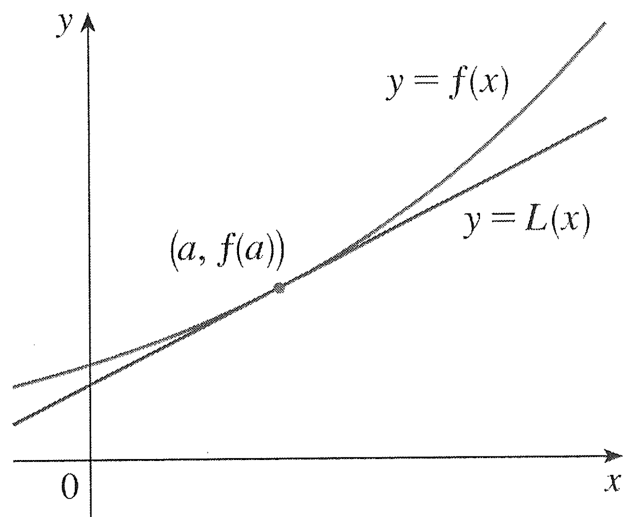
is the tangent line approximation, or **linearization**, of f at $x = a$.

Geometrically, the linearization of f at $x = a$ is the equation of the tangent line to the graph of $f(x)$ at the point $(a, f(a))$.

If $|x - a|$ is sufficiently small, then $f(x)$ can be linearly approximated by $L(x)$; that is,

$$f(x) \approx f(a) + f'(a)(x - a).$$

This approximation is illustrated in the picture on the right:



Example 1: (Nuehauser, Example # 1, p. 194)

(a) Find the linear approximation of $f(x) = \sqrt{x}$ at $x = a$.

(b) use your answer in (a) to find an approximate value of $\sqrt{26}$.

(a) $f(x) = \sqrt{x}$ and $f'(x) = \frac{1}{2\sqrt{x}}$

Thus the linearization of f at $x = a$ is

$$L(x) = f(a) + f'(a)\cdot(x-a)$$

$$\boxed{L(x) = \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a)}$$

(b) In order to approximate $\sqrt{26}$ notice that 26 is close to 25; and we know that $\sqrt{25} = 5$

Thus we choose to linearize $f(x) = \sqrt{x}$ at $\boxed{a = 25}$

$$L(x) = 5 + \frac{1}{10}(x-25)$$

Thus $\sqrt{26} \approx L(26) = 5 + \frac{1}{10}(\underbrace{26-25}_{=1}) = \boxed{5.1}$

Example 2: (Online Homework HW16, # 18)

Find the linearization $L(x)$ of the function $g(x) = x\,f(x^2)$ at $x = 2$ given the following information:

$$f(2) = 1 \qquad f'(2) = 10 \qquad f(4) = 5 \qquad f'(4) = -2$$

We want the linearization of $g(x) = x\,f(x^2)$ at $x = 2$. Thus we need

$$g(2) = 2 \cdot f(2^2) = 2 \cdot f(4) = 2 \cdot 5 = \underline{\underline{10}}$$

and $g'(2)$. For this we need first $g'(x)$:

$$g'(x) = 1 \cdot f(x^2) + x \cdot \underbrace{f'(x^2) \cdot 2x}_{\text{chain rule}}$$

$$= f(x^2) + 2x^2 \cdot f'(x^2)$$

So $g'(2) = f(4) + 8 \cdot f'(4) = 5 + 8(-2) = \underline{\underline{-11}}$

Thus : $L(x) = 10 - 11(x-2)$

$$\overset{!}{=} \underline{\underline{-11x + 32}}$$

Example 3: (Nuehauser, Problem # 34, p. 199)

Plant Biomass: Suppose that a certain plant is grown along a gradient ranging from nitrogen-poor to nitrogen-rich soil.

Experimental data show that the average mass per plant grown in a soil with a total nitrogen content of 1000 mg nitrogen per kg of soil is 2.7 g and the rate of change of the average mass per plant at this nitrogen level is 1.05×10^{-3} g per mg change in total nitrogen per kg soil.

Use a linear approximation to predict the average mass per plant grown in a soil with a total nitrogen content of 1100 mg nitrogen per kg of soil.

We know that $B(1{,}000) = 2.7$ and $\left.\frac{dB}{dt}\right|$

and $B'(1{,}000) = 1.05 \times 10^{-3}$

Thus the linearization is

$$L(x) = 2.7 + 1.05 \times 10^{-3}(x - 1{,}000)$$

Hence the value at $x = 1{,}100$ is

$$B(1{,}100) \approx L(1{,}100) = 2.7 + 1.05 \times 10^{-3}(1{,}100 - 1{,}000)$$

$$= 2.7 + 1.05 \times 10^{-3}(100)$$

$$= 2.7 + 0.105 = \underline{\underline{2.805}}$$

Example 4: (Nuehauser, Example # 3, p. 195)

Suppose $N = N(t)$ represents a population size at time t and the rate of growth as a function of N is $g(N)$.

Find the linear approximation of the growth rate at $N = 0$.

[**Hint:** We can assume that $g(0) = 0$. Indeed, when the population has size $N = 0$, its grow rate will be zero.]

[**Remark:** Your answer should show that for small population sizes, the population grows approximately exponentially.]

$$\frac{dN}{dt} = \text{growth rate} = g(N)$$

We want to find the linearization of the growth rate at $N=0$:

$$L(N) = g(0) + g'(0)\cdot(N-0)$$
$$= g(0) + g'(0)\cdot N$$

By assumption $g(0) = 0$; so $\underline{L(N) = g'(0)N}$

If we set $g'(0) = r$ then

$$\frac{dN}{dt} \approx L(N) = rN$$, which describes an exp. growth

Example 5: (Neuhauser, Problem # 33, p. 199)

Plant Biomass: Suppose that the specific growth rate of a plant is 1%; that is, if $B(t)$ denotes the biomass at time t, then

$$\frac{1}{B(t)}\frac{dB}{dt} = 0.01$$

Suppose that the biomass at time $t = 1$ is equal to 5 grams.

Use a linear approximation to compute the biomass at time $t = 1.1$.

We know that $B(1) = 5$

and that $\dfrac{1}{B(t)} \dfrac{dB}{dt} = 0.01$

so $\left.\dfrac{dB}{dt}\right|_{t=1} = 0.01 \cdot B(1) = 0.01 \cdot 5 = 0.05$

Thus the linearization of the biomass at $t=1$

is: $L(t) = B(1) + \left.\dfrac{dB}{dt}\right|_{t=1} \cdot (t-1)$

$= 5 + 0.05(t-1)$

Hence $B(1.1) \cong L(1.1) = 5 + 0.05\,(1.1-1)$

$= 5 + 0.05\,(0.1) = \boxed{5.005}$ //

Higher Order Approximations

The tangent linear approximation $L(x) = f(a) + f'(a)(x-a)$ is the best first-degree (linear) approximation to $f(x)$ near $x = a$ because $f(x)$ and $L(x)$ have the same value and the same rate of change at a

$$L(a) = f(a) \qquad L'(a) = f'(a).$$

For a better approximation than a linear one, let's try to find better approximations by looking for an nth-degree polynomial

$$T_n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

such that T_n and its first n derivatives have the same value at $x = a$ as f and its first n derivatives at $x = a$.
We can show that the resulting polynomial is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

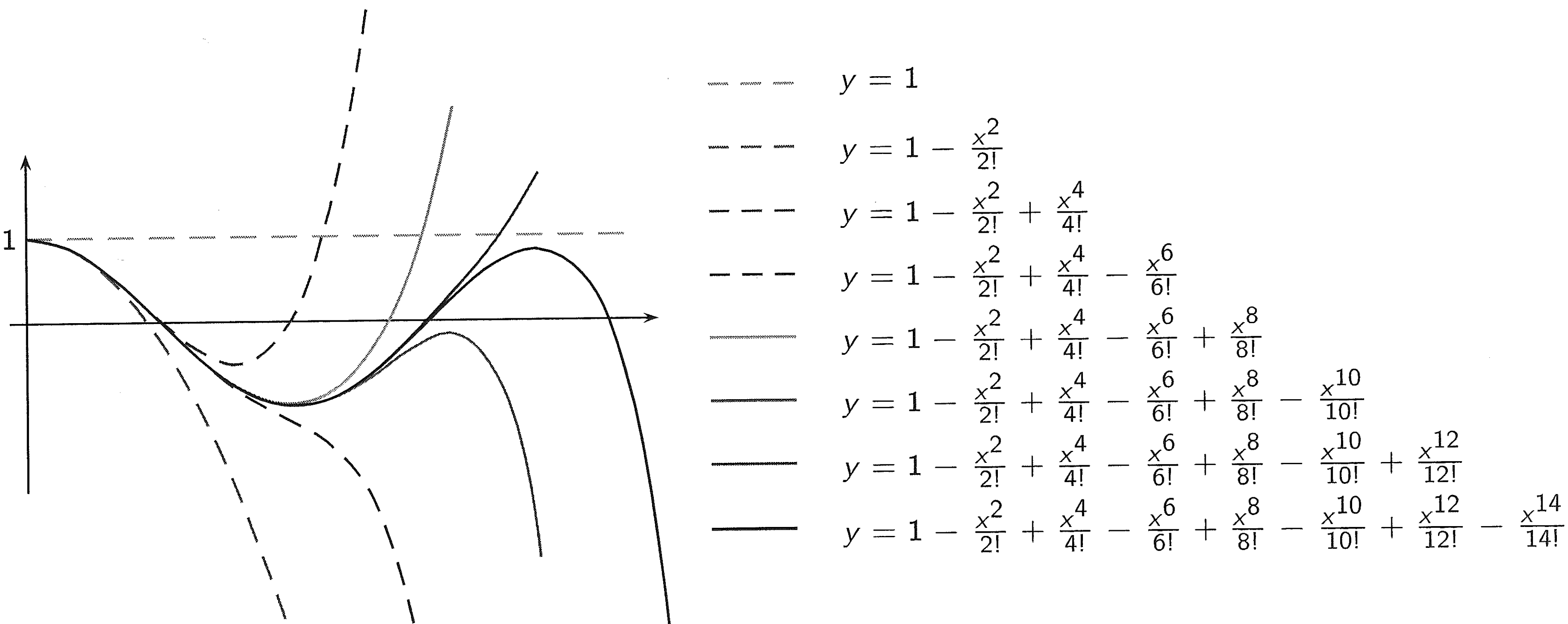
It is called the ***n*****th-degree Taylor polynomial** of f centered at $x = a$.

Approximation of $\cos x$ centered at $a = 0$

Consider the graph of the polynomial

$$T_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1}\frac{x^{2(n-1)}}{(2n-2)!} + (-1)^n \frac{x^{2n}}{(2n)!}.$$

As n increases, the graph of $T_{2n}(x)$ appears to approach the one of $\cos x$. This suggests that we can approximate $\cos x$ with $T_{2n}(x)$ as $n \to \infty$.

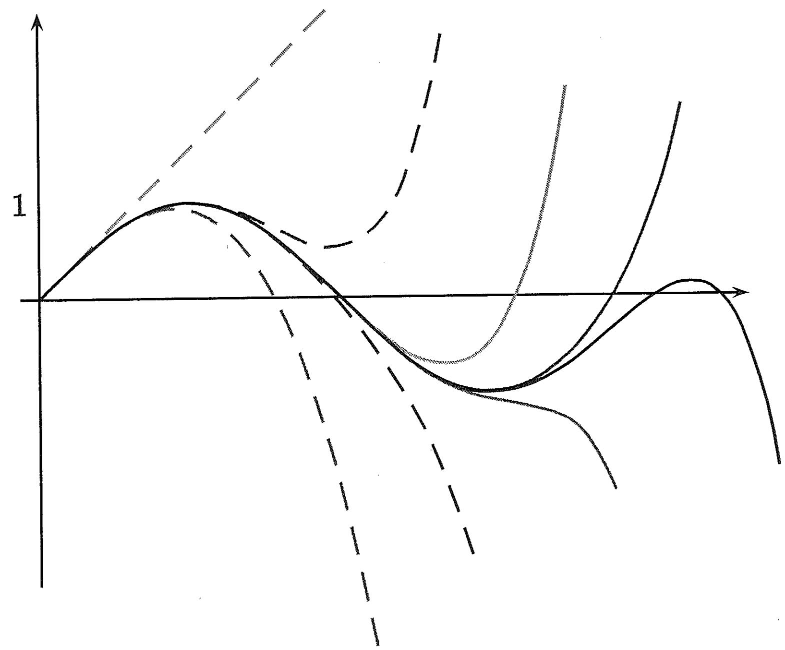


Approximation of $\sin x$ centered at $a = 0$

Consider the graph of the polynomial

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

As n increases, the graph of $T_{2n+1}(x)$ appears to approach the one of $\sin x$. This suggests that we can approximate $\sin x$ with $T_{2n+1}(x)$ as $n \to \infty$.



$y = x$

$y = x - \frac{x^3}{3!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$

$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!}$

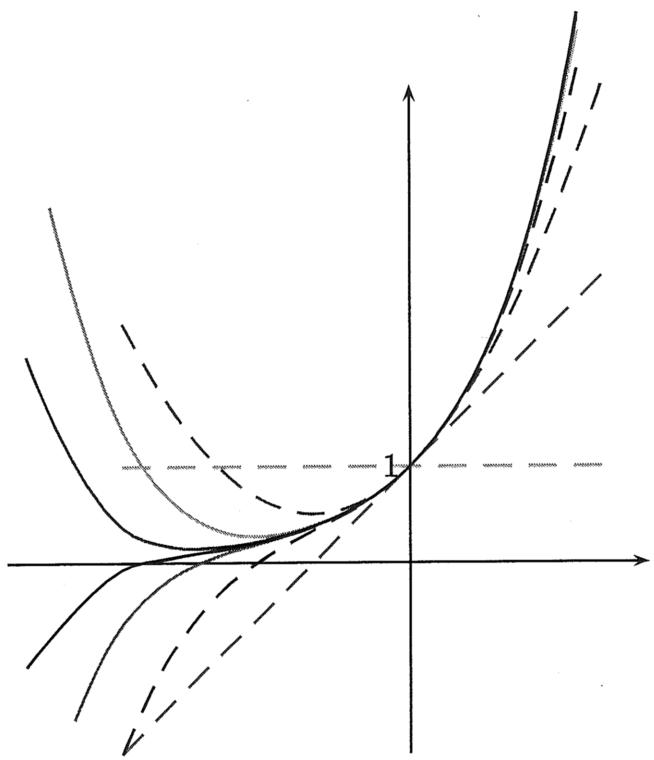
$y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!}$

Approximation of e^x centered at $a = 0$

Consider the graph of the polynomial

$$T_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}.$$

As n increases, the graph of $T_n(x)$ appears to approach the one of e^x.
This suggests that we can approximate e^x with $T_n(x)$ as $n \to \infty$.



$y = 1$

$y = 1 + x$

$y = 1 + x + \frac{x^2}{2!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$

$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

Example 6: (Bloomberg Business, 10/23/15)

Google parent Alphabet Inc. reached a record share price a day after reporting better-than-projected quarterly revenue and profit fueled by increased ad sales and a tighter lid on costs. [...] The actual figure that the company announced for the share buyback was unusually specific: $5,099,019,513.59. Turns out, those numbers correspond to the square root of 26, or the number of letters in the English alphabet.

— . . . —

Let $f(x) = \sqrt{x}$ and $a = 25$. The 5th-degree Taylor polynomial of f centered at 25 can be shown to be

$$T_5(x) = 5 + \frac{1}{10}(x-25) - \frac{1}{1,000}(x-25)^2 + \frac{1}{50,000}(x-25)^3 - \frac{1}{2,000,000}(x-25)^4 + \frac{1}{71,428,571.43}(x-25)^5$$

We can then check that

$$\sqrt{26} \approx T_5(26) = 5 + \frac{1}{10} - \frac{1}{1,000} + \frac{1}{50,000} - \frac{1}{2,000,000} + \frac{1}{71,428,571.43} = 5.099019514$$

This means that we overestimated Alphabet Inc. buyback by 41¢.