MA 137 — Calculus 1 with Life Science Applications
Extrema and The Mean Value Theorem
(Section 5.1)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

November 2, 2016

Finding the largest profit, or the smallest possible cost, or the shortest possible time for performing a given procedure or task are some examples of practical real-world applications of Calculus.

The basic mathematical question underlying such applied problems is how to find (if they exist) the largest or smallest values of a given function on a given interval.

This procedure depends on the nature of the interval.

Global (or Absolute) Extreme Values

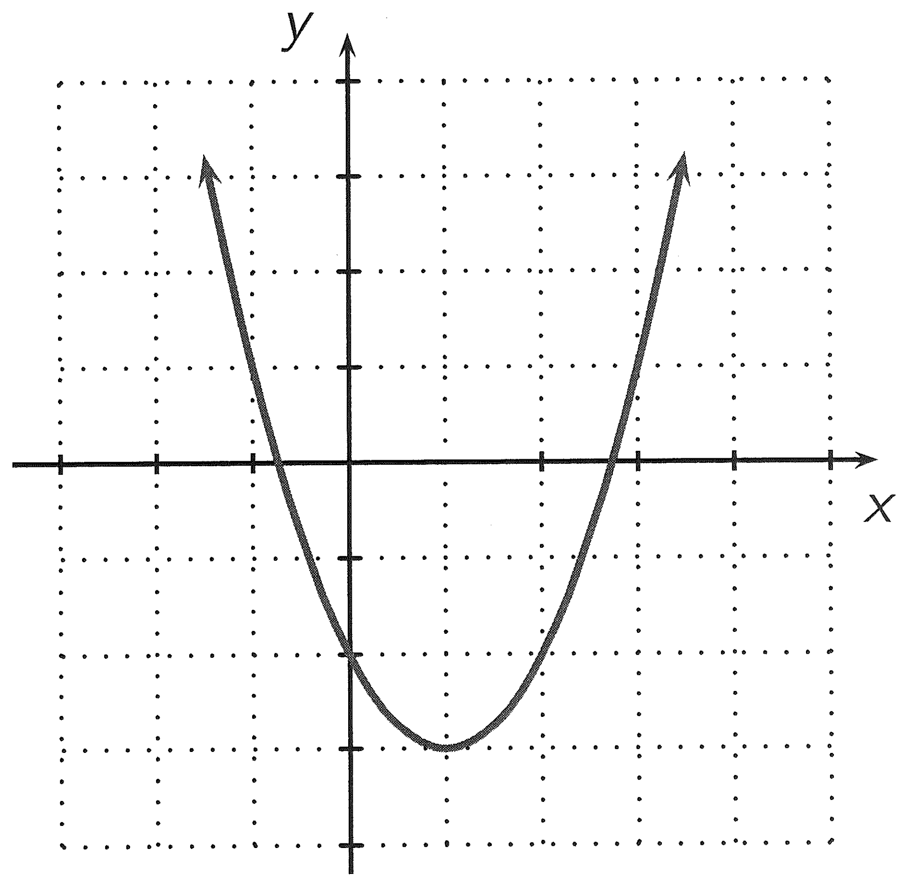
The largest value a function (possibly) attains on an interval is called its **global (or absolute) maximum value**.

The smallest value a function (possibly) attains on an interval is called its **global (or absolute) minimum value**.

Both maximum and minimum values (if they exist) are called **global (or absolute) extreme values**.

Example 1:

Find the maximum and minimum values for the function $f(x) = (x-1)^2 - 3$, if they exist.



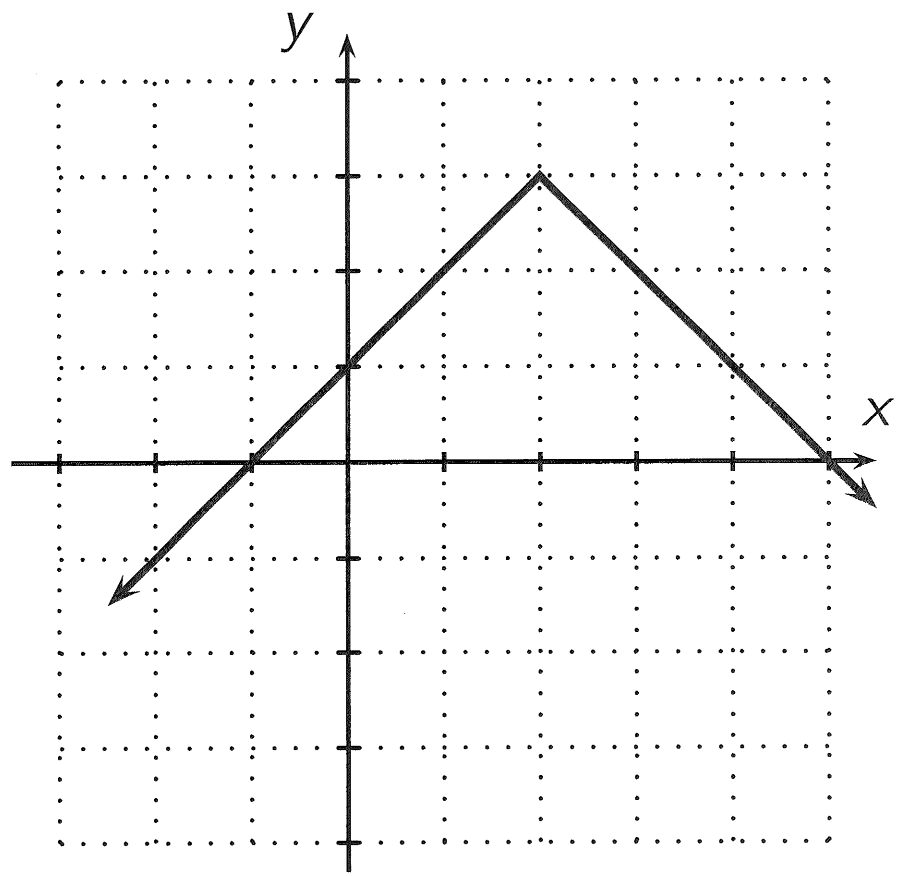
The function has no global maximum since it grows without any bound.

It has a global minimum of -3, which is attained when $x = 1$. It corresponds to the vertex of the parabola.

(Notice that the vertex corresponds to the point where $f'(x) = 0$)

Example 2:

Find the maximum and minimum values for the function $f(x) = -|x-2|+3$, if they exist.



The function has no global minimum

It has a global maximum of 3,
which is attained when $x = 2$.

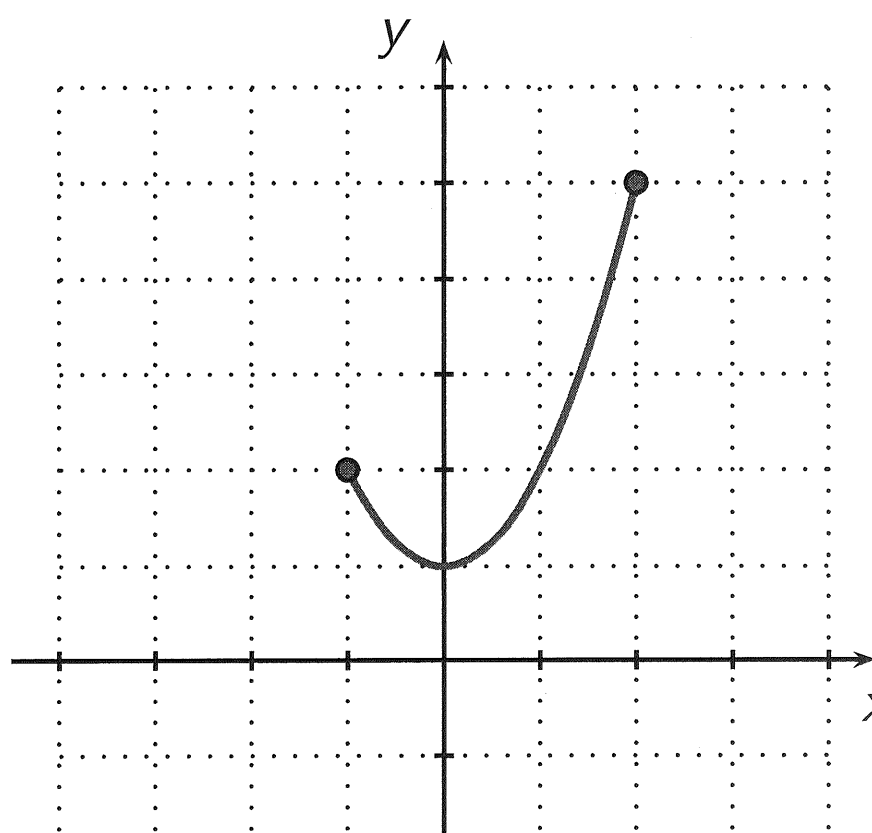
(notice that the global max is attained
at the point where $f'(x)$ does not exist)

Example 3:

Find the maximum and minimum values for the function

$$f(x) = x^2 + 1, \qquad x \in [-1, 2]$$

if they exist.



The function has a global maximum of 5, which is attained when $x = 2$.

The function has a global minimum of 1, which is attained when $x = 0$.

The Extreme Value Theorem (EVT)

We first focus on continuous functions on a closed and bounded interval.

The question of largest and smallest values of a continuous function f on an interval that is not closed and bounded requires us to pay more attention to the behavior of the graph of f, and specifically to where the graph is rising and where it is falling.

Closed and bounded intervals

An interval is **closed and bounded** if it has finite length and contains its endpoints.

For example, the interval $[-2, 5]$ is closed and bounded.

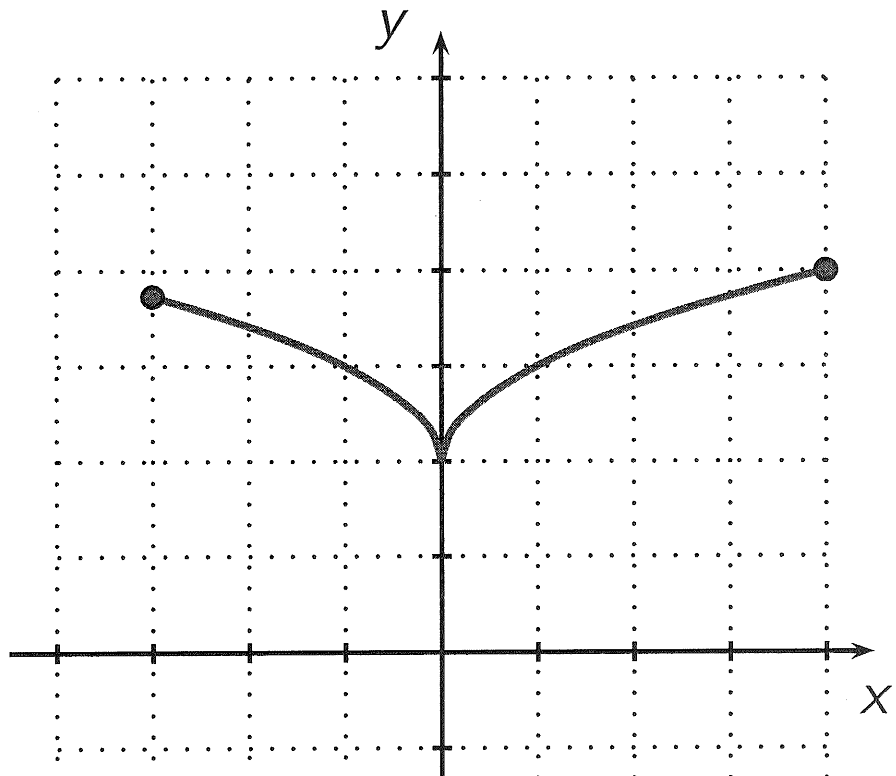
Theorem (The Extreme Value Theorem)

If a function f is continuous on a closed, bounded interval $[a, b]$, then the function f attains a global maximum and a global minimum value on $[a, b]$.

Example 4:

Let $f(x) = \begin{cases} 2 + \sqrt{x} & \text{if } x > 0 \\ 2 + \sqrt{-x} & \text{if } x \leq 0. \end{cases}$

Does $f(x)$ have a maximum and a minimum value on $[-3, 4]$?
How does this example illustrate the Extreme Value Theorem?



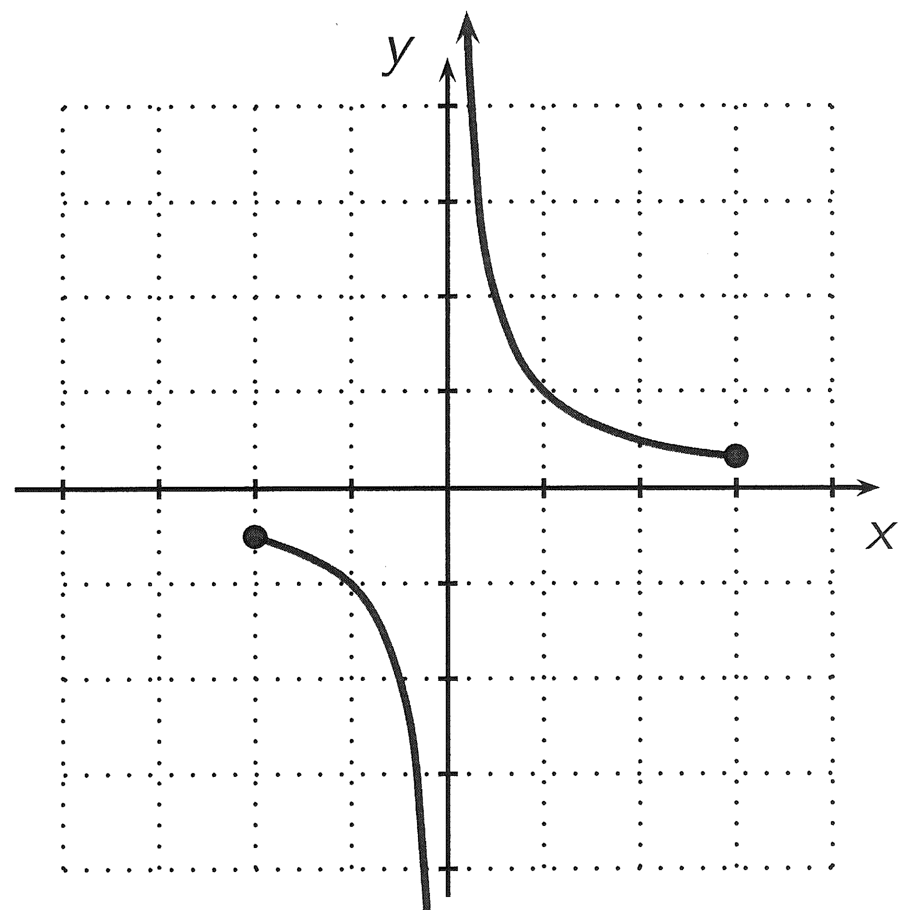
Yes the function is continuous on a closed interval hence by the Extreme Value Theorem it has a global max. and a global minimum.

the global max is 4, which is attained when $x=4$.

The global min is 2, which is attained when $x=0$.

Example 5:

Let $g(x) = \frac{1}{x}$. Does $g(x)$ have a maximum value and a minimum value on $[-2, 3]$? Does this example contradict the Extreme Value Theorem? Why or why not?

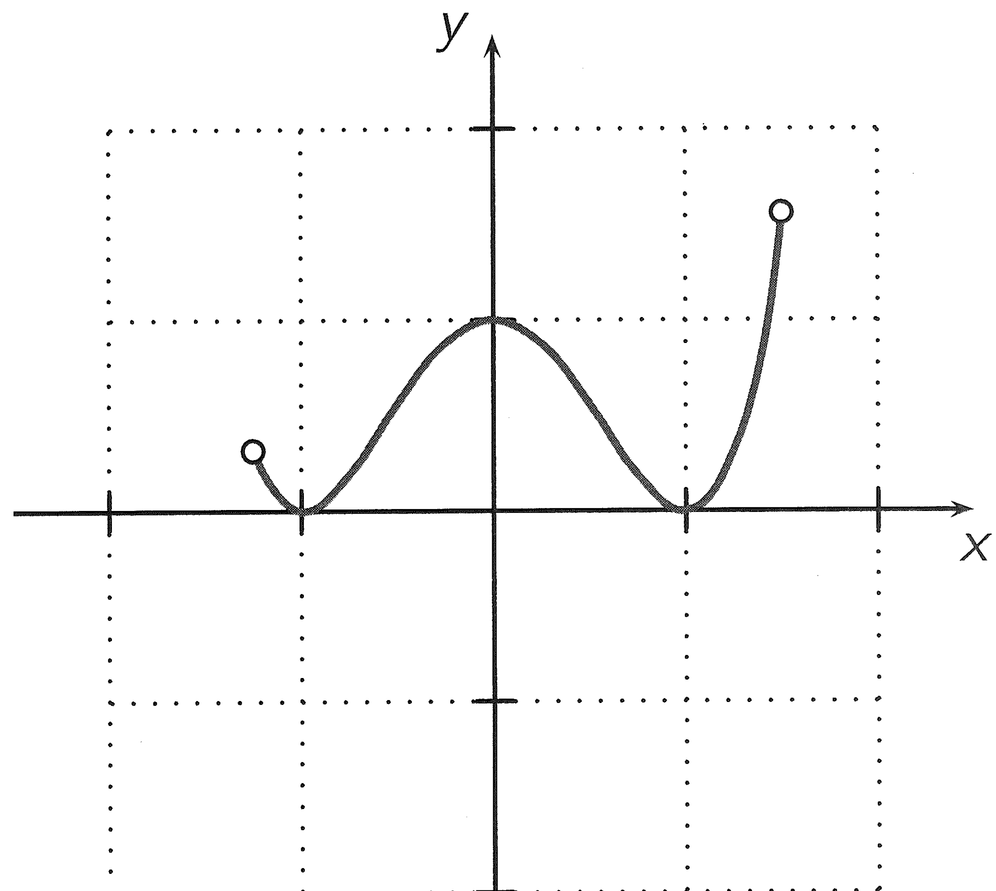


The function is <u>not</u> continuous on the closed interval $[-2, 3]$. Hence the Extreme Value Theorem <u>does not</u> apply.

The function has no global max nor a global minimum.

Example 6:

Let $h(x) = x^4 - 2x^2 + 1$. Does $h(x)$ have a maximum value and a minimum value on $(-1.25, 1.5)$? Does this example contradict the Extreme Value Theorem? Why or why not?



The function $h(x)$ has a global minimum on the interval $(-1.25, 1.5)$ <u>but</u> has no global maximum.

In fact, the function $h(x)$ is continuous but the interval over which we are considering it is <u>not closed</u>. Hence the EVT does not apply.

Local Extreme Points

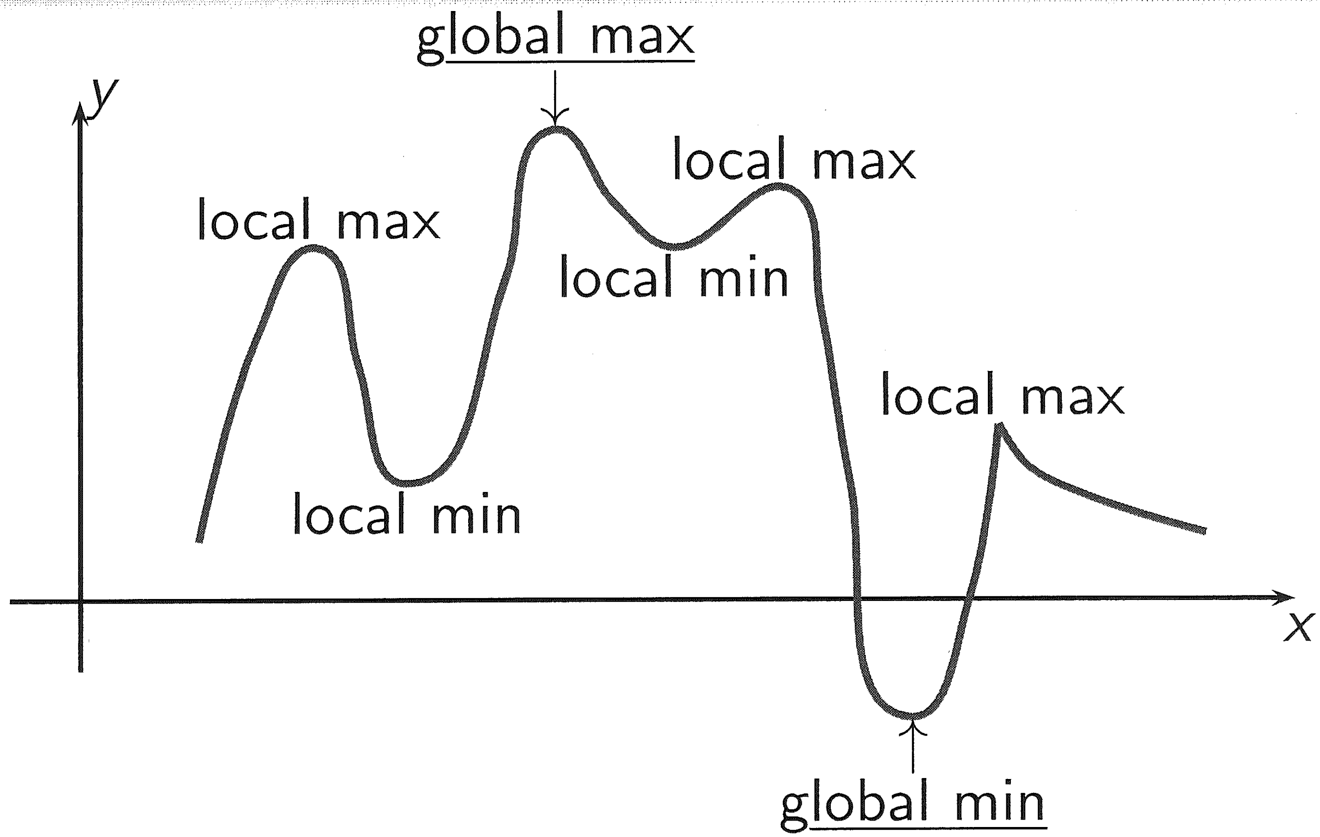
The EVT is an existence statement; it doesn't tell you how to locate the maximum and minimum values of f.

We need to narrow down the list of possible points on the given interval where the function f *might* have an extreme value to (usually) just a few possibilities. You can then evaluate f at these few possibilities, and pick out the smallest and largest value.

For this we need to discuss local (or relative) extrema, which are points where a graph is higher or lower than all *nearby* points.

Local (or relative) extreme points

A function f has a **local** (or **relative**) **maximum** at a point $(c, f(c))$ if there is some interval about c such that $f(c) \geq f(x)$ for all x in that interval. A function f has a **local** (or **relative**) **minimum** at a point $(c, f(c))$ if there is some interval about c such that $f(c) \leq f(x)$ for all x in that interval.



If you thought of the graph of the function as the profile of a landscape, the global maximum could represent the highest hill in the landscape, while the minimum could represent the deepest valley. The other points indicated in the graph, which look like tops of hills (although not the highest hills) and bottom of valleys (although not the deepest valleys), are the **local** (or **relative**) **extreme values**.

Fermat's Theorem

Theorem (Fermat's Theorem)

Let $f(x)$ be a continuous function. If f has an extreme value at an interior point c and if f is differentiable at $x = c$, then $f'(c) = 0$.

This results provide the following guidelines for finding candidates for local extrema:

Corollary

Let $f(x)$ be a continuous function on the closed, bounded interval $[a, b]$. If f has an extreme value at c in the interval, then either

- $c = a$ *or* $c = b$;
- $a < c < b$ *and* $f'(c) = 0$;
- $a < c < b$ *and* f' *is not defined at* $x = c$.

Remark: If f is defined at the point $x = c$ and either $f'(c) = 0$ or $f'(c)$ is undefined then the point c is called a **critical point** of f.

Example 7:

Find the maximum and minimum values of $f(x) = x^3 - 3x^2 - 9x + 5$ on the interval $[0, 4]$. For which values x are the maximum and minimum values attained?

The function $f(x) = x^3 - 3x^2 - 9x + 5$ is continuous for every $x \in \mathbb{R}$. Hence it is continuous on $[0,4]$.
Moreover $[0,4]$ is a closed and bounded interval.
Hence by the EVT it has a global max and a global min. We need to check the value of f at the end point and where $f'(x)=0$ (since f' exists everywhere).

$$f'(x) = 3x^2 - 6x - 9 = 0 \iff x^2 - 2x - 3 = 0$$

$$\iff (x-3)(x+1) = 0. \quad \therefore x = 3, -1$$

Notice that only $3 \in [0,4]$.

	x	$f(x)$
endpoints	0	$f(0) = 5$
	4	$f(4) = -15$
critical #	3	$f(3) = -22$

Hence f has a global maximum of 5 attained at $x=0$. f has a global minimum of -22 attained at $x=3$.

Example 8: (Online Homework HW17, # 7)

Find the maximum and minimum values of $f(x) = \dfrac{4x}{x^2+1}$ on the interval $[-4, 0]$. For which values x are the maximum and minimum values attained?

The function $f(x) = \frac{4x}{x^2+1}$ is continuous for every $x \in \mathbb{R}$ as the denominator is never 0, and it is the quotient of two continuous functions. In particular it is continuous on the closed interval $[-4,0]$.
Hence by the EVT it has a global max and a global minimum.

$$f'(x) = \frac{4(x^2+1) - 4x(2x)}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{4-4x^2}{(x^2+1)^2}$$

f' is defined for every x. $f'(x) = 0$

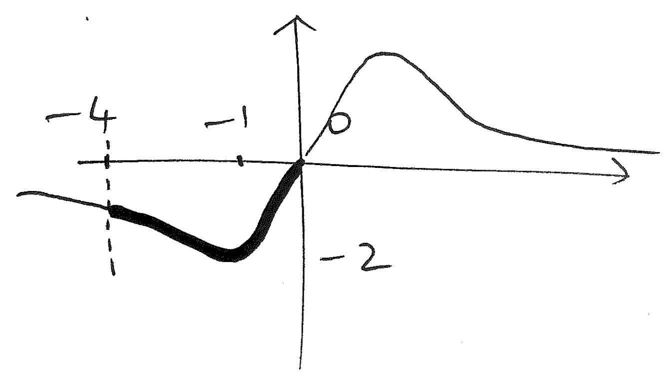
$$\iff 4-4x^2 = 0 \iff x^2 = 1 \iff x = \pm 1$$

Notice that only $-1 \in [-4,0]$.

	x	$f(x)$
endpoints of interval	-4	$f(-4) = -16/17$
	0	$f(0) = 0$
critical #	-1	$f(-1) = -2$

Hence f has a global maximum of 0, attained at $x = 0$.

f has a global minimum of -2, attained at $x = -1$.



Example 9:

Find the maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-1, 8]$. For which values x are the maximum and minimum values attained?

The function is continuous for every x; in particular it is continuous on the closed interval $[-1, 8]$. Thus by the EVT it has a global max and a global min.

Notice that $f'(x) = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-\frac{1}{3}} = \underline{\underline{\frac{2}{3\sqrt[3]{x}}}}$

Thus $f'(x) = 0$ cannot occur.

However f' is not differentiable at $x = 0$.

	end points		critical #
x	-1	8	0
$f(x)$	1	4	0

$\circledast$ global max 4 attained at $x = 8$

$\circledast$ global min 0 attained at $x = 0$

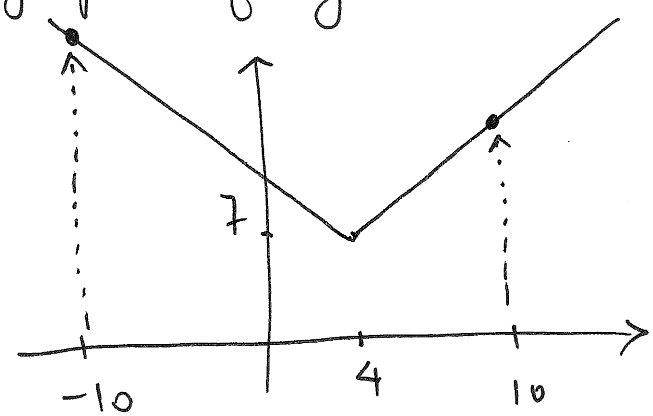
Example 10:

Find the t values on the interval $[-10, 10]$ where $g(t) = |t-4| + 7$ takes its maximum and minimum values. What are the maximum and minimum values?

The function $g(t)$ is continuous everywhere; in particular it is continuous on the closed interval $[-10,10]$. Hence by the EVT it has a global max and a global min.

We need to test the value of g at the end points of the intervals and where $g'=0$ or does not exist. Notice that g' is never 0, but it does not exist at $t=4$.

The graph of g is:



	t	$g(t)$	
end points	-10	21	← global max
	10	13	
critical #	4	7	← global min

Example 11: (Online Homework HW17, # 9)

Find the absolute maximum and minimum values of the function

$$f(x) = \frac{10\cos x}{4 + 2\sin x}$$

over the interval $[0, 2\pi]$. If there are multiple points in a single category list the points in increasing order in x value.

$$f(x) = \frac{10\cos x}{4 + 2\sin x}$$

is a continuous function over the closed interval $[0, 2\pi]$. Hence by the EVT it has a global max and a global minimum.

We need to find the values where $f'(x) = 0$.
Notice that f is differentiable everywhere:

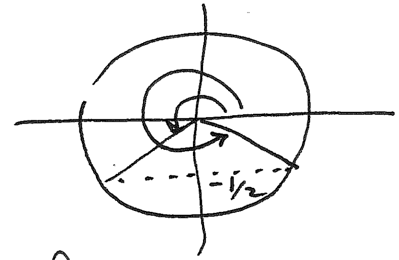
$$f'(x) = \frac{10(-\sin x)(4 + 2\sin x) - 10\cos x\,(+2\cos x)}{(4 + 2\sin x)^2}$$

$$= \frac{-40\sin x - 20\sin^2 x - 20\cos^2 x}{(4 + 2\sin x)^2}$$

$$= \frac{-20 - 40\sin x}{(4 + 2\sin x)^2}$$ as $\underline{\sin^2 x + \cos^2 x = 1}$

Thus $f'(x) = 0 \iff -20 - 40\sin x = 0$

$\iff \sin x = -\frac{1}{2}$.



Notice that this occurs for

$x = \pi + \frac{\pi}{6} = 7/6\,\pi$ and $x = 2\pi - \pi/6 = \frac{11}{6}\pi$.

Hence :

	x	$f(x)$
endpoints	0	$5/2 = 2.5$
	2π	$5/2 = 2.5$
critical #s	$7/6\,\pi$	$f(7/6\,\pi) = \dfrac{10\cos(7/6\,\pi)}{4 + 2\sin(7/6\,\pi)} = \dfrac{10(-\sqrt{3}/2)}{4+2(-1/2)} = \dfrac{-5\sqrt{3}}{3} \cong -2.8868$
	$\frac{11}{6}\pi$	$f(11/6\,\pi) = \dfrac{10(+\sqrt{3}/2)}{4+2(-1/2)} = \dfrac{5}{3}\sqrt{3} \cong 2.8868$

Hence the global max is 2.8868 at $x = 11/6\,\pi$ and the global min is -2.8868 at $x = 7/6\,\pi$

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Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

November 4, 2016

The Mean Value Theorem (MVT)

The Mean Value Theorem is a very important in calculus. Its consequences are far reaching, and we will use it to derive important results that will help us to analyze functions.

Theorem (Mean Value Theorem)

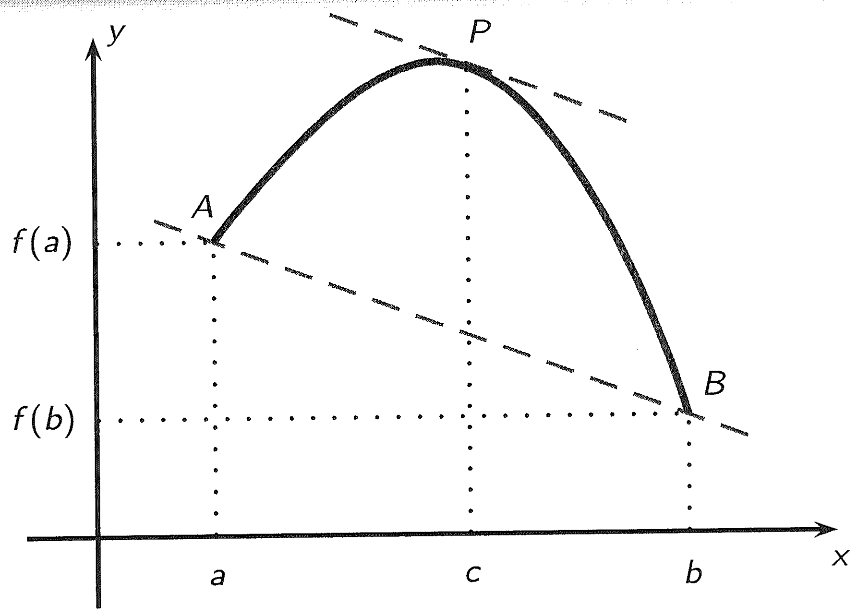
If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one number $c \in (a, b)$ such that

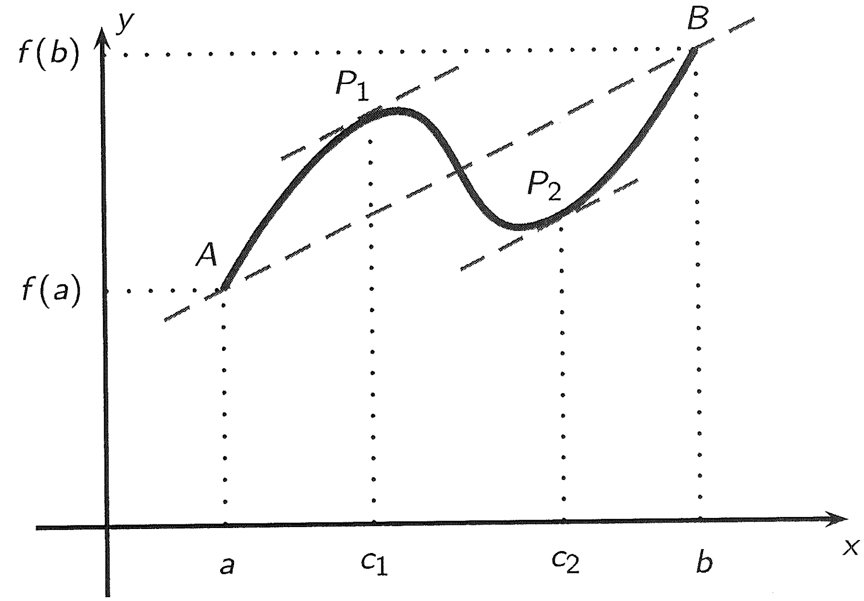
$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Geometrically, it says that there exists a point $P(c, f(c))$ on the graph where the tangent line at this point is parallel to the secant line through $A(a, f(a))$ and $B(b, f(b))$.

The MVT is an "existence" result: It tells us neither how many such points there are nor where they are in the interval (a, b).

Geometric Interpretation and a Special Case





The proof of the MVT is typically done by first showing a special case of the theorem called Rolle's Theorem.
You can read its proof on p. 211 of the Neuhauser book.

Theorem (Rolle's Theorem – 1691)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and if $f(a) = f(b)$, then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.

The MVT follows from Rolle's theorem and is a "tilted" version of that theorem. The secant and tangent lines in the MVT are no longer necessarily horizontal, as in Rolle's theorem, but are "tilted"; they are still parallel, though.

Proof of the MVT: We define the following function:

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The function F is continuous on $[a, b]$ and differentiable on (a, b). Furthermore, $F(a) = f(a) = F(b)$. Hence, we can apply Rolle's theorem to the function $F(x)$. There exists a $c \in (a, b)$ with $F'(c) = 0$. Since

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

it follows that, for this value of c,

$$0 = F'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and hence

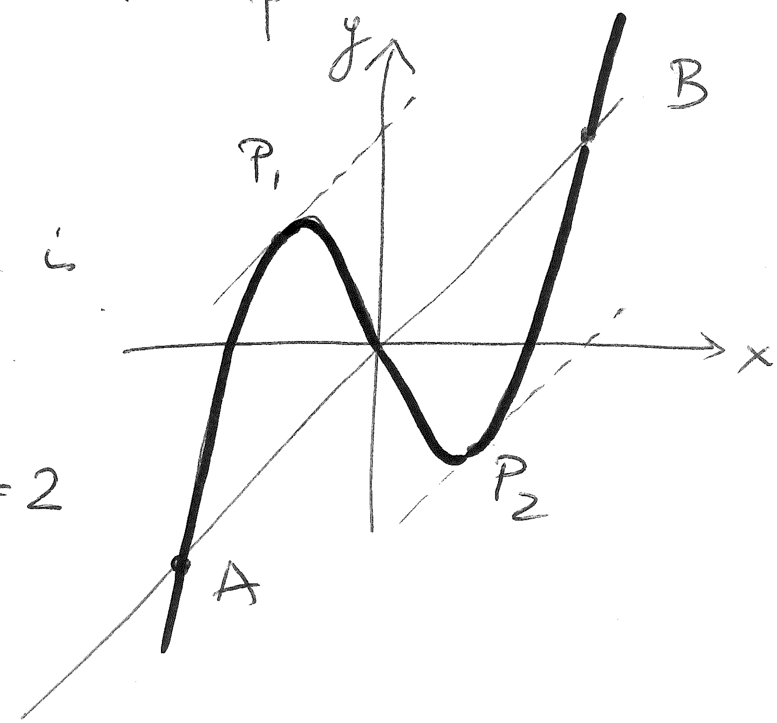
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example 1: (Online Homework HW17, # 10)

Graph the function $f(x) = x^3 - 2x$ and its secant line through the points $(-2, -4)$ and $(2, 4)$. Use the graph to estimate the x-coordinate of the points where the tangent line is parallel to the secant line.

Find the exact value of the numbers c that satisfy the conclusion of the Mean Value Theorem for the interval $[-2, 2]$.

Consider $f(x) = x^3 - 2x$ and the points $A(-2,-4)$ and $B(2,4)$



the slope of the secant line is

$$\frac{f(b)-f(a)}{b-a} = \frac{4-(-4)}{2-(-2)} = \frac{8}{4} = 2$$

Now, $f'(x) = 3x^2 - 2$

To find $(c, f(c))$ as in the Mean Value Theorem we need to solve:

$$f'(c) = 2 \iff 3x^2 - 2 = 2 \iff x^2 = 4/3$$

$$\iff \boxed{x = \pm \tfrac{2}{3}\sqrt{3}}$$

$\therefore$ $P_1(-1.1547,\ 0.7698)$
$P_2(1.1547,\ -0.7698)$

Example 2: (Online Homework HW17, # 12)

Find all numbers c that satisfy the conclusion of Rolle's Theorem for the following function

$$f(x) = 9x\sqrt{x+2}$$

on the interval $[-2, 0]$.

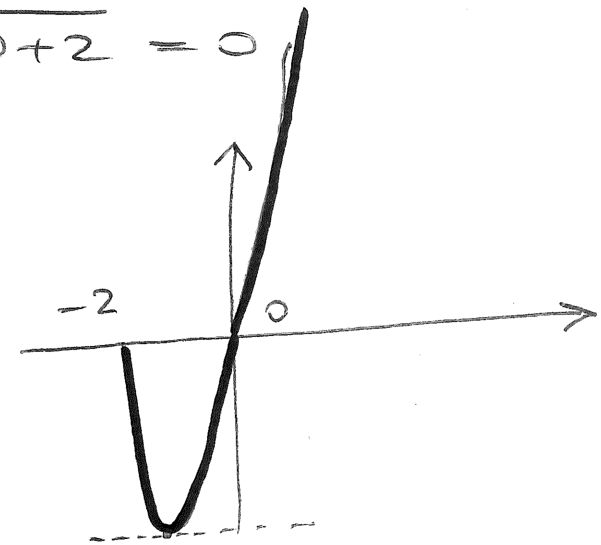
$f(x) = 9x\sqrt{x+2}$ on $[-2, 0]$

is continuous on $[-2,0]$ and differentiable on $(-2,0)$

Notice that $f(-2) = 9(-2)\sqrt{-2+2} = 0$

$f(0) = 9\cdot 0\sqrt{0+2} = 0$

We want to find c in $[-2,0]$ such that $f'(c) = 0$.



$$f'(x) = 9\cdot 1\cdot\sqrt{x+2} + \underbrace{9\cdot x\cdot \frac{1}{2\sqrt{x+2}}\cdot 1}_{\text{chain rule}}$$

$$f'(x) = \frac{18(\sqrt{x+2})^2 + 9x}{2\sqrt{x+2}} = \frac{27x+36}{2\sqrt{x+2}}$$

not differentiable at $\underline{x=-2}$

Hence $f'(c) = 0 \iff \dfrac{27c+36}{2\sqrt{c+2}} = 0 \iff$

$27c + 36 = 0$ $\quad \boxed{c = -\frac{36}{27} \cong -1.334}$

Example 3: (Online Homework HW17, # 13)

Consider the function $f(x) = 3 - 3x^{2/3}$ on the interval $[-1, 1]$. Which of the three hypotheses of Rolle's Theorem fails for this function on the interval?

(a) $f(x)$ is continuous on $[-1, 1]$.
(b) $f(x)$ is differentiable on $(-1, 1)$.
(c) $f(-1) = f(1)$.

$f(x) = 3 - 3x^{2/3}$

* is continuous on $[-1, 1]$

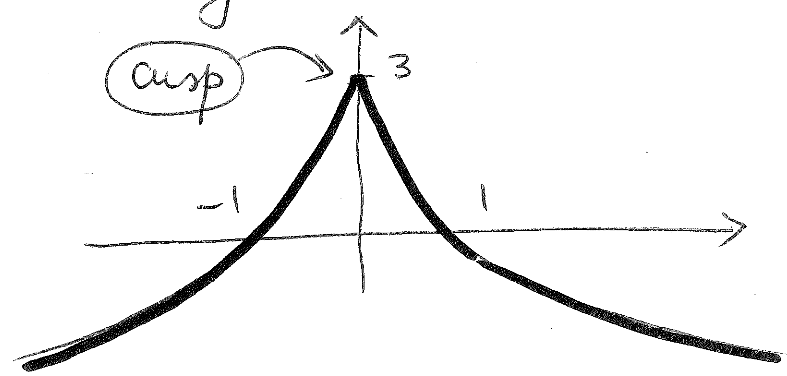
* $f(-1) = f(1) = 0$

* **But** $f(x)$ is not differentiable at $x = 0$

in fact $f'(x) = -3 \cdot \frac{2}{3} x^{2/3 - 1} = -2x^{-1/3}$

$= -\dfrac{2}{\sqrt[3]{x}}$

hence at $x = 0$ the tangent line is vertical



Consequences of the MVT

We discuss two consequences of the MVT.

The first corollary is useful in obtaining information about a function on the basis of its derivative. The importance of the second corollary will become more apparent in Example 7 and Section 5.8.

Corollary 1

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) such that

$$m \leq f'(x) \leq M \qquad \text{for all } \; x \in (a, b)$$

then

$$m(b-a) \leq f(b) - f(a) \leq M(b-a)$$

Corollary 2

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), with $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

Example 4: (Online Homework HW17, # 14)

Suppose $f(x)$ is continuous on $[3,5]$ and

$$-5 \le f'(x) \le 2$$

for all x in $(3,5)$.
Use the Mean Value Theorem to estimate $f(5) - f(3)$.

$f(x)$ is continuous on $[3,5]$ and

$$-5 \leq f'(x) \leq 2$$

for all $x \in (3,5)$. By the MVT there exists $c \in (3,5)$ such that $f'(c) = \dfrac{f(5)-f(3)}{5-3}$

Hence for that particular c:

$$-5 \leq f'(c) \leq 2$$

$$\Longleftrightarrow$$

$$-5 \leq \frac{f(5)-f(3)}{2} \leq 2$$

$$\Longleftrightarrow$$

$$\boxed{-10 \leq f(5)-f(3) \leq 4}$$

Example 5: (Neuhauser, Example # 8, p. 212)

Denote the population size at time t by $N(t)$, and assume that $N(t)$ is continuous on the interval $[0,10]$ and differentiable on the interval $(0,10)$ with $N(0)=100$ and $\left|\frac{dN}{dt}\right| \leq 3$ for all $t \in (0,10)$.

What can you say about $N(10)$?

We know that $N(t)$ is continuous on $[0,10]$

and differentiable on $(0,10)$.

Moreover $N(0) = 100$ and $-3 \leq N'(t) \leq 3$ for all $t \in (0,10)$

By the MVT there exists $c \in (0,10)$ such that

$$N'(c) = \frac{N(10) - N(0)}{10 - 0}$$

For that c we have the estimate

$$-3 \leq N'(c) \leq 3$$

$$\Longleftrightarrow$$

$$-3 \leq \frac{N(10) - N(0)}{10} \leq 3$$

$$\Longleftrightarrow \quad -30 \leq N(10) - N(0) \leq 30 \quad \Longleftrightarrow$$

$$\boxed{N(0) - 30 \leq N(10) \leq N(0) + 30} \quad \Longleftrightarrow \quad \boxed{70 \leq N(10) \leq 130}$$

Example 6: (Online Homework HW17, # 15)

Let $f(x) = 8\sin(x)$.

(a) $|f'(x)| \leq$ ____

(b) By the Mean Value Theorem,

$$|f(b) - f(a)| \leq \_\_\_\_ |a - b|$$

for all a and b.

[**Remark:** This problem is also a variation of Example 9, Neuhauser, p. 212]

Let $f(x) = 8\sin x$. Then $f'(x) = 8\cos x$.

Since $-1 \leq \cos x \leq 1$ for all x, then

$$-8 \leq f'(x) = 8\cos x \leq 8$$

for all x. Or $|f'(x)| \leq 8$.

In particular this is true for all $x \in [a,b]$.

By the MVT there is a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Hence

$$-8 \leq f'(c) \leq 8 \iff$$

$$-8 \leq \frac{f(b)-f(a)}{b-a} \leq 8$$

or

$$\left| \frac{f(b)-f(a)}{b-a} \right| \leq 8$$

or $\boxed{|f(b)-f(a)| \leq 8|x-a|}$

Example 7: (Neuhauser, Problem # 56, p. 256)

We have seen that $f(x) = f_0 e^{rx}$ satisfies the differential equation $\dfrac{df}{dx} = r\,f(x)$ with $f(0) = f_0$.

This exercise will show that $f(x)$ is in fact the only solution.

Suppose that r is a constant and f is a differentiable function with

$$\frac{df}{dx} = r\,f(x) \tag{1}$$

for all $x \in \mathbb{R}$, and $f(0) = f_0$. The following steps will show that $f(x) = f_0 e^{rx}$, $x \in \mathbb{R}$, is the only solution of (1).

(a) Define the function $F(x) = f(x)e^{-rx}$, $x \in \mathbb{R}$. Use the product rule to show that $F'(x) = e^{-rx}[f'(x) - rf(x)]$.

(b) Use **(a)** and (1) to show that $F'(x) = 0$ for all $x \in \mathbb{R}$.

(c) Use Corollary 2 to show that $F(x)$ is a constant and, hence, $F(x) = F(0) = f_0$.

(d) Show that (c) implies that $f_0 = f(x)e^{-rx}$ and therefore, $f(x) = f_0 e^{rx}$.

Suppose that f is a solution of $\boxed{\dfrac{df}{dx} = rf}$

and satisfies $f(0) = f_0$.

(a) Define a new function $\boxed{F(x) = f(x) \cdot e^{-rx}}$

Then the derivative $F'(x)$ is:

$$F'(x) = f'(x)\, e^{-rx} + f(x) \cdot \underbrace{e^{-rx} \cdot (-r)}_{\text{chain rule}}$$

$$= e^{-rx} \cdot \left(f'(x) - r f(x) \right)$$

(b) But $\dfrac{df}{dx} = rf \iff f'(x) - rf(x) = 0$

Hence $F'(x) = e^{-rx} \cdot [\,0\,] = 0$

for all $x \in \mathbb{R}$.

(c) Since $F(x)$ is continuous for all x in any closed interval and differentiable for all x in the same open interval, with $F'(x) = 0$.

Then by Corollary 2, $F(x)$ is constant.

$$\boxed{F(x) = f(x)e^{-rx} = \text{Constant}}$$

Evaluate it at 0 : $F(0) = f_0 \cdot e^{-r\cdot 0} = f_0$

$= \text{constant}$

(d) Thus $\boxed{F(x) = f(x)e^{-rx} = f_0}$

or $\boxed{f(x) = f_0 e^{rx}}$ ///

MA 137 — Calculus 1 with Life Science Applications
Monotonicity and Concavity
(Section 5.2)
Extrema, Inflection Points, and Graphing
(Section 5.3)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

November 7 & 9, 2016

Increasing and Decreasing Functions

A function f is said to be increasing when its graph rises and decreasing when its graph falls. More precisely, we say that

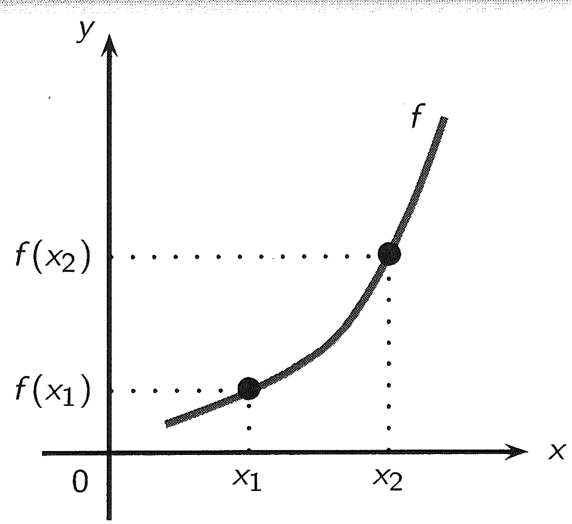
Definition

f is **(strictly) increasing** on an interval I if

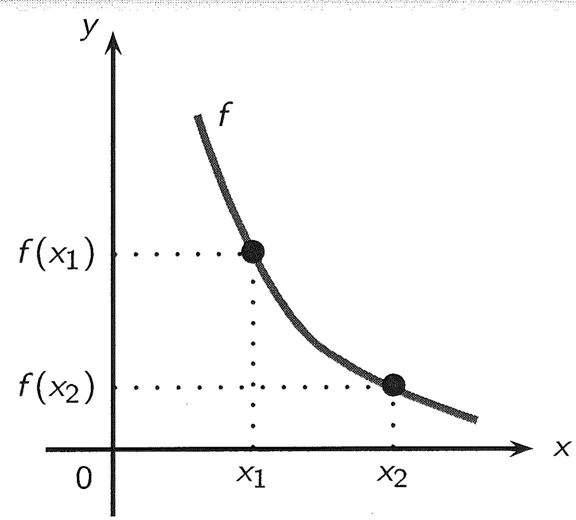
$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I$$

f is **(strictly) decreasing** on an interval I if

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 < x_2 \text{ in } I$$



f is increasing



f is decreasing

First Derivative Test for Monotonicity

Theorem (First Derivative Test for Monotonicity)

Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

(a) *If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.*

(b) *If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.*

Proof: Suppose $f'(x) > 0$ on an interval I. We wish to show that $f(x_1) < f(x_2)$ for any pair $x_1 < x_2$ in $[a, b]$.

Let x_1 and x_2 be any pair of point in $[a, b]$ satisfying $x_1 < x_2$. Then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). We can therefore apply the MVT to f defined on $[x_1, x_2]$: There exists a number $c \in (x_1, x_2)$ such that

$$\frac{f(x_1) - f(x_2)}{x_2 - x_1} = f'(c)$$

Now, $f'(c) > 0$ as $c \in [x_1, x_2] \subset [a, b]$; so

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

so $f(x_2) - f(x_1) > 0$, since $x_2 - x_1 > 0$. Therefore, $f(x_1) < f(x_2)$.
Because x_1 and x_2 are arbitrary numbers in $[a, b]$ satisfying $x_1 < x_2$, it follows that f is increasing on the whole interval.

The proof of part (b) is similar.

First Derivative Test for (Local) Extrema

Theorem (First Derivative Test for (Local) Extrema)

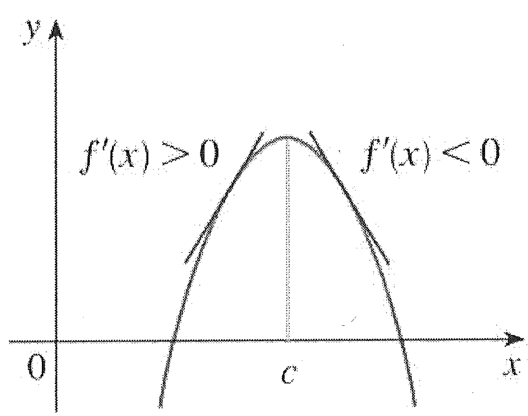
If f has a critical value at $x = c$, then

- *f has a local maximum at $x = c$ if the sign of f' around c is*

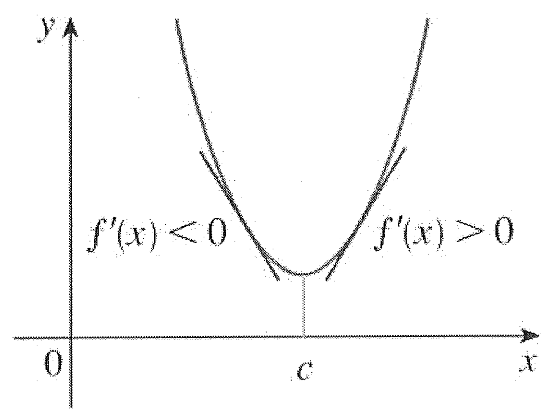
$$\begin{array}{ccc} +++ & & --- \\ \hline & c & \end{array}$$

- *f has a local minimum at $x = c$ if the sign of f' around c is*

$$\begin{array}{ccc} --- & & +++ \\ \hline & c & \end{array}$$



(a) Local maximum

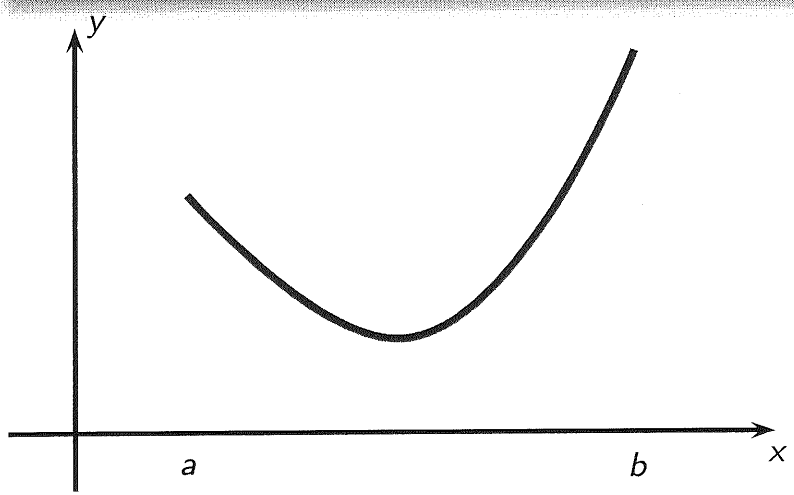


(b) Local minimum

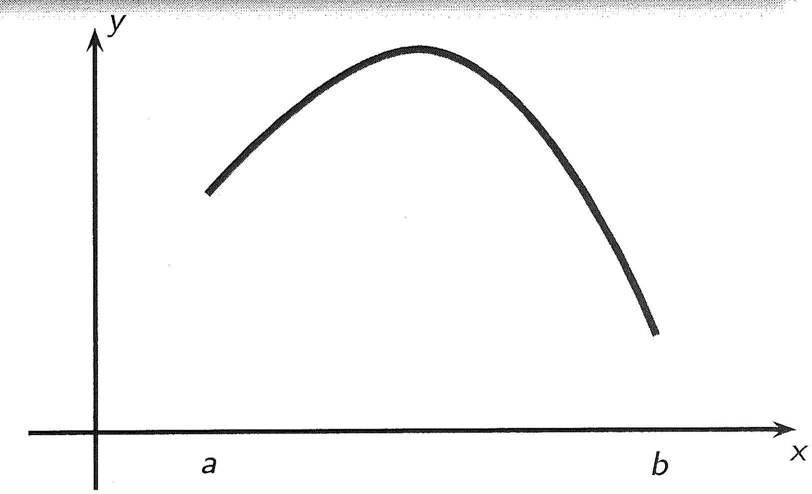
Concavity

The second derivative can also be used to help sketch the graph of a function. More precisely, the second derivative can be used to determine when the graph of a function is concave upward or concave downward.

The graph of a function $y = f(x)$ is **concave upward** on an interval $[a, b]$ if the graph lies above each of the tangent lines at every point in the interval $[a, b]$. The graph of a function $y = f(x)$ is **concave downward** on an interval $[a, b]$ if the graph lies below each of the tangent lines at every point in the interval $[a, b]$.



graph of function concave upward on $[a, b]$



graph of function concave downward on $[a, b]$

Second Derivative Test for Concavity

Consider a function $f(x)$.

If $f''(x) > 0$ over an interval $[a, b]$, then the derivative $f'(x)$ is increasing on the interval $[a, b]$. That means the slopes of the tangent lines to the graph of $y = f(x)$ are increasing on the interval $[a, b]$. From this it can be seen that the graph of the function $y = f(x)$ is concave upward.

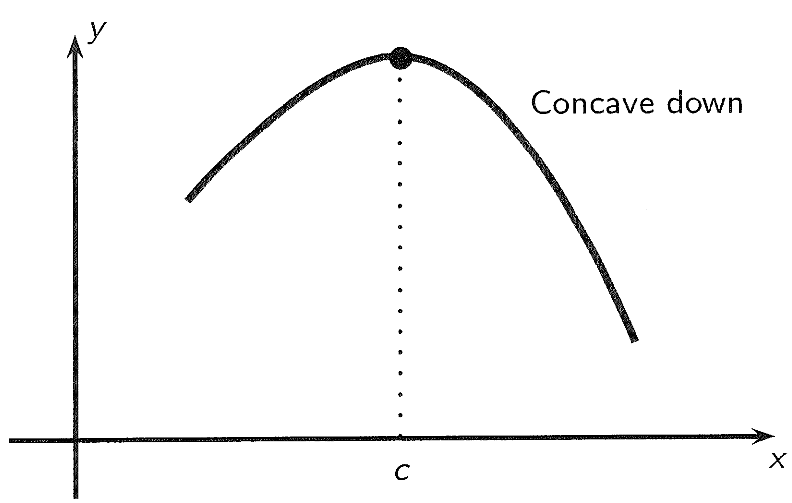
If $f''(x) < 0$ over an interval $[a, b]$. Then the derivative $f'(x)$ is decreasing on the interval $[a, b]$. That means the slopes of the tangent lines to the graph of $y = f(x)$ are decreasing on the interval $[a, b]$. From this it can be seen that the graph of the function $y = f(x)$ is concave downward.

Second Derivative Test for (Local) Extrema

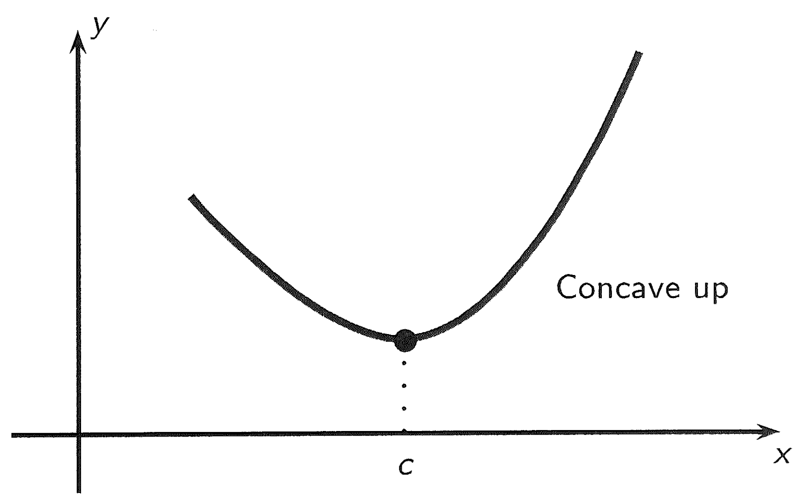
Theorem (Second Derivative Test for (Local) Extrema)

Suppose that f is twice differentiable on an open interval containing c.

- *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max. at $x = c$.*
- *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min. at $x = c$.*



f has a local max at c



f has a local min at c

Inflection Points

A point $(c, f(c))$ on the graph is called a **point of inflection** if the graph of $y = f(x)$ changes concavity at $x = c$. That is, if the graph goes from concave up to concave down, or from concave down to concave up.

If $(c, f(c))$ is a point of inflection on the graph of $y = f(x)$ and if the second derivative is defined at this point, then $f''(c) = 0$.

Thus, points of inflection on the graph of $y = f(x)$ are found where either $f''(x) = 0$ or the second derivative is not defined.

However, if either $f''(x) = 0$ or the second derivative is not defined at a point, it is not necessarily the case that the point is a point of inflection. Care must be taken.

About Graphing a Function

Using the first and the second derivatives of a twice-differentiable function, we can obtain a fair amount of information about the function.

We can determine intervals on which the function is increasing, decreasing, concave up, and concave down. We can identify local and global extrema and find inflection points.

To graph the function, we also need to know how the function behaves in the neighborhood of points where either the function or its derivative is not defined, and we need to know how the function behaves at the endpoints of its domain (or, if the function is defined for all $x \in \mathbb{R}$, how the function behaves for $x \to \pm\infty$).

A line $y = b$ is a horizontal asymptote if either

$$\lim_{x \to +\infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b$$

A line $x = c$ is a vertical asymptote if

$$\lim_{x \to c^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to c^-} f(x) = \pm\infty$$

Example 1:

Find the intervals where the function $f(x) = x^3 - 3x^2 + 1$ is increasing and the ones where it is decreasing. Use this information to sketch the graph of $f(x) = x^3 - 3x^2 + 1$.

$f(x) = x^3 - 3x^2 + 1$

* We want to find the intervals when f is increasing and decreasing

$f'(x) = 3x^2 - 6x = 3x(x-2)$

		0		2	
$3x$	$-----$		$++++$		
$x-2$	$-------$				$+++$
$f'(x) = 3x(x-2)$	$++++$		$--$		$+++$
	$\nearrow$	0	$\searrow$	2	$\nearrow$

$\therefore$ f is increasing on $(-\infty, 0)$ and $(2, +\infty)$

$\therefore$ f is decreasing on $(0, 2)$

There is a local max at $x=0$; a local min at $x=2$

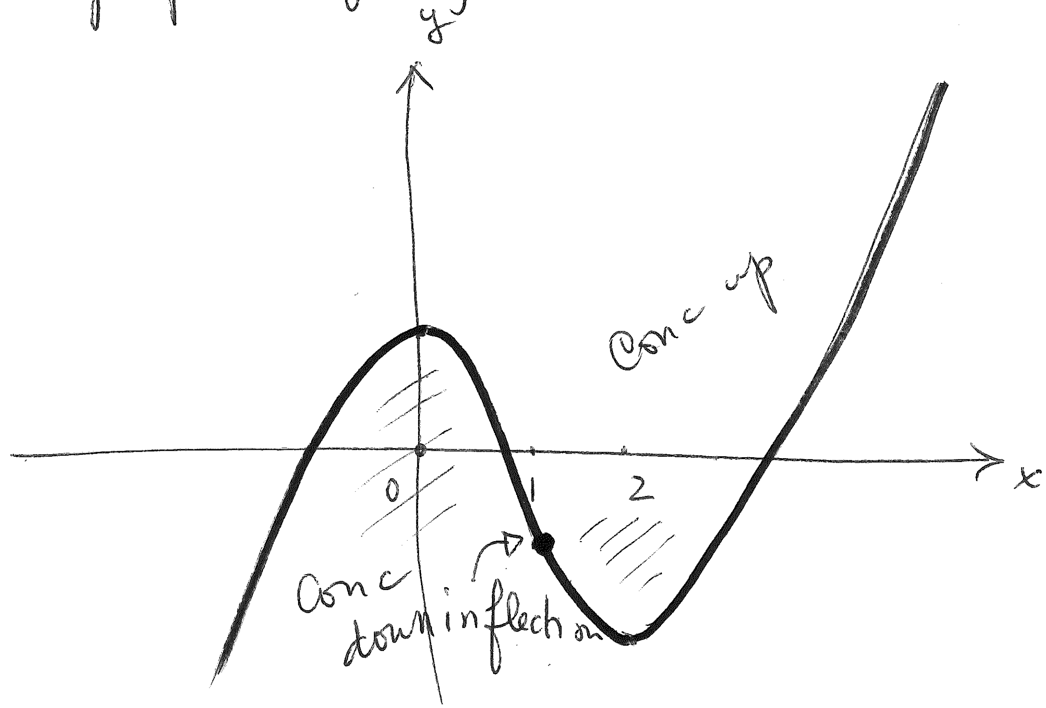
* About concavities : $f''(x) = 6x - 6 = 6(x-1)$

		1	
sign f''	$-----$		$++++$
	conc down		conc up

Hence f is concave down on $(-\infty, 1)$ and it is concave up on $(1, +\infty)$

There is an inflection point at $x = 1$.

The graph of f looks like



	x	$f(x)$
local max	0	1
inflection	1	-1
local min $\rightarrow$	2	-3

<u>Note</u>: it is hard to find where the graph meets precisely the x-axis!

Example 2:

Let $f(x) = \dfrac{x+4}{x+7}$. Find the intervals over which the function is increasing.

$f(x) = \dfrac{x+4}{x+7}$ not defined at $x = -7$

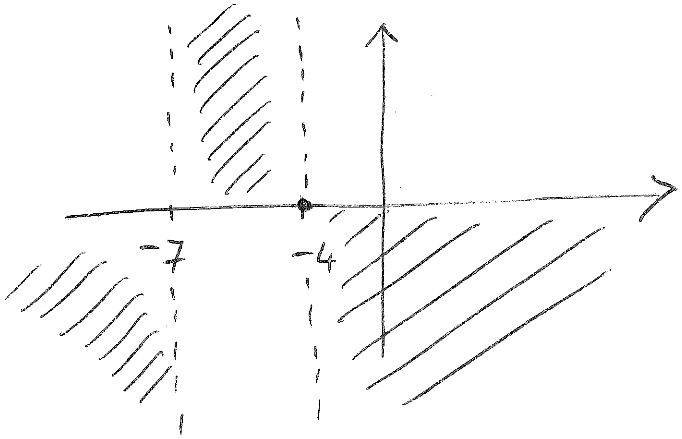
* Notice that the sign of $f(x)$ is as follows

$x+4$: $- - - -$ (-4) $+ + + + +$

$x+7$: $- -$ (-7) $+ + + + + + + + +$

$f(x) = \dfrac{x+4}{x+7}$: $+ +$ (-7) $- - -$ (-4) $+ + + + + +$

hence the graph of f lies in:



* Let's find $f'(x)$.

$$f'(x) = \frac{1\cdot(x+7) - (x+4)(1)}{(x+7)^2} = \frac{\cancel{x}+7-\cancel{x}-4}{(x+7)^2} = \frac{3}{(x+7)^2}$$

The sign of f is always strictly positive;
Notice f' is not differentiable at -7.

sign f': $+ + +$ (-7) $+ + + +$

Hence f' is always increasing

* Let's also look at the concavities of f.

We need $f''(x)$.

$$f'(x) = \frac{3}{(x+7)^2} = 3(x+7)^{-2} \Longrightarrow f''(x) = -6(x+7)^{-3} \cdot (1)$$

$$\therefore \; f'' = \frac{-6}{(x+7)^3}$$

-6 − − − − − − − − − − − − −

$(x+7)^3$ − − − − ∘ + + + + + (at -7)

$f'' = \dfrac{-6}{(x+7)^3}$ + + + ∘ − − − − − − − (at -7)

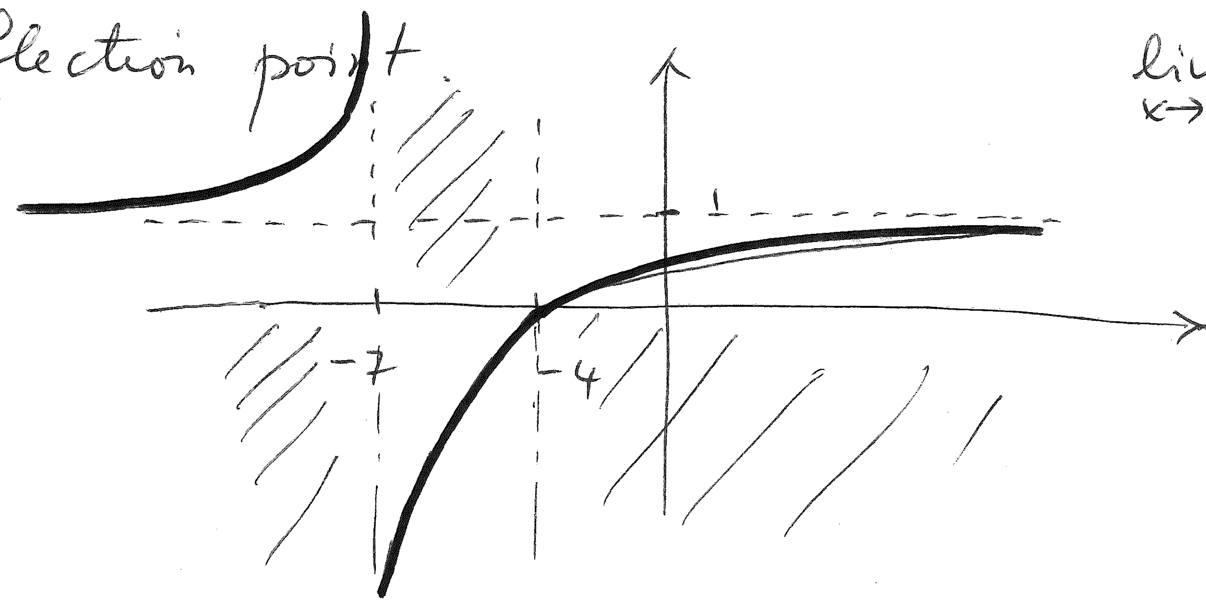
Conc up   conc down

Notice that there is

no inflection point

$$\lim_{x\to\infty} \frac{x+4}{x+7} = 1$$

graph



Example 3:

Let $h(x) = x^2 e^{-x}$.

(a) On what intervals is h increasing or decreasing?
(b) At what values of x does h have a local maximum or minimum?
(c) On what intervals is h concave upward or downward?
(d) State the x-coordinate of the inflection point(s) of h.
(e) Use the information in the above to sketch the graph of h.

$h(x) = x^2 e^{-x}$

* Notice that $x^2 \geq 0$ for all x and $e^{-x} > 0$ for all x

Thus $h(x) = x^2 e^{-x} \geq 0$ for all x. Hence the graph of h is in the 1^{st} and 2^{nd} quadrant.

* $h'(x) = 2x\, e^{-x} + x^2 \cdot \left[e^{-x}(-1)\right] = \underline{e^{-x}\left[2x - x^2\right]} = e^{-x}\, x(2-x)$

sign of h':

	$(-\infty, 0)$	0	$(0, 2)$	2	$(2, +\infty)$
$2-x$	$+$		$+$	0	$-$
x	$-$	0	$+$		$+$
e^{-x}	$+$		$+$		$+$
$h'(x) = e^{-x} \cdot x(2-x)$	$-$	0	$+$	0	$-$
	$\searrow$		$\nearrow$		$\searrow$

$\therefore$ $h(x)$ is increasing on $(0, 2)$

$\therefore$ $h(x)$ is decreasing on $(-\infty, 0)$ and $(2, +\infty)$

$\therefore$ local min at $x = 0$; local max at $x = 2$

$$* \quad h''(x) = (-e^{-x})(2x - x^2) + e^{-x}(2-2x) =$$

$$= e^{-x}\left[-2x + x^2 + 2 - 2x\right] = e^{-x}(x^2 - 4x + 2)$$

$$h''(x) = 0 \iff e^{-x}(x^2 - 4x + 2) = 0 \iff x^2 - 4x + 2 = 0$$

$$x_{1,2} = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2} \begin{cases} 2+\sqrt{2} \cong 3.41 \\ 2-\sqrt{2} \cong 0.59 \end{cases}$$

sign of h'':

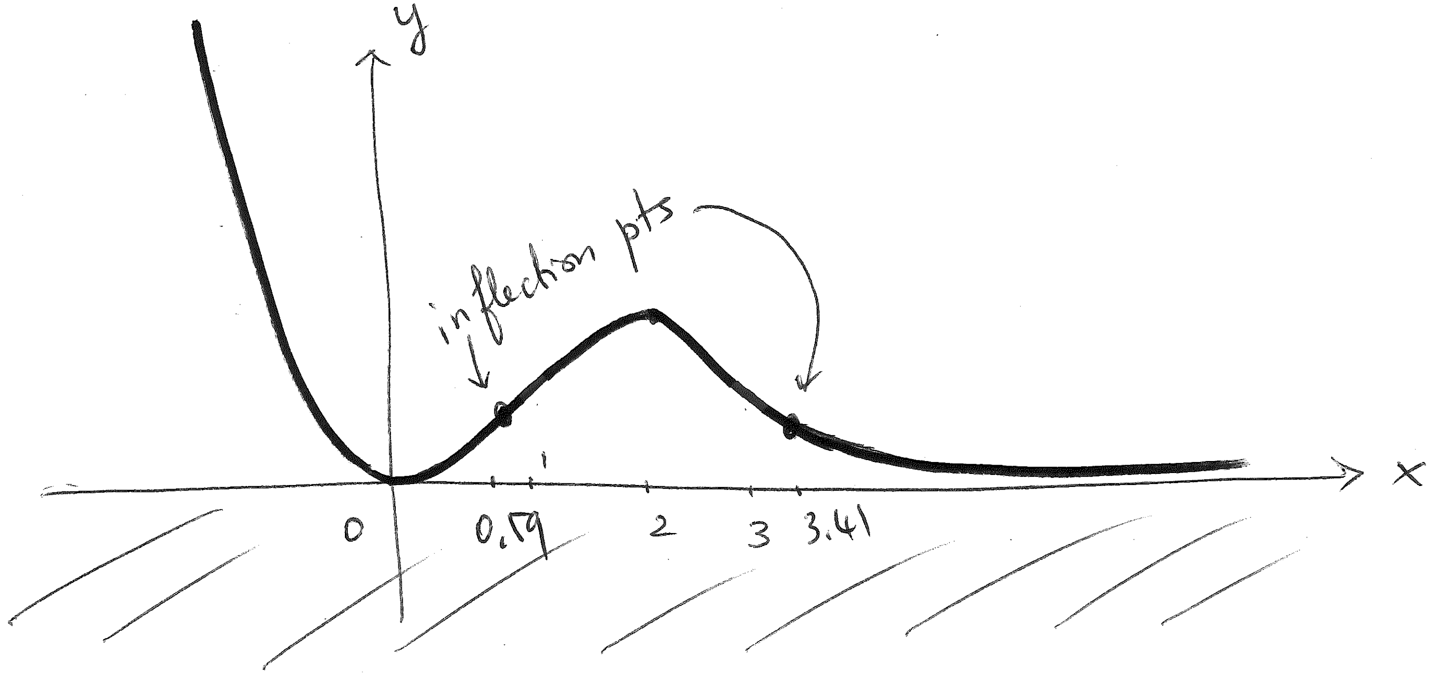
	$(-\infty, 0.59)$	0.59	$(0.59, 3.41)$	3.41	$(3.41, +\infty)$
$x^2 - 4x + 2$	+ + +		- - - -		+ + +
e^{-x}	+ + +		+ + + +		+ + +
$e^{-x}(x^2-4x+2)$	+ + +		- - - -		+ + +

$\therefore$ h is concave up on $(-\infty, 0.59)$ and $(3.41, +\infty)$

h is concave down on $(0.59, 3.41)$

Notice that $x_1 = 0.59$ and $x_2 = 3.41$ are both inflection points as there is a change of concavity

The graph of $h(x)$ looks like :



notice that it is reasonable to expect that

$\lim_{x \to \infty} x^2 e^{-x} = 0$. We will see this with L'Hospital rule in Section 5.5

Moreover the function has a <u>global min</u> at $x = 0$; there is just a <u>local max</u> at $x = 2$

Example 4

Find the inflection points of the function $g(x) = e^{-x^2}$.

$g(x) = e^{-x^2}$

* Notice that this function is always positive. Hence the graph will be in the first and second quadrant. The graph is also symmetric w.r.t. the y-axis.

* $g'(x) = e^{-x^2} \cdot (-2x) = -2x\, e^{-x^2}$

hence the sign of g' is:

$-2x$: $+ + +$ 0 $- - -$

e^{-x^2} : $+ + + + + +$

$g'(x) = -2x e^{-x^2}$: $+ + +$ 0 $- - - -$

$\nearrow$ 0 $\searrow$

$\therefore$ the function g is increasing on $(-\infty, 0)$ and decreasing on $(0, +\infty)$. There is a local max at $x = 0$. (It is actually a <u>global</u> max!)

$$* \quad g''(x) = -2(1)e^{-x^2} - 2x\left[e^{-x^2}(-2x)\right]$$

$$= e^{-x^2}\left[-2 + 4x^2\right]$$

Hence $g''(x) = 0 \iff -2 + 4x^2 = 0 \iff x^2 = \frac{1}{2}$

$\iff \quad x_{1,2} = \pm \frac{\sqrt{2}}{2} = \pm\, 0.707$

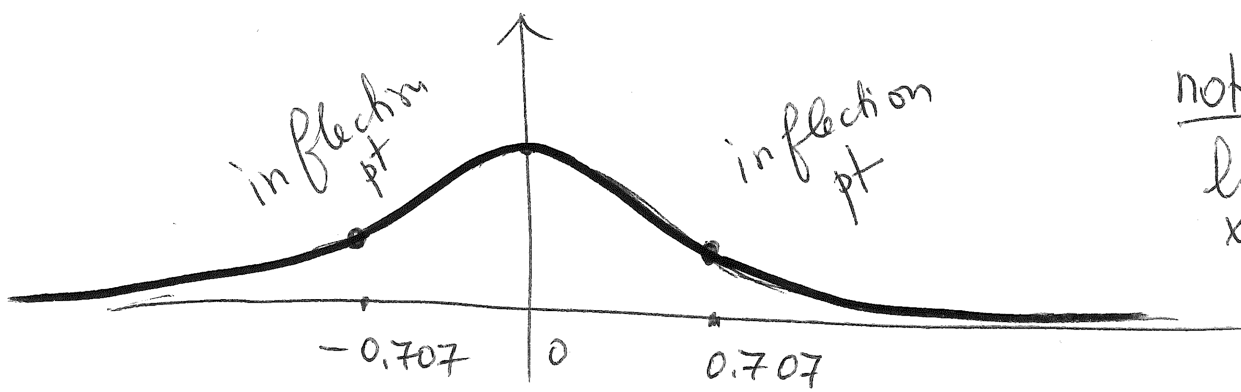
Sign of g'' :

	$(-\infty, -0.707)$	-0.707	$(-0.707, 0.707)$	0.707	$(0.707, +\infty)$
e^{-x^2}	+ +		+ + +		+ +
$4x^2 - 2$	+ +	0	− − −	0	+ + +
g''	+ +	0	− − −	0	+ + +

$g(x)$ is conc. up on $(-\infty, -0.707)$ and $(0.707, +\infty)$

$g(x)$ is conc down on $(-0.707, 0.707)$

$\therefore \quad \underline{x_1 = -0.707}$ and $\underline{x_2 = 0.707}$ are $\underline{\text{inflection pts}}$



notice :

$$\lim_{x \to \pm\infty} e^{-x^2} = 0$$

Example 5:

Suppose $g(x) = \dfrac{\sqrt{x-3}}{x}$. Find the value of x in the interval $[3, +\infty)$ where $g(x)$ takes its maximum.

The function $g(x) = \dfrac{\sqrt{x-3}}{x}$ is defined on $[3, +\infty)$

Notice that in that interval $g(x) \geq 0$ for all x.

We need to find the intervals of increase and decrease

For that we need to study the sign of $g'(x)$:

$$g'(x) = \frac{\frac{1}{2\sqrt{x-3}} \cdot (1) \cdot x - \sqrt{x-3} \cdot (1)}{x^2} =$$

$$= \frac{\frac{x - 2\left(\sqrt{x-3}\right)^2}{2\sqrt{x-3}}}{x^2} = \frac{x - 2(x-3)}{2x^2\sqrt{x-3}}$$

$$= \frac{x - 2x + 6}{2x^2\sqrt{x-3}} = \frac{6-x}{2x^2\sqrt{x-3}}$$

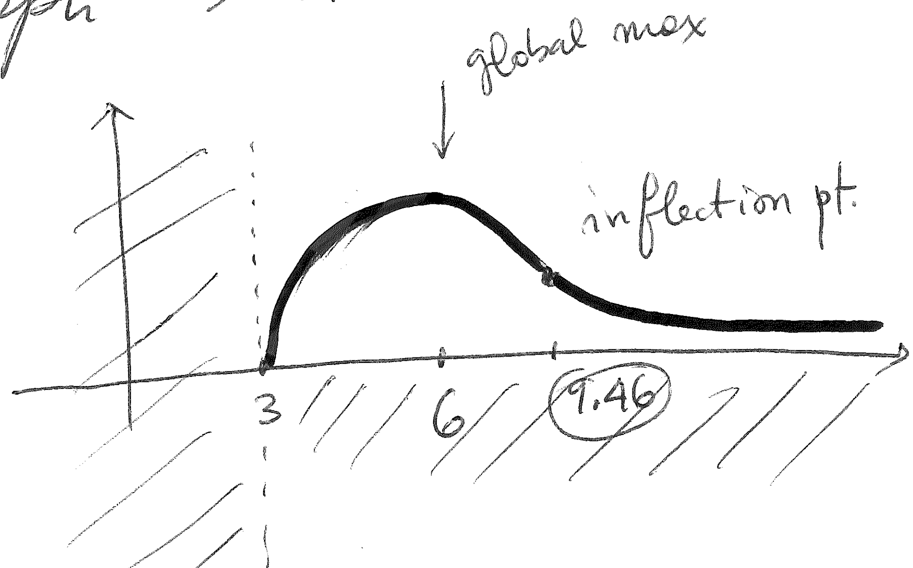
Hence: $g'(x)$

x	3		6	
$g'(x)$		$+++$		$----$
g		$\nearrow$		$\searrow$

Thus $g(x)$ is increasing on $[3,6)$ and decreasing on $(6,+\infty)$.

Thus $x=6$ is the point where g has a local max. However, because of the behavior of g, this is actually a <u>global</u> max.

The graph looks like:



note $\lim_{x\to\infty} \frac{\sqrt{x-3}}{x} = 0$

From the graph we see that there must be inflection point(s). To find them we need $g''(x)$.

$$g''(x) = \frac{(-1)\left[2x^2\sqrt{x-3}\right] - (6-x)\cdot\left[4x\sqrt{x-3} + 2x^2\frac{1}{2\sqrt{x-3}}\right]}{\left(2x^2\sqrt{x-3}\right)^2}$$

$$= \frac{-4x^2(x-3) - (6-x)\,8x(x-3) - (6-x)(2x^2)}{4x^4(x-3)\cdot\left[2\sqrt{x-3}\right]}$$

$$= \dots = \frac{3x(x^2-12x+24)}{4x^4(x-3)\sqrt{x-3}}$$

$g''(x)=0$ in $[3,+\infty)$ $\iff$ $x^2-12x+24=0$ $\iff$ $x_{1,2} = \frac{12 \pm \sqrt{12^2-4\cdot 24}}{2}$

$= 6 \pm 2\sqrt{3}$

$=$ 9.46

2.54

Example 6: (Exam 3, Fall 13, # 3)

Let $f(x) = \ln(x^2+1)$. You are given that

$$f'(x) = \frac{2x}{x^2+1} \qquad \text{and} \qquad f''(x) = \frac{2-2x^2}{(x^2+1)^2}.$$

(a) On what intervals is f increasing or decreasing?
(b) At what values of x does f have a local maximum or minimum?
(c) On what intervals is f concave upward or downward?
(d) State the x-coordinate of the inflection point(s) of f.
(e) Use the information in the above to sketch the graph of f.

$$f'(x) = \frac{2x}{x^2+1}$$

sign of $f'(x)$:

$2x$ $- - - \overset{0}{\cdot} + + + +$

x^2+1 $+ + + + + + +$

$\frac{2x}{x^2+1}$ $- - - \underset{0}{\bullet} + + + + + +$

$\searrow \quad \nearrow$

hence f is <u>decreasing</u> on $(-\infty, 0)$ and <u>increasing</u> on $(0, +\infty)$

$\therefore$ <u>local min</u> at $x=0$

$$f'' = \frac{2-2x^2}{(x^2+1)^2}$$

$2-2x^2$ $- - - \underset{-\sqrt{1/2}}{\bullet} + + + + + \underset{\sqrt{1/2}}{\bullet} - - -$

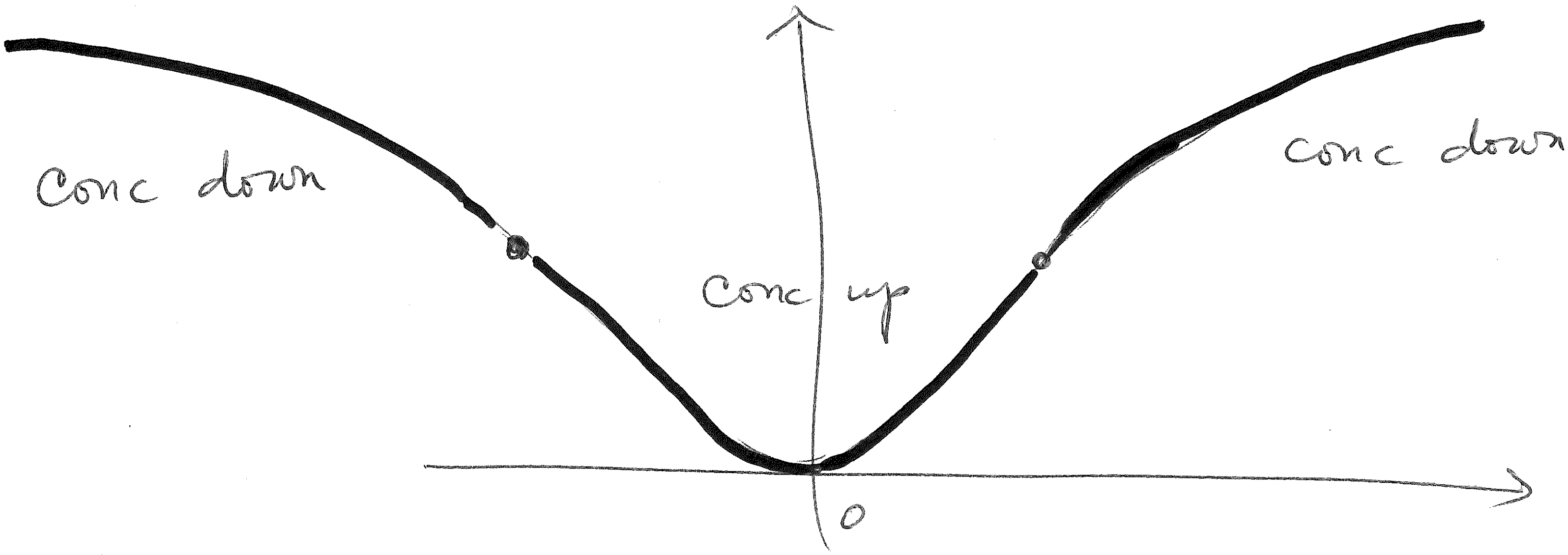
$(x^2+1)^2$ $+ + + + + + + + +$

$\frac{2-2x^2}{(x^2+1)^2}$ $- - \underset{-\sqrt{1/2}}{\bullet} + + + \underset{\sqrt{1/2}}{\bullet} - - -$

$\therefore$ f <u>concave down</u> on $(-\infty, -\sqrt{\tfrac{1}{2}})$ and $(\sqrt{\tfrac{1}{2}}, +\infty)$

f is concave up on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

There are inflection points at $x_1 = -\frac{1}{\sqrt{2}}$ and $x_2 = \frac{1}{\sqrt{2}}$



notice that $\lim_{x \to \pm\infty} f(x) = +\infty$

also $x = 0$ is a <u>global</u> minimum

Example 7:

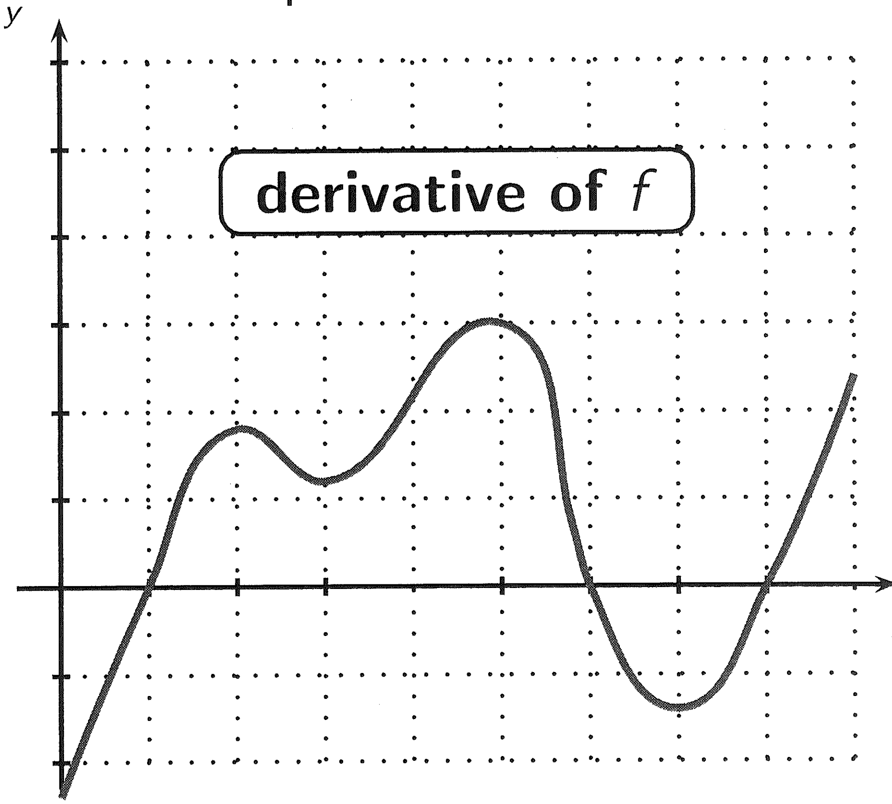
The graph of the derivative f' of a function f is shown.

(a) On what intervals is f increasing or decreasing?

(b) At what values of x does f have a local maximum or minimum?

(c) On what intervals is f concave upward or downward?

(d) State the x-coordinate of the inflection points of f.



Recall that what we are given is the graph of the <u>derivative</u> of f:

$f' = 0$ from the graph at $x = 1, 6, 8$

the sign of f':

sign f': $\quad \underset{0}{|} \; - \; - \; \underset{1}{\bullet} \; + + + + + \; \underset{6}{|} \; - - - \; \underset{8}{|} \; + + \; \underset{9}{|}$

$\qquad\qquad \searrow \qquad \nearrow \qquad \searrow \quad \nearrow$

f is <u>increasing</u> on $(1, 6)$ and $(8, 9)$

f is <u>decreasing</u> on $(0, 1)$ and $(6, 8)$

<u>local min</u> at $x = 1$; $x = 8$

<u>local max</u> at $x = 6$

To find when f is concave up or concave down we need to know when f' is increasing ($\equiv f$ conc. up) and when f' is decreasing ($\equiv f$ conc. down)

f conc up on $(0,2)$, $(3,5)$, $(7,9)$

f conc down on $(2,3)$, $(5,7)$

inflection points at $x = 2, 3, 5, 7$

Example 8: (Online Homework HW18, #14)

Suppose that on the interval I, $f(x)$ is positive and concave up. Furthermore, assume that $f''(x)$ exists and let $g(x) = (f(x))^2$. Use this information to answer the following questions.

(a) $f''(x) >$ ____ on I.
(b) $g''(x) = 2(A^2 + Bf''(x))$, where $A =$ ____ and $B =$ ____
(c) $g''(x) >$ ____ on I.
(d) $g(x)$ is ____ on I.

$f(x)$ is positive and concave up

Hence: $f(x) \geq 0$ and $f''(x) \geq 0$

Consider $g(x) = [f(x)]^2$

$$g'(x) = 2[f(x)]^{2-1} \cdot f'(x) = 2f(x)f'(x)$$

$$g''(x) = 2f'(x) \cdot f'(x) + 2f(x) \cdot f''(x)$$

product rule

$$= 2[f'(x)]^2 + 2f(x) \cdot f''(x)$$

(a) $f''(x) \geq 0$ on I because f is concave up

(b) $A = f'(x)$ and $B = f(x)$

(c) Since $g''(x) = 2\left[(f'(x))^2 + f(x) f''(x) \right]$

and $f(x) \geq 0$, $f''(x) \geq 0$ and $(f'(x))^2 \geq 0$

then we have that $\underline{g''(x) \geq 0}$

(d) So g is <u>concave up</u>

MA 137 — Calculus 1 with Life Science Applications
Optimization
(Section 5.4)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

November 11 & 14, 2016

Introduction

There are many situations in which we wish to maximize or minimize certain quantities. For instance,

- in a chemical reaction, you might wish to know under which conditions the reaction rate is maximized;
- in an agricultural setting, you might be interested in finding the amount of fertilizer that would maximize the yield of some crops;
- in a medical setting, you might wish to optimize the dosage of a drug for maximum benefit;
- optimization problems also arise in the study of the evolution of life histories and involve questions such as when an organism should begin reproduction in order to maximize the number of surviving offspring.

In each case, we are interested in finding global extrema.

Suggestions

The most important **skill** in solving a word problem is reading comprehension. The most important **attitude** to have in attacking word problems is to be willing to think about what you are reading and to give up on hoping to *mechanically* apply a set of steps.

MAX-MIN PROBLEMS

All max-min problems ask you to find the largest or smallest value of a function on an interval. Usually, the hard part is reading the English and finding the formula for the function. Once you have found the function, then you can use the techniques from Sections 5.1, 5.2, and 5.3 to find the largest or smallest values.

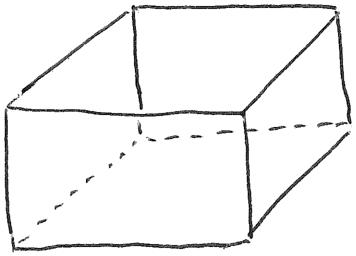
Max-Min Guideline

Nevertheless – despite our previous remarks – we will present some useful strategies to employ that are often helpful.

1. Read the problem
2. Define your variables. If possible, draw a picture and label it.
3. Determine exactly what needs to be maximized or minimized.
4. Write the *general* formula for what you are trying to maximize or minimize. If this formula only involves *one* variable, then skip to step **8**.
5. Find the relationship(s) (i.e., equation(s)) between the variables.
6. Do the algebra to solve for one variable in the equation(s) as a function of the other(s).
7. Use your formula from step **4** to rewrite the formula that you want to maximize or minimize as a function of one variable only.
8. Write down the interval over which the above variable can vary, for the particular word problem you are solving.
9. Take the derivative and find the critical points.
10. Use the techniques from Chapter 5 to find the maximum or the minimum.

Example 1: (Online Homework HW19, # 1)

Find the dimensions of an open rectangular box with a square base that holds 7000 cubic cm and is constructed with the least building material possible.



Let x be the size of one of the sides of the square base. Let y be the height of the box.

We know that $\boxed{x^2 \cdot y \overset{\text{MUST}}{=} 7000}$

Hence $y = \dfrac{7\,000}{x^2}$

The box has no top and we need to minimize the surface area:

$$\text{Surface area} = S = \underbrace{4xy}_{\text{4 sides}} + \underbrace{x^2}_{\text{base}} = 4x\frac{7000}{x^2} + x^2$$

$$\therefore \quad \boxed{S(x) = \frac{28{,}000}{x} + x^2}$$

There are no constraints on x other than

$$0 < x < +\infty$$

it is an <u>open</u> interval.

Since $x > 0$ then $S(x)$ only takes positive values

That is $S(x) > 0$.

$$S'(x) = -\frac{28{,}000}{x^2} + 2x = \frac{2x^3 - 28{,}000}{x^2}$$

$$S'(x) = 0 \iff 2x^3 - 28{,}000 = 0 \iff x^3 = 14{,}000$$

$$\iff x = \sqrt[3]{14000} = \underline{\underline{24.101}}$$

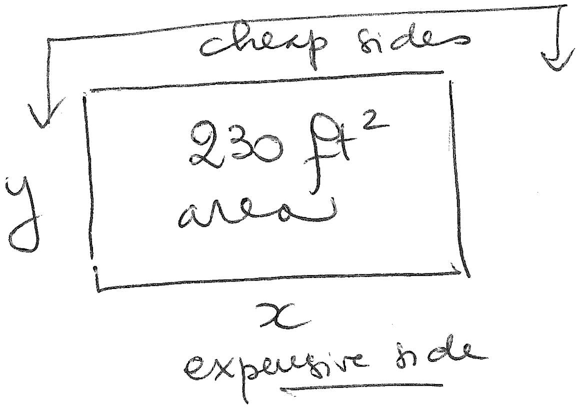
sign of $S'(x)$

$$\begin{array}{ccccc} & - - - - - & & + + + + + & \\ 0 & & 24.101 & & \end{array}$$

Hence $S(x)$ is decreasing before 24.101 and increasing after 24.101. Hence the value $\boxed{x = 24.101}$ is a point of a local min. Since there is just this value, it is actually a <u>global min for</u> $S(x)$

Example 2: (Online Homework HW19, # 4)

A fence is to be built to enclose a rectangular area of 230 square feet. The fence along three sides is to be made of material that costs 3 dollars per foot, and the material for the fourth side costs 13 dollars per foot. Find the dimensions of the enclosure that is most economical to construct.



Let x and y be the sizes of the sides of the rectangular area: $xy = 230$

So $y = \frac{230}{x}$ $\qquad$ $0 < x < \infty$

$$\text{Cost of enclosure} = \underbrace{3 \cdot (2y + x)}_{\text{cheaper side}} + \underbrace{13 \cdot x}_{\text{expensive side}}$$

$$C(x) = 6\left(\frac{230}{x}\right) + 16x = \frac{1380}{x} + 16x \qquad 0 < x < \infty$$

We need to minimize $C(x)$.

$$C'(x) = -\frac{1380}{x^2} + 16 = \frac{16x^2 - 1380}{x^2}$$

only positive

$$C'(x) = 0 \iff 16x^2 - 1380 = 0 \iff x = \pm\sqrt{\frac{1380}{16}} \cong \pm 9.287 \text{ feet}$$

sign of C' $\qquad$ $- - - \;\big|\; + + +$

9.287

Thus $C(x)$ has a local minimum at

$x = 9.287$

$$\begin{array}{ccc} - - - - & \mid & + + + \\ \hline \searrow & 9.287 & \nearrow \end{array}$$

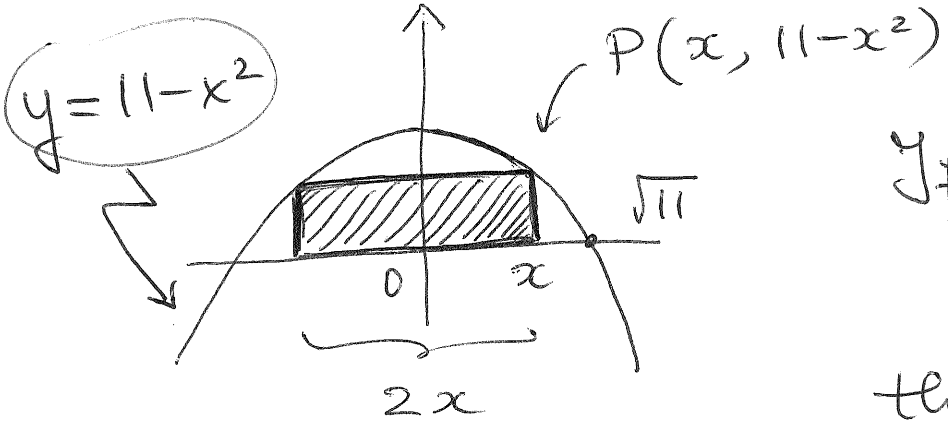
Since this is the only extremum, then $C(x)$ has a global minimum.

The other dimension of the enclosure is

$$y = \frac{230}{9.287} = \underline{24.766 \text{ ft}}$$

Example 3: (Online Homework HW19, # 5)

A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y = 11 - x^2$. What are the dimensions of such a rectangle with the greatest possible area?



If x denotes a point between 0 and $\sqrt{11}$ then the height of the rectangle at $P(x, 11-x^2)$ is exactly $11-x^2$

Thus the area of the whole rectangle is

$$A(x) = \underbrace{2x}_{\text{base}} \cdot \underbrace{(11-x^2)}_{\text{height}} \qquad 0 \le x \le \sqrt{11}$$

Hence we need to maximize $A(x) = 22x - 2x^3$ on the closed interval $[0, \sqrt{11}]$. Since $A(x)$ is continuous on the interval, we know that a global max exists by the Extreme Value Theorem

We need to check the value of $A(x)$ at

- end points of $[0, \sqrt{11}]$
- critical points of $A(x)$

Obviously, at the end points of $[0, \sqrt{11}]$ the area of the rectangle is 0.

$A'(x) = 22 - 6x^2 = 0 \iff x^2 = \frac{22}{6}$

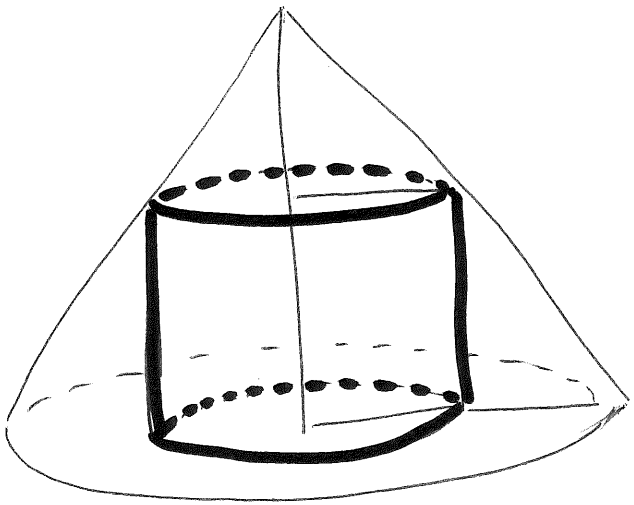
$\iff \underline{x = \pm 1.915}$ But x is positive for our choice so $\underline{x_0 = 1.915}$

$A(1.915) \cong 28.0845 \longleftarrow$ global max at this value

The dimensions are : $\underline{\underline{2x_0 \text{ times } y_0}}$: 3.83 times 7.333

Example 4: (Online Homework HW19, # 6)

A cylinder is inscribed in a right circular cone of height 6 and radius (at the base) equal to 2.5. What are the dimensions of such a cylinder which has maximum volume?



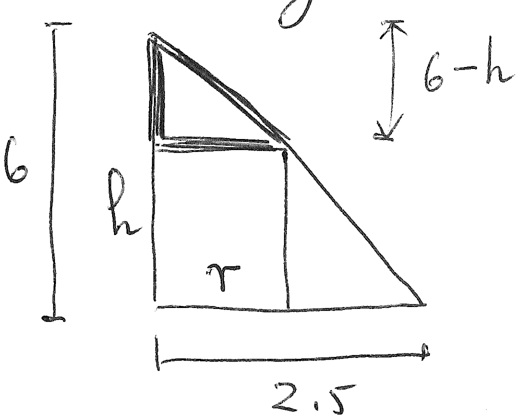
The height of the cone is 6 and the radius of the cone is 2.5

Let r be the radius of the cylinder and h its height.

$0 \leq r \leq 2.5$

We need to maximize the volume of the cylinder

$V = \pi r^2 \cdot h$. We need to eliminate h from the expression. We use the similar triangles from the picture:



$$\frac{6}{2.5} = \frac{6-h}{r}$$ OR

$$2.4r = 6 - h$$ OR

$$\boxed{h = 6 - 2.4r}$$

Thus : $V(r) = \pi r^2 (6 - 2.4r) \qquad 0 \le r \le 2.5$

$= 6\pi r^2 - 2.4\pi r^3$

Since $V(r)$ is continuous on the closed interval $[0, 2.5]$, by the EVT the function $V(r)$ has a global maximum. The candidates when the global max is attained are: end points or critical points.

- $V(0) = 0 = V(2.5)$
- $V'(r) = 0 \iff 12\pi r - 7.2\pi r^2 = 0$

$\iff r = 0$ OR $12\pi - 7.2\pi r = 0$

$r = 1.667$

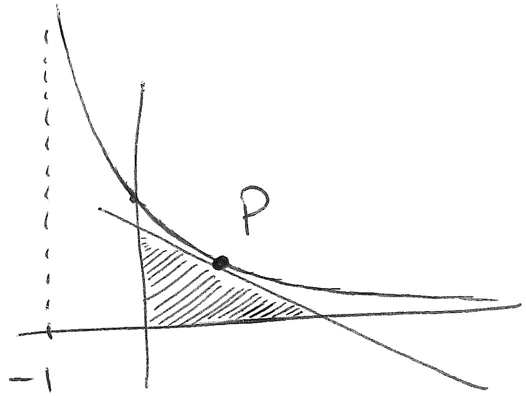
$h = 1.9992$

$V(1.667) = 17.453$ unit^3 global max

Example 5: (Online Homework HW19, # 8)

Find the maximum area of a triangle formed in the first quadrant by the x-axis, y-axis and a tangent line to the graph of $f(x) = (x+1)^{-2}$.

$f(x) = \frac{1}{(x+1)^2}$ has the following graph:



Let P a point on graph of f.

Let $x = a$ be its x-coordinate; that $y = \frac{1}{(a+1)^2}$.

We need to find the equation of the tg line at P.

$$f'(x) = -2(x+1)^{-3} = \frac{-2}{(x+1)^3} \quad \text{so} \quad f'(a) = \frac{-2}{(a+1)^3}$$

$$\therefore \quad \boxed{y - \frac{1}{(a+1)^2} = \frac{-2}{(a+1)^3}(x-a)}$$

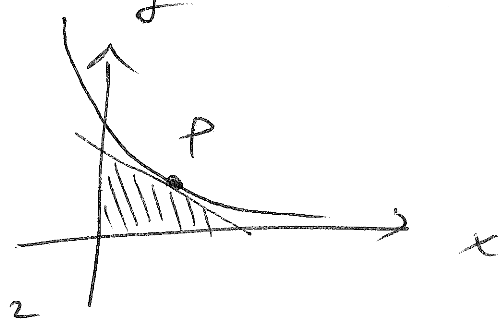
OR $$y = -\frac{2}{(a+1)^3}x + \frac{2a}{(a+1)^3} + \frac{1}{(a+1)^2}$$

$$= -\frac{2}{(a+1)^3}x + \frac{3a+1}{(a+1)^3}$$

The y-intercept of the tangent line is

$y = \dfrac{3a+1}{(a+1)^3}$; the x-intercept of the tangent line

is $\dfrac{3a+1}{2}$



Thus the area of the triangle is

$$A(a) = \frac{3a+1}{2} \cdot \frac{(3a+1)}{(a+1)^3} \cdot \frac{1}{2} = \frac{(3a+1)^2}{4(a+1)^3}$$

the x-coordinate of the point P varies between -1 and ∞.

Thus we need to maximize

$$\boxed{A(a) = \frac{(3a+1)^2}{4(a+1)^3} \quad \text{on} \quad -1 < a < \infty}$$

For simplicity, let's change the name of the variable:

$$A(x) = \frac{(3x+1)^2}{4(x+1)^3} \qquad -1 < x < \infty$$

We need to study the sign of $A'(x)$:

$$A'(x) = \frac{2(3x+1)\cdot(3)\left[4(x+1)^3\right] - (3x+1)^2\cdot\left[4\cdot 3(x+1)^2\cdot(1)\right]}{16(x+1)^6}$$

$$= \frac{12(3x+1)\cancel{(x+1)^2}\left[2(x+1) - (3x+1)\right]}{16(x+1)^{\cancel{6}\,4}} = \underline{\underline{\frac{12(3x+1)(1-x)}{16(x+1)^4}}}$$

$A'(x) = 0 \iff \underline{\underline{x = -1/3}}$ OR $\underline{\underline{x = 1}}$

sign of A':

-1		$-1/3$		1	
	$-$		$+$		$- - -$

thus there is $\underline{\text{a maximum}}$ at $\underline{x = 1}$

$$\boxed{A(1) = 1/2}$$

Example 6: (Neuhauser, Example 2, p. 238)

Let $Y(N)$ be the yield of an agricultural crop as a function of nitrogen level N in the soil. A model that is used for this relationship is

$$Y(N) = \frac{N}{1+N^2} \qquad \text{for} \quad N \geq 0$$

(where N is measured in appropriate units). Find the nitrogen level that maximizes yield.

$$Y(N) = \frac{N}{1+N^2}, \quad \text{with } N \geq 0$$

[If you do not like the name of the variable switch to

$$y(x) = \frac{x}{1+x^2}.$$]

Notice that $Y(N) \geq 0$ for $N \geq 0$, as it should be since $Y(N)$ denotes a yield.

Let's find $Y'(N)$. $$Y'(N) = \frac{1 \cdot (1+N^2) - N(2N)}{(1+N^2)^2} =$$

$$= \frac{1+N^2-2N^2}{(1+N^2)^2} = \frac{1-N^2}{(1+N^2)^2}$$

Notice that $Y'(N) = 0 \iff 1-N^2 = 0 \iff$

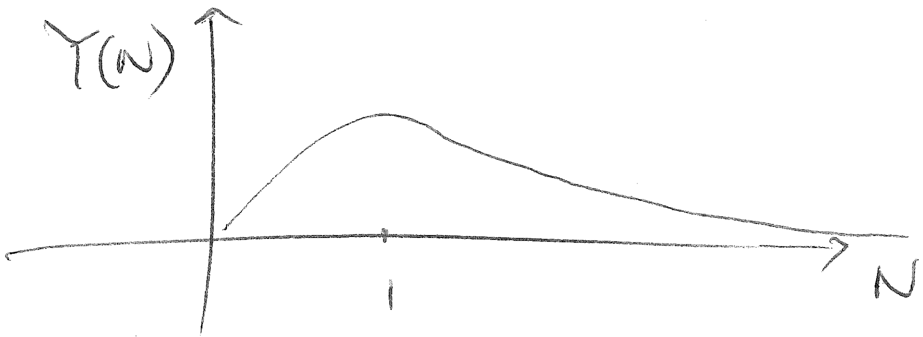
$N = \pm 1$. Hence $N=1$ since $Y(N)$ is defined

for $N \geq 0$. Also sign Y': $\quad ++ \;\underset{1}{|}\; ---$

Hence $N=1$ is a point when $Y(N)$ has a local maximum. It is actually a global max, since there is only one critical value.

Notice $\lim\limits_{N\to\infty} Y(N) = \lim\limits_{N\to\infty} \dfrac{N}{1+N^2} = 0$

graph of $Y(N)$ looks like



Example 7:

Suppose that a patient is given a dosage x of some medication, and the probability of a cure is

$$P(x) = \frac{\sqrt{x}}{1+x}.$$

What dosage maximizes the probability of a cure?

$$P(x) = \frac{\sqrt{x}}{1+x} \qquad 0 < x < +\infty$$

We need to find the critical numbers and study the sign of $P'(x)$:

$$P'(x) = \frac{\frac{1}{2\sqrt{x}}(1+x) - \sqrt{x}\,(1)}{(1+x)^2} = \frac{\frac{1+x - (2\sqrt{x})\sqrt{x}}{2\sqrt{x}}}{(1+x)^2}$$

$$= \frac{1+x-2x}{2\sqrt{x}\,(1+x)^2} = \frac{1-2x}{2\sqrt{x}\,(1+x)^2}$$

$P'(x)=0 \iff 1-2x=0$ so $x = 1/2$

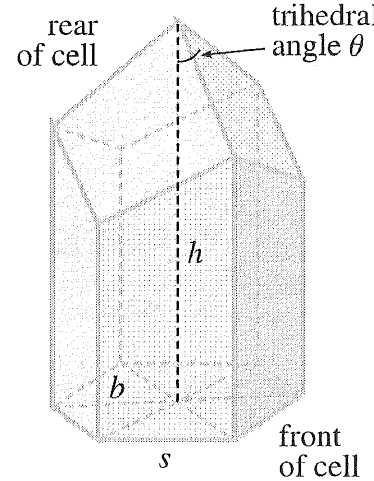
sign P': $\quad + + \;\; \underset{1/2}{|} \;\; - - - -$

$\nearrow \quad 1/2 \quad \searrow$

Thus $x=1/2$ is a local max. Actually it is a global max.

Example 8: (Online Homework HW19, # 11)

In a beehive, each cell is a regular hexagonal prism, open at one end with a trihedal angle at the other end. It is believed that bees form their cells in such a way as to minimize the surface area for a given volume, thus using the least amount of wax in cell construction. Examination of these cells has shown that the measure of the apex angle θ is amazingly consistent.



Based on the geometry of the cell, it can be shown that the surface area S is given by

$$S = 6sh - \frac{3}{2}s^2 \cot\theta + \frac{3\sqrt{3}}{2}s^2 \csc\theta$$

where s, the length of the sides of the hexagon, and h, the height, are constants.

(a) Calculate $dS/d\theta$.

(b) What angle should bees prefer (in radians)?

(c) Determine the minimum surface area of the cell.

$$S = 6sh - \frac{3}{2}s^2 \underline{\underline{\cot\theta}} + \frac{3\sqrt{3}}{2}s^2 \underline{\underline{\csc\theta}}$$

s, h are constant

$$\frac{dS}{d\theta} = -\frac{3}{2}s^2 \frac{d}{d\theta}\left(\frac{\cos\theta}{\sin\theta}\right) + \frac{3\sqrt{3}}{2}s^2 \frac{d}{d\theta}\left(\frac{1}{\sin\theta}\right)$$

$$= -\frac{3}{2}s^2 \frac{(-\sin\theta)\sin\theta - \cos\theta(\cos\theta)}{\sin^2\theta} + \frac{3\sqrt{3}}{2}s^2 \frac{(-\cos\theta)}{\sin^2\theta}$$

$$= -\frac{3}{2}s^2 \frac{(-1)}{\sin^2\theta} - \frac{3\sqrt{3}s^2\cos\theta}{2\sin^2\theta}$$

$$= \boxed{\frac{3s^2(1-\sqrt{3}\cos\theta)}{2\sin^2\theta}}$$

$$\frac{dS}{d\theta} = 0 \iff 1-\sqrt{3}\cos\theta = 0 \iff \cos\theta = \frac{1}{\sqrt{3}}$$

Using your calculator $\cos\theta = \frac{1}{\sqrt{3}} \iff$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.9553$$

sign of $\frac{dS}{d\theta}$ $\quad - - - - - \cdot + + +$

$\qquad\qquad\qquad\qquad \searrow \quad \nearrow$

Hence this is the value that gives a local min.

Again this is a global min.

when $\cos\theta = \frac{1}{\sqrt{3}}$ $\quad \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{3}} = \sqrt{\frac{2}{3}}$

Hence

$$S = \underset{\text{area}}{\text{minimal surface}} = 6sh - \frac{3}{2}s^2 \frac{1/\sqrt{3}}{\sqrt{2}/\sqrt{3}} + \frac{3\sqrt{3}}{2}s^2 \cdot \frac{1}{\sqrt{2/3}}$$

$\therefore$ the minimal surface area is

$$S = 6sh - \frac{3}{2}s^2\frac{1}{\sqrt{2}} + \frac{3\sqrt{3}s^2}{2}\cdot\frac{\sqrt{3}}{\sqrt{2}}$$

$$= 6sh - \frac{3}{2}s^2\frac{1}{\sqrt{2}} + \frac{9s^2}{2\sqrt{2}}$$

$$= 6sh + \frac{3s^2}{\sqrt{2}}$$

Example 9: (Online Homework HW19, # 12)

One question for fishery management is how to control fishing to optimize profits for the fishermen. One DE describing the population dynamics for a population of fish F with harvesting is given by the equation,

$$\frac{dF}{dt} = rF\left(1 - \frac{F}{K}\right) - xF$$

where r is the growth rate of this species of fish at low density, K is the carrying capacity of this population, and x is the harvesting effort of the fishermen. The non-zero equilibrium of this equation is given by

$$F_e = K\frac{(r-x)}{r}.$$

One formula for profitability is computed by the equation

$$P(x) = xF_e = Kx\frac{(r-x)}{r}.$$

Find the level of harvesting x that produces the maximum profit possible $x_{\max}$ with this dynamics.

What is the equilibrium population F_e at this optimal profitability?

Also, determine the maximum possible fish population for this model and at what harvesting level this occurs.

$$P(x) = Kx\left(\frac{r-x}{r}\right) = Kx - \frac{K}{r}x^2$$

$$P'(x) = K - \frac{K}{r}\cdot 2x = 0$$

$$\iff K = \frac{K}{r}2x \iff x = \frac{r}{2}$$

sign of $P'(x)$: $+++ \;|\; ---$ at $r/2$

$\therefore$ there is a local max when $x = r/2$

Since this is the only critical value it gives a global max at $x = r/2$

F_e when $x = r/2$ is $K \cdot \frac{\left(r - (r/2)\right)}{r} = K\frac{r/2}{r} = \frac{K}{2}$

Of course the maximum population in this model would occur when there is no harvesting: $\underline{x = 0}$. In this case

$\frac{dF}{dt} = rF(1 - F/K)$ is the logistic growth model. The max occurs when we reach the carrying capacity K.

Example 10: (Online Homework HW19, # 14)

[From: D. A. Roff, The Evolution of Life Histories, Chapman and Hall, 1992.]

Semelparous organisms breed only once during their lifetime. Examples of this type of reproduction strategy can be found with Pacific salmon and bamboo. The per capita rate of increase, r, can be thought of as a measure of reproductive fitness. The greater r, the more offspring an individual produces. The intrinsic rate of increase is typically a function of age, x. Models for age-structured populations of semelparous organisms predict that the intrinsic rate of increase as a function of x is given by

$$r(x) = \frac{\ln[L(x)M(x)]}{x},$$

where $L(x)$ is the probability of surviving to age x and $M(x)$ is the number of female births at age x. Suppose that

$$L(x) = e^{-0.17x} \qquad \text{and} \qquad M(x) = 3x^{0.7}.$$

Find the optimal age of reproduction.

$$r(x) = \frac{\ln(L(x)M(x))}{x}$$

$$= \frac{\ln(e^{-0.17x} \cdot 3x^{0.7})}{x}$$

$$= \frac{\ln e^{-0.17x} + \ln(3x^{0.7})}{x}$$

$$= \frac{-0.17x + (\ln 3) + \ln(x^{0.7})}{x}$$

$$= \boxed{-0.17 + \frac{\ln(3) + 0.7 \ln x}{x}} \qquad 0 < x < \infty$$

We need to find $r'(x) = 0$ and study the sign of $r'(x)$.

$$r(x) = -0.17 + \frac{\ln(3) + 0.7 \ln x}{x}$$

$$r'(x) = 0 + \frac{\left(0.7 \frac{1}{x}\right) x - \left(\ln(3) + 0.7 \ln x\right) \cdot (1)}{x^2}$$

$$= \frac{0.7 - \ln(3) - 0.7 \ln x}{x^2}$$

$$r'(x) = 0 \iff 0.7 - \ln(3) - 0.7 \ln x = 0$$

$$\iff \ln x = \frac{0.7 - \ln(3)}{0.7} \cong -0.5694$$

$$\iff x = e^{-0.5694} \cong 0.5658$$

optimal reproduction age

sign $r'(x)$

$$\begin{array}{ccc} + + & | & - - - \\ \hline \nearrow & 0.5658 & \searrow \end{array}$$

MA 137 — Calculus 1 with Life Science Applications
L'Hôpital's Rule
(Section 5.5)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

November 16, 2016

Heuristics

We have often encountered the situation in which we had to compute $\lim_{x \to a} \frac{f(x)}{g(x)}$ and we had that both the following limits were zero

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0.$$

Using a linear approximation at $x = a$, we find that, for x close to a

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)}$$

Since $f(a) = g(a) = 0$ and $x \neq a$, the right-hand side is equal to

$$\frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}$$

provided that $f'(a)/g'(a)$ is defined. We therefore hope that something like

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

holds when $f(a)/g(a)$ is of the form $0/0$ and $f'(a)/g'(a)$ is defined. In fact, something like this does hold; it is called **l'Hôpital's rule.**

L'Hôpital's Rule

Theorem

Suppose that f and g are differentiable functions and that

$$\lim_{x \longrightarrow a} f(x) = 0 = \lim_{x \longrightarrow a} g(x) \qquad \textit{or} \qquad \lim_{x \longrightarrow a} f(x) = \infty = \lim_{x \longrightarrow a} g(x)$$

Then

$$\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} = \lim_{x \longrightarrow a} \frac{f'(x)}{g'(x)}$$

provided the second limit exists.

L'Hôpital's rule can actually be applied to calculate limits for seven kinds of indeterminate expressions

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0 \cdot \infty \qquad \infty - \infty \qquad 0^0 \qquad 1^\infty \qquad \infty^0.$$

(Note that l'Hôpitals rule works for $a = +\infty$ or $-\infty$ as well.)

Reduction to 0/0 or ∞/∞ Form

$0 \cdot \infty$ Suppose we have to compute $\lim_{x\to a} f(x)g(x)$ where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$. To apply l'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{1/g(x)} = \lim_{x\to a} \frac{g(x)}{1/f(x)}$$

In the first case the ratio is $0/0$, whereas in the second case the ratio is ∞/∞. Usually only one of the two expressions is easy to evaluate.

$\infty - \infty$ Suppose we have to compute $\lim_{x\to a} [f(x) - g(x)]$ where $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. To apply l'Hôpital's rule to this kind of limit write it in one of the two forms

$$\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x)\left(1 - \frac{g(x)}{f(x)}\right) = \lim_{x\to a} g(x)\left(\frac{f(x)}{g(x)} - 1\right)$$

and hope that the limit is of the form $0 \cdot \infty$.

$0^0 \; 1^\infty \; \infty^0$ Suppose we have to compute $\lim_{x\to a} [f(x)]^{g(x)}$, which becomes of the form 0^0, 1^∞ or ∞^0. The key to solving these limits is to write them as exponentials

$$\lim_{x\to a} [f(x)]^{g(x)} = \lim_{x\to a} \exp\left\{ \ln\, [f(x)]^{g(x)} \right\} = \lim_{x\to a} \exp\left\{ g(x) \cdot \ln f(x) \right\} = \exp\left[\lim_{x\to a} \left(g(x) \cdot \ln f(x)\right) \right].$$

The last step, in which we interchanged lim and exp, uses the fact that the exponential function is continuous.

Example 1: (Nuehauser, p. 247)

Evaluate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$.

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = \frac{0}{0} \quad \text{if we use direct evaluation}$$

Hence we can apply l'Hôpital's rule.

We obtain:

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{2x}{1} = \frac{6}{1} = \boxed{6}$$

Note: in Chapter 3 we solved the problem by factoring and simplifying the expression

$$\lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} = \lim_{x \to 3} x+3 = 3+3 = 6$$

Example 2: (Nuehauser, p. 247)

Evaluate $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$.

$$\lim_{x\to 0} \frac{e^x-1}{x} = \frac{e^0-1}{0} = \frac{0}{0}$$

Hence we can use l'Hôpital's rule:

$$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{e^x}{1} = e^0 = \boxed{1}$$

Example 3:

Evaluate $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1-1}{0} = \frac{0}{0}$$ by direct substitution

Hence we can use l'Hôpital's rule

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{-(-\sin x)}{2x} = \lim_{x \to 0} \frac{\sin x}{2x}$$

$= \frac{0}{0}$ again. Hence we use l'Hôpital's rule again

$$= \lim_{x \to 0} \frac{\cos x}{2} = \boxed{\frac{1}{2}}$$

<u>Note</u> that in Section 3.4 we gave a geometric argument for $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Example 4: (Neuhauser, Problem # 25, p. 252)

Evaluate $\lim_{x \to \infty} x \cdot e^{-x}$.

What about $\lim_{x \to \infty} x^{13} \cdot e^{-x}$? (Online Homework HW20, # 5)

$$\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$$

We can use l'Hôpital rule. Hence

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = \boxed{0}$$

<u>HW20, #5</u> : $\lim_{x\to\infty} \frac{x^{13}}{e^x} = \frac{\infty}{\infty}$ = use l'Hôpital's

rule $= \lim_{x\to\infty} \frac{13x^{12}}{e^x} = \frac{\infty}{\infty} = \ldots =$ use

<u>many</u> <u>more</u> <u>times</u> l'Hôpital's rule to get

$$= \lim_{x\to\infty} \frac{13!}{e^x} = \frac{13!}{\infty} = \boxed{0}$$

Example 5: (Online Homework HW20, # 3)

Evaluate $\lim_{x \to 0^+} 7\sqrt{x} \cdot \ln x.$

$$\lim_{x \to 0^+} 7\sqrt{x} \cdot \ln x = 0 \cdot (-\infty)$$

Hence we can use l'Hôpital's rule, where we rewrite $7\sqrt{x} \ln x$ as $\dfrac{7 \ln x}{1/\sqrt{x}} = \dfrac{7 \ln x}{x^{-1/2}}$

Hence:

$$\lim_{x \to 0^+} 7\sqrt{x} \cdot \ln x = \lim_{x \to 0^+} \frac{7 \ln x}{x^{-1/2}} = \frac{-\infty}{+\infty} =$$

$$= \text{l'Hôpital} = \lim_{x \to 0^+} \frac{7\,1/x}{-\frac{1}{2}\,x^{-3/2}} = \lim_{x \to 0^+} 7\frac{1}{x}\left[-2x^{3/2}\right]$$

$$= \lim_{x \to 0^+} -14\,\frac{x\sqrt{x}}{x} = \lim_{x \to 0^+} -14\sqrt{x} = 0$$

Hence $\boxed{\lim_{x \to 0^+} 7\sqrt{x} \ln x = 0}$

Example 6: (Neuhauser, Example # 10, p. 250)

Evaluate $\lim_{x \to \infty} x - \sqrt{x^2 + x}$.

$$\lim_{x\to\infty} x - \sqrt{x^2+x} = \infty - \infty$$

Let's rewrite the limit as follows:

$$\lim_{x\to\infty} x - \sqrt{x^2+x} = \lim_{x\to\infty} x\left[1 - \frac{\sqrt{x^2+x}}{x}\right] =$$

$$= \lim_{x\to\infty} \frac{1-\sqrt{1+\frac{1}{x}}}{\frac{1}{x}} = \frac{0}{0} =$$ hence we can apply l'Hôpital's rule

$$= \lim_{x\to\infty} \frac{-\frac{1}{2}\left(1+\frac{1}{x}\right)^{-1/2} \cdot \cancel{\left(-1/x^2\right)}}{\cancel{-1/x^2}} = \lim_{x\to\infty} \frac{-1/2}{\sqrt{1+1/x}} = \boxed{-\frac{1}{2}}$$

Example 7: (Online Homework HW20, # 4)

Evaluate $\lim_{x \to 0^+} x^x$.

$$\lim_{x\to 0^+} x^x = 0^0$$

Hence we can rewrite the limit as:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = \lim_{x\to 0^+} e^{x \ln x}$$

$$= e^{\lim_{x\to 0^+} x \ln x} = e^0 = \boxed{1}$$

<u>But</u> $\lim_{x\to 0^+} x \ln x = 0\cdot(-\infty) = \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \dfrac{-\infty}{\infty}$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \frac{1}{x}\cdot(-x^2) = \lim_{x\to 0^+}(-x) = \boxed{0}$$

Example 8: (Neuhauser, Problem # 62, p. 253)

Use l'Hôpital's rule to find $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x$ where c is a constant.

What about $\lim_{x\to\infty} 3x\big(\ln(x+3)-\ln x\big)$? (Online Homework HW20, #

$$\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = 1^\infty$$

Hence we can rewrite the limit as

$$\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{c}{x}\right)^x} =$$

$$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{c}{x}\right)} = e^{\lim\limits_{x\to\infty}\frac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}}} = \boxed{e^c}$$

<u>Note</u> that $\lim\limits_{x\to\infty}\dfrac{\ln\left(1+\frac{c}{x}\right)}{\frac{1}{x}} = \dfrac{0}{0} =$ by l'Hôpital's rule

$$= \lim_{x\to\infty}\frac{\frac{1}{1+c/x}\cdot\left(-\frac{c}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{c}{1+c/x} = \lim_{x\to\infty}\frac{cx}{x+c} = \boxed{c}$$

$$\lim_{x\to\infty} 3x\left[\ln(x+3)-\ln x\right] = \infty(\infty-\infty)$$

$$= \lim_{x\to\infty} 3x \cdot \ln\left(\frac{x+3}{x}\right) = \lim_{x\to\infty} 3 \cdot \ln\left(\frac{x+3}{x}\right)^x =$$

$$= 3 \cdot \ln\left[\underbrace{\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x}_{e^3}\right] = 3 \ln e^3 = 3 \cdot 3 = 9$$

(by the first part)

MA 137 — Calculus 1 with Life Science Applications
Difference Equations: Stability
(Section 5.6)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

November 18, 2016

First-Order Recursions (Review)

In Chapter 2 we saw that an important biological application of sequences consists of models of seasonally breeding populations with nonoverlapping generations where the population size at one generation depends only on the population size of the previous generation.

The discrete exponential growth model fits into this category.

To this end, we introduced first-order recursions [$\equiv$ difference equations or iterated maps] by setting

$$x_{t+1} = f(x_t), \qquad t = 0, 1, 2, \ldots$$

where $f(x)$ is a function ($\equiv$ updating function) that describes the density dependence of the population dynamics.

The name *difference equation* comes from writing the dynamics in the form

$$\frac{x_{t+1} - x_t}{(t+1) - t} = g(x_t)$$

[where $g(x) = f(x) - x$], which allows us to track population size changes from one time step to the next.

The name *iterated map* refers to the recursive definition.

Fixed Points ($\equiv$ Equilibria)

In Chapter 2, we were able to analyze difference equations only numerically (except for equations describing exponential growth, which we were able to solve).

We saw that fixed points (or equilibria) played a special role.

A **fixed point** $\widehat{x}$ satisfies the equation

$$\widehat{x} = f(\widehat{x})$$

and has the property that if $x_0 = \widehat{x}$, then $x_t = \widehat{x}$ for $t = 1, 2, 3, \ldots$.

We also saw in a number of applications that, under certain conditions, x_t converged to the fixed point as $t \to \infty$ even if $x_0 \neq \widehat{x}$.

However, back in Chapter 2, we were not able to predict when such behavior would occur.

Example 1: (Neuhauser, Example # 1, p. 257)

Find the equilibria of the recursive sequence

$$x_{t+1} = \frac{1}{4} - \frac{5}{4}x_t^2, \qquad t = 0, 1, 2, \ldots$$

What happens to x_t as $t \to \infty$ if $x_0 = -0.9$?

(You could use for example an Excel spreadsheet.)

$$x_{t+1} = \frac{1}{4} - \frac{5}{4} x_t^2$$

To find the fixed points we need to solve

$$x = \frac{1}{4} - \frac{5}{4} x^2 \iff 4x = 1 - 5x^2 \iff$$

$$5x^2 + 4x - 1 = 0$$

We can factor it as: $(5x \quad)(x \quad) = 0$

and we obtain after inspection :

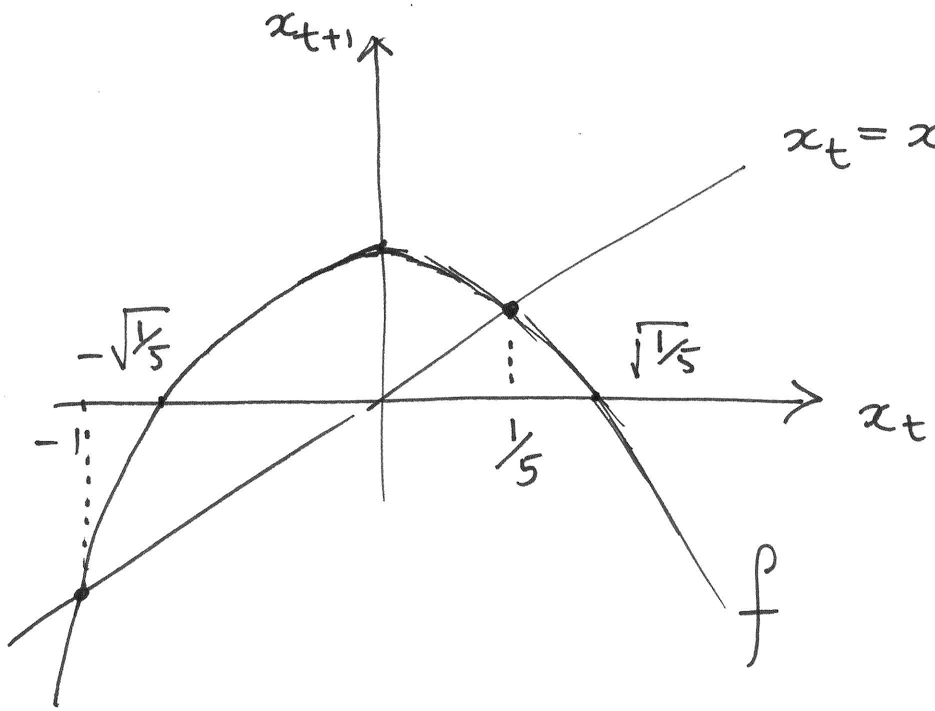
$$(5x - 1)(x + 1) = 0$$

$$\therefore \boxed{\hat{x}_1 = 1/5} \text{ and } \boxed{\hat{x}_2 = -1}$$

Notice that $x_{t+1} = \frac{1}{4} - \frac{5}{4} x_t^2$

$$= f(x_t)$$

where $y = f(x) = 1/4 - 5/4\, x^2$ parabola



graphic interpretation of fixed points

Note that $x_{t+1} = 1/4 - 5/4\, x_t^2$ implies

$x_0 = -0.9$ $\quad x_1 = -0.7625$ $\quad x_2 = -0.4767$

$x_3 = -0.0341$ $\quad x_4 = 0.2485$ $\quad x_5 = 0.1728$

$x_6 = 0.21268$ $\quad x_7 = 0.1935$ $\quad$ etc....

Exponential Growth

Exponential growth in discrete time is given by the recursion

$$N_{t+1} = R\,N_t, \qquad t = 0, 1, 2, \ldots$$

where N_t is the population size at time t and $R > 0$ is the growth rate.

We assume throughout that $N_0 \geq 0$, which implies that $N_t \geq 0$.

The fixed point of our recursion can be found by solving $N = R\,N$.

The only solution of this equation is $\widehat{N} = 0$, unless $R = 1$.
If $R = 1$, then the population size never changes, regardless of N_0.

What happens if we start with $N_0 > 0$ and $R \neq 1$?

In Chapter 2, we found that

$$N_t = N_0 R^t$$

is a solution of our recursion. Using this fact, we concluded that

$$N_t \longrightarrow \begin{cases} 0 & \text{if } 0 < R < 1 \\ \infty & \text{if } R > 1, \end{cases}$$

as $t \to \infty$.

We can interpret the behavior of N_t as follows:

If $0 < R < 1$ and $N_0 > 0$, then N_t will return to the equilibrium $\widehat{N} = 0$;

if $R \geq 1$ and $N_0 > 0$, then N_t will not return to the equilibrium $\widehat{N} = 0$ (more precisely, if $R = 1$, N_t will stay at N_0; if $R > 1$, N_t will go to ∞).

Terminology

We say that $\widehat{N} = 0$ is **stable** if $0 < R < 1$ and **unstable** if $R > 1$.

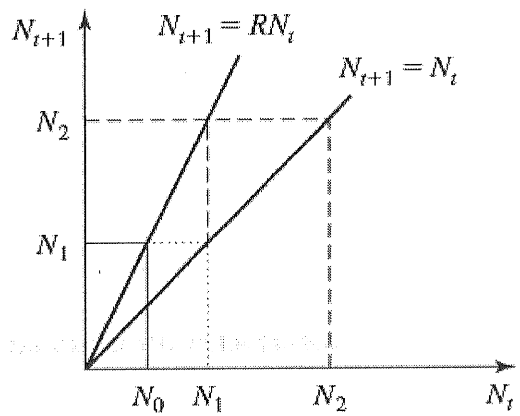
The case $R = 1$ is called **neutral**, since, no matter what the value of N_0 is, $N_t = N_0$ for $t = 1, 2, 3, \ldots$

Cobwebbing

We can determine **graphically** whether a fixed point is stable or unstable.

The fixed points of exponential growth recursive sequence are found graphically where the graphs of $N_{t+1} = RN_t$ and $N_{t+1} = N_t$ intersect.

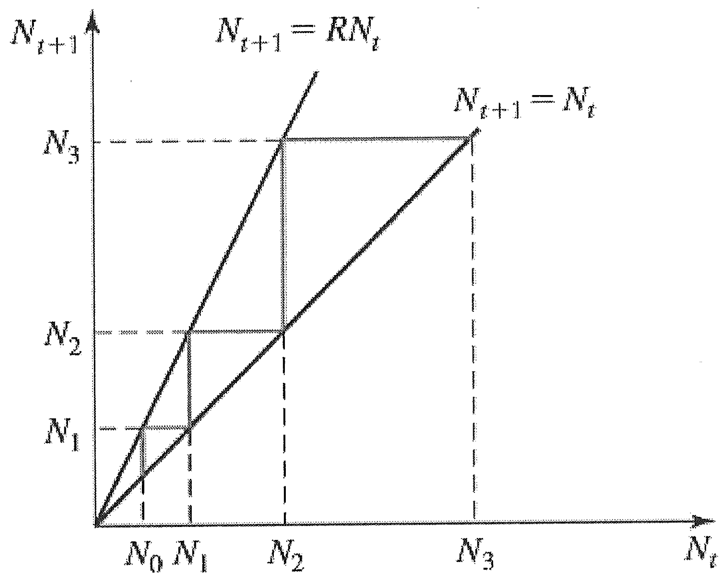
We see that the two graphs intersect where $N_t = 0$ only when $R \neq 1$.

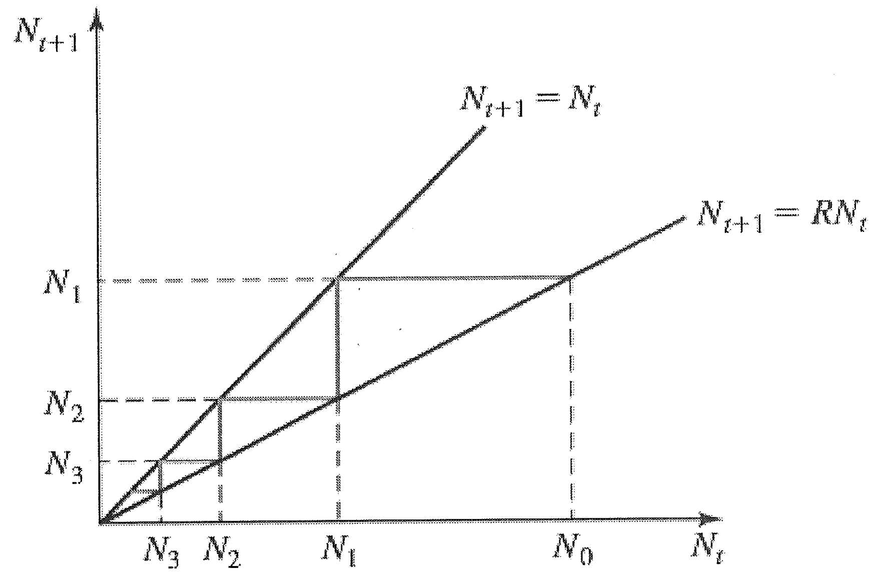


We can use the two graphs on the left to follow successive population sizes. Start at N_0 on the horizontal axis. Since $N_1 = RN_0$, we find N_1 on the vertical axis, as shown by the solid vertical and horizontal line segments. Using the line $N_{t+1} = N_t$, we can locate N_1 on the horizontal axis by the dotted horizontal and vertical line segments.

Using the line $N_{t+1} = RN_t$ again, we can find N_2 on the vertical axis, as shown in the figure by the broken horizontal and vertical line segments. Using the line $N_{t+1} = N_t$ once more, we can locate N_2 on the horizontal axis and then repeat the preceding steps to find N_3 on the vertical axis, and so on.

This procedure is called **cobwebbing**.





In the **figure on the left**, $R > 1$, and we see that if $N_0 > 0$, then N_t will not converge to the fixed point $\widehat{N} = 0$, but instead will move away from 0 (and, in fact, will go to infinity as t tends to infinity).

In the **figure on the right**, $0 < R < 1$, we see that if $N_0 > 0$, then N_t will return to the fixed point $\widehat{N} = 0$.

General Case

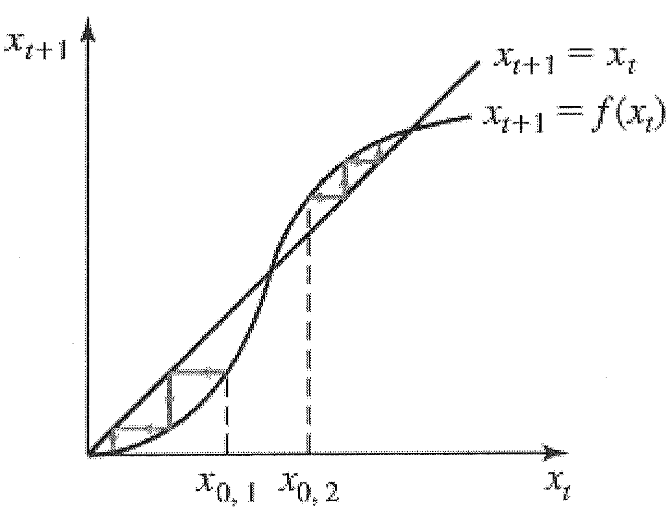
The general form of a first-order recursion is

$$x_{t+1} = f(x_t), \qquad t = 0, 1, 2, \ldots$$

We assume that the function f is differentiable in its domain.

- To find fixed points **algebraically**, we solve $x = f(x)$.
- To find them **graphically**, we look for points of intersection of the graphs of $x_{t+1} = f(x_t)$ and $x_{t+1} = x_t$.

The graphs in the picture intersect more than once, which means that there are multiple equilibria. We can use the cobwebbing procedure from the previous subsection to graphically investigate the behavior of the difference equation for different initial values.



Two cases are shown in the picture, one starting at $x_{0,1}$ and the other at $x_{0,2}$. We see that x_t converges to different values, depending on the initial value.

Stability Criterion

To determine the stability of an equilibrium — that is, whether it is stable or unstable — we will start at a value that is different from the equilibrium and check whether the solution will return to the equilibrium. We allow only initial values that are close to the equilibrium (we call it a **small perturbation**). The reason for looking only at small perturbations is that if there are multiple equilibria and if we start too far away from the equilibrium of interest, we might end up at a different equilibrium, not because the equilibrium of interest is unstable, but simply because we are drawn to another equilibrium.

If we are concerned only with small perturbations, we can approximate the function $f(x)$ by its linearization at the equilibrium $\widehat{x}$. Since the slope of the tangent-line approximation of $f(x)$ at $\widehat{x}$ is given by $f'(\widehat{x})$, we are led to the following criterion,

Theorem (Stability Criterion)

An equilibrium $\widehat{x}$ of $x_{t+1} = f(x_t)$ is locally stable if $|f'(\widehat{x})| < 1$.

Proof:

We look at the linearization of $f(x)$ about the equilibrium $\widehat{x}$ and investigated how a small perturbation affects the future of the solution. We denote a small perturbation at time t by z_t and write

$$x_t = \widehat{x} + z_t$$

Then $$x_{t+1} = f(x_t) = f(\widehat{x} + z_t)$$

Now, the linear approximation of $f(\widehat{x} + z_t)$ at $\widehat{x}$ is $L(\widehat{x} + z_t) = f(\widehat{x}) + f'(\widehat{x})\, z_t$. Taking this into account, we can approximate $x_{t+1}[= \widehat{x} + z_{t+1}]$ by

$$\widehat{x} + z_{t+1} \approx f(\widehat{x}) + f'(\widehat{x})\, z_t.$$

Since $f(\widehat{x}) = \widehat{x}$ ($\widehat{x}$ is an equilibrium), we find that

$$z_{t+1} \approx f'(\widehat{x})\, z_t$$

This approximation reminds of the equation $y_{t+1} = R\, y_t$ for exponential growth, where we identify y_t with z_t and R with $f'(\widehat{x})$. Since the solution of $y_{t+1} = R\, y_t$ is $y_t = y_0 R^t$ and $R^t \to 0$ as $t \to \infty$ for $|R| < 1$, we obtain the criterion $|f'(\widehat{x})| < 1$ for local stability. That is, if $|f'(\widehat{x})| < 1$, then the perturbation z_t will converge to $\widehat{z} = 0$ or, equivalently, $x_t \to \widehat{x}$ as $t \to \infty$.

(Again) Example 1: (Neuhauser, Example # 1, p. 257)

Use the stability criterion to characterize the stability of the equilibria of

$$x_{t+1} = \frac{1}{4} - \frac{5}{4}x_t^2, \qquad t = 0, 1, 2, \ldots$$

We already found the equilibria (fixed points) of

$$x_{t+1} = \frac{1}{4} - \frac{5}{4} x_t^2 = f(x_t)$$

where

$$f(x) = \frac{1}{4} - \frac{5}{4} x^2$$

We saw that $\hat{x}_1 = \frac{1}{5} = 0.2$

$$\hat{x}_2 = -1$$

Now, let's use the <u>stability criterion</u>:

$$f(x) = \frac{1}{4} - \frac{5}{4} x^2 \qquad f' = -\frac{5}{2} x$$

$$f'(-1) = -\frac{5}{2}(-1) = 5/2 > 1 \quad \therefore \quad \boxed{\hat{x}_2 = -1 \text{ is unstable}}$$

$$f'\left(\frac{1}{5}\right) = -\frac{5}{2}\left(\frac{1}{5}\right) = -\frac{1}{2} \quad \text{and} \quad \left|f'\left(\frac{1}{5}\right)\right| = \frac{1}{2} < 1$$

$$\therefore \quad \boxed{\hat{x}_1 = 1/5 \text{ is locally stable}}$$

What about the cobwebbing, say with starting point $x_0 = -0.9$?

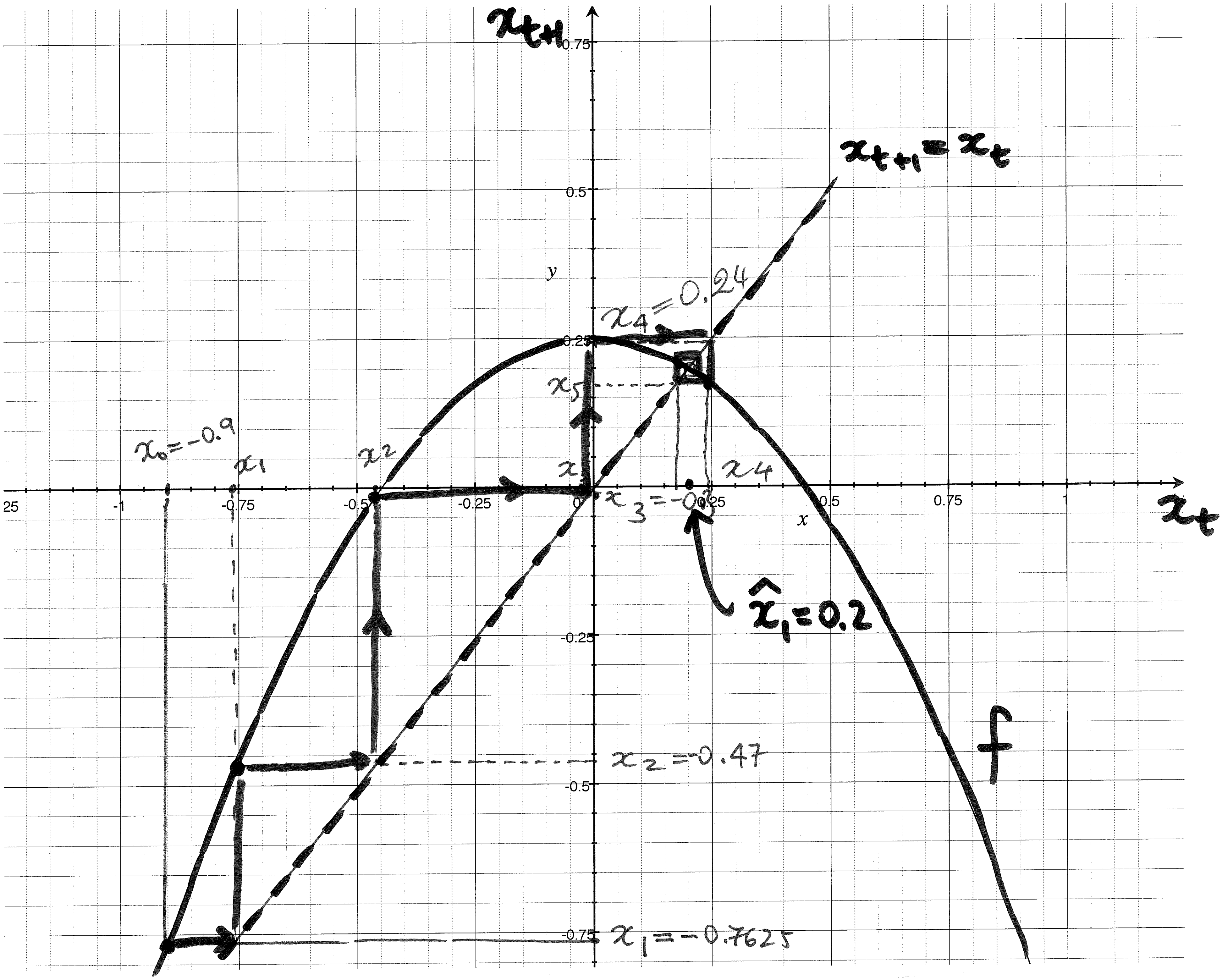
We already computed some values

$x_0 = -0.9$; $x_1 = -0.7625$; $x_2 = -0.4767$

$x_3 = -0.0341$; $x_4 = 0.2485$; $x_5 = 0.1728$

$x_6 = 0.2127$; $x_7 = 0.1935$; ....

Here is how the picture of the cobwebbing looks like :

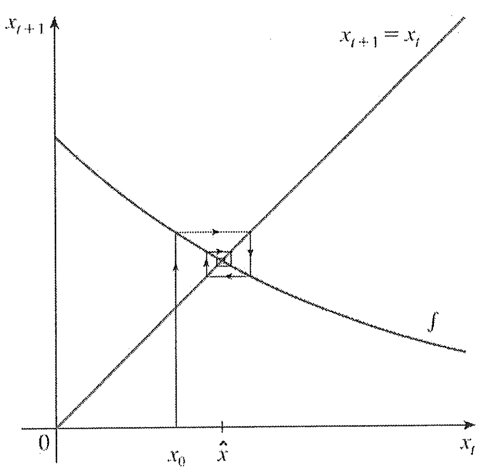


x_{t+1}
$x_{t+1} = x_t$
$x_4 = 0.24$
x_5
$x_0 = -0.9$
x_1
x_2
x_3
x_4
$x_3 = -0.03$
x_t
$\hat{x}_1 = 0.2$
f
$x_2 = -0.47$
$x_1 = -0.7625$

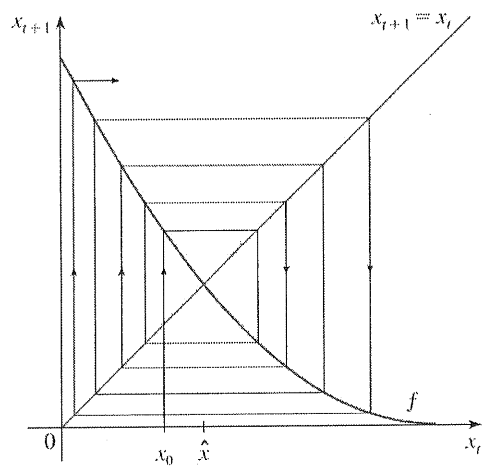
Geometric Considerations

We know from the Stability Criterion that when the slope of the tangent line to f at the equilibrium $\widehat{x}$ is between -1 and 1, x_t converges to the equilibrium $\widehat{x}$.

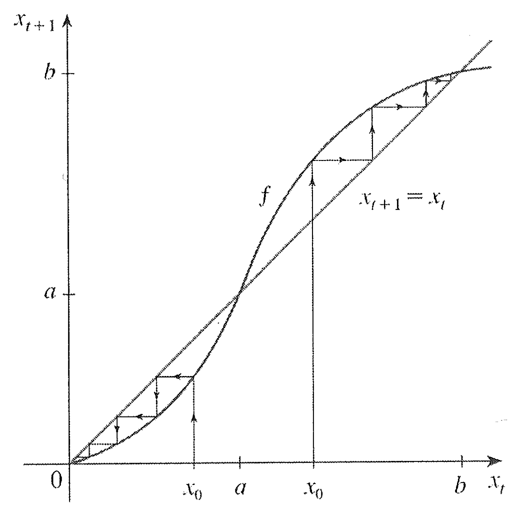
The solution x_t approaches the equilibrium in a **spiral** (thus exhibiting **oscillatory** behavior) when the slope of the tangent line at the equilibrium is negative, whereas it approaches it in **one direction** (thus exhibiting **nonoscillatory** behavior) when the slope of the tangent line at the equilibrium is positive.



(a) Stable spiral



(b) Unstable spiral



Example 2: (Neuhauser, Example # 2, p. 257)

Use the stability criterion to characterize the stability of the equilibria of

$$x_{t+1} = \frac{x_t}{0.1 + x_t}, \qquad t = 0, 1, 2, \ldots$$

We need to find the fixed points of

$$x_{t+1} = \frac{x_t}{0.1 + x_t} \qquad t = 0, 1, 2, \ldots \qquad f(x) = \frac{x}{0.1+x}$$

i.e. we need to solve $x = \dfrac{x}{0.1+x}$

$$\iff x(0.1+x) = x \iff$$

$$x\left[0.1+x-1\right] = 0 \iff x\left[x-0.9\right] = 0$$

$$\therefore \boxed{\hat{x}_1 = 0} \text{ and } \boxed{\hat{x}_2 = 0.9}$$

Now to use the stability criterion we need $f'(x)$:

$$f(x) = \frac{x}{0.1+x} \implies f'(x) = \frac{1\cdot(0.1+x) - x(1)}{(0.1+x)^2} = \frac{0.1}{(0.1+x)^2}$$

Hence :

$$\hat{x}_1 = 0 \Longrightarrow f'(0) = \frac{0.1}{(0.1+0)^2} = \frac{0.1}{0.01} = \underline{\underline{10}}$$

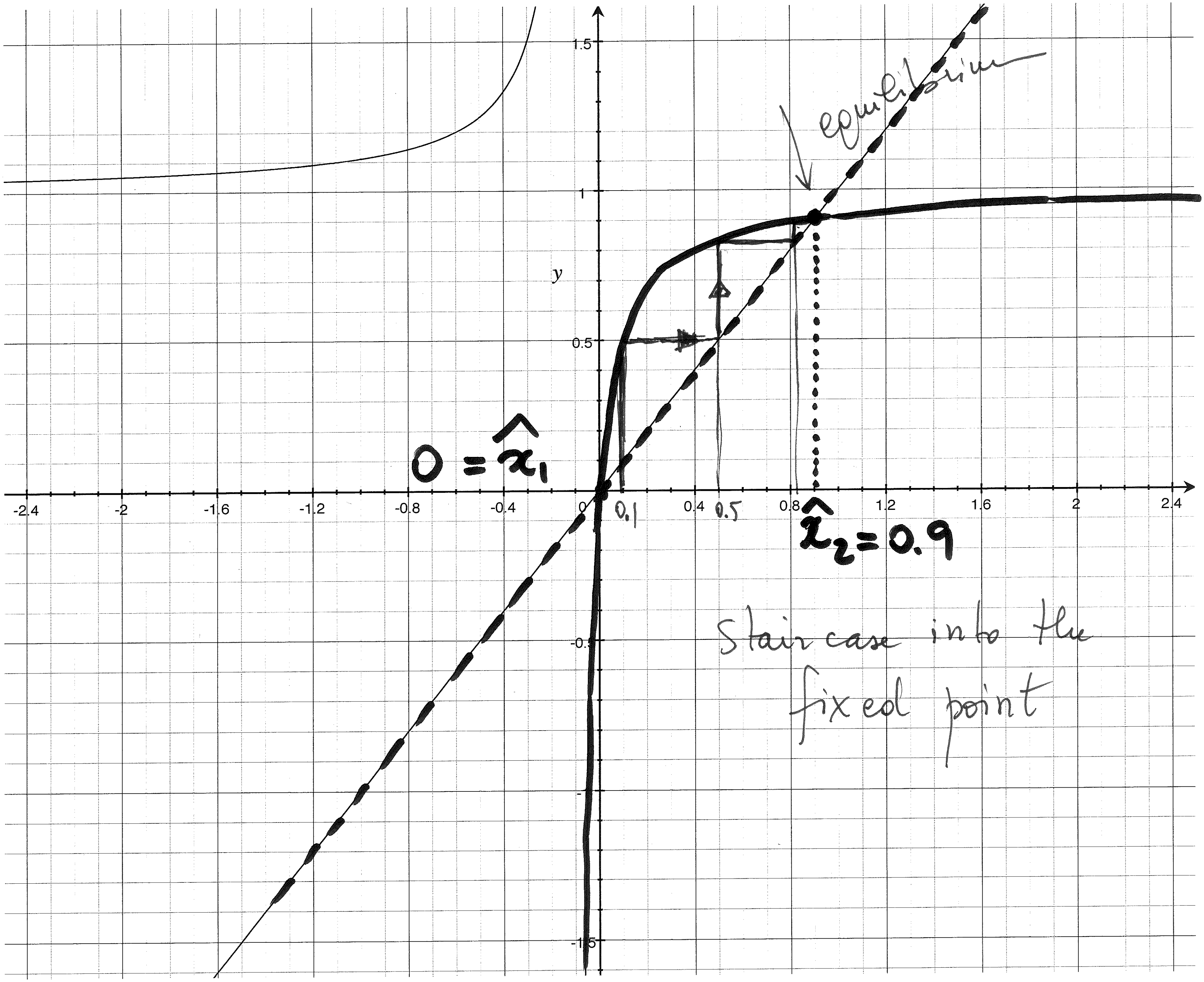
$$\hat{x}_2 = 0.9 \Longrightarrow f'(0.9) = \frac{0.1}{(0.1+0.9)^2} = \frac{0.1}{1^2} = 0.1$$

$\therefore$ $\boxed{\hat{x}_2 = 0.9 \text{ is locally stable}}$

$\boxed{\hat{x}_1 = 0 \text{ is unstable}}$

Here is an example of cobwebbing with

$x_0 = 0.1$



equilibrium
$0 = \hat{x}_1$
0.1
0.5
$\hat{x}_2 = 0.9$
Staircase into the fixed point
y
1.5
1
0.5
-0.5
-1.5
-2.4
-2
-1.6
-1.2
-0.8
-0.4
0
0.4
0.8
1.2
1.6
2
2.4

Example 3: (Neuhauser, Example # 4, p. 259)

Denote by N_t the size of a population at time t, $t = 0, 1, 2, \ldots$ Find all equilibria and determine their stability for the **discrete logistic growth sequence**

$$N_{t+1} = N_t \left[1 + R \left(1 - \frac{N_t}{K} \right) \right]$$

where we assume that the parameters R and K are both positive.

Hence $f'(N) = 1 + R - \frac{2R}{K} N$

Now when $\hat{N}_1 = 0 \implies f'(0) = 1 + R - 0 = 1+R$

which is strictly > 1

$\therefore \hat{N}_1 = 0$ is <u>unstable</u>

Now when $\hat{N}_2 = K \implies f'(K) = 1+R - \frac{2R}{K} \cdot K$

$$= 1 + R - 2R$$

$$= \underline{\underline{1 - R}}$$

Hence $\hat{N}_2 = K$ is locally stable $\iff |1-R| < 1$

$\iff -1 < 1 - R < 1 \iff \boxed{0 < R < 2}$

$$N_{t+1} = N_t\left(1 + R\left(1 - \frac{N_t}{K}\right)\right) = f(N_t)$$

to find the equilibria we need to solve

$$N = N\left[1 + R\left(1 - \frac{N}{K}\right)\right] \iff \boxed{\hat{N}_1 = 0} \quad \underline{\underline{\text{OR}}}$$

$$1 = 1 + R\left(1 - \frac{N}{K}\right) \iff 0 = R\left(1 - \frac{N}{K}\right)$$

$$\iff 1 - N/K = 0 \iff \boxed{\hat{N}_2 = K}$$

Now to use the stability criterion we need $f'(N)$ from

$$f(N) = N\left(1 + R\left(1 - \frac{N}{K}\right)\right)$$

$$= N + RN - \frac{R}{K}N^2$$

Another Idea for a Possible Project?

Biologist T.S. Bellows investigated the ability of several difference equations to describe the population dynamics of insects. He found that the so called *Generalized Beverton-Holt model* provided the best description. If x_n denotes the population density in the n-th generation, then the model is of the form

$$x_{n+1} = \frac{r\,x_n}{1 + x_n^b}$$

where r is the intrinsic fitness of population and b measures the abruptness of density dependence.

For three insect species, Bellows found the following parameter estimates:

* Budworm moth: $r = 3.5$ and $b = 2.7$;
* Colorado potato beetle: $r = 75$ and $b = 4.8$;
* Meadow plant bug: $r = 2.2$ and $b = 1.4$.

(a) Use these parameter estimates to determine which population supports a stable equilibrium.

(b) For the species that do not support a stable equilibrium simulate their dynamics.

MA 137 — Calculus 1 with Life Science Applications
Antiderivatives
(Section 5.8)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

November 21 & 28, 2016

From Differential to Integral Calculus

Roughly speaking, Calculus has two parts:

differential calculus and integral calculus

At the core of **differential calculus** (which we have been studying so far) is the concept of the instantaneous rate of change of a function. We have seen how this concept can be used to locally approximate functions, to identify maxima and minima, to decide stability of equilibria, etc.

Integral calculus, on the other hand, deals with accumulated change, and, thereby, recovering a function from a mathematical description of its instantaneous rate of change. This recovery process, interestingly enough, is related to the concept of finding the area enclosed by a curve. This will be studied in Chapter 6 (and in the follow up course, MA 138).

Antiderivatives

Many mathematical operations have an inverse. For example, to undo addition we use subtraction. To undo undo exponentiation we take logarithms. The process of differentiation can be undone by a process called *antidifferentiation*.

To motivate antidifferentiation, suppose we know the rate at which a bacteria population is growing and want to know the size of the population at some future time. The problem is to find a function F whose derivative is a known function f.

Definition

A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Warning: Although we will learn rules that allow us to compute antiderivatives, this process is typically **much more** difficult than finding derivatives; in addition, there are even cases where it is impossible to find an expression for an antiderivative.

Corollaries of MVT

Two corollaries of the Mean Value Theorem will help us in finding antiderivatives. The first one is Corollary 2 from Section 5.1 (p. 212 of Neuhauser's textbook):

Corollary 2

If f is continuous on $[a, b]$ and differentiable on (a, b), with $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

Corollary 2 is the converse of the fact that $f'(x) = 0$ whenever $f(x)$ is a constant function. Corollary 2 tells us that all antiderivatives of a function that is identically 0 are constant functions.

Corollary 3 says that functions with identical derivative differ only by a constant; that is, to find all antiderivatives of a given function, we need only find one.

Corollary 3

If $F(x)$ and $G(x)$ are antiderivatives of the continuous function $f(x)$ on an interval I, then there exists a constant c such that $G(x) = F(x) + c$ for all $x \in I$.

Proof: Since $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, it follows that $F'(x) = f(x) = G'(x)$ for all $x \in I$. Thus

$$[F(x) - G(x)]' = F'(x) - G'(x) = f(x) - f(x) = 0.$$

It follows from Corollary 2, applied to the function $F - G$, that $F(x) - G(x) = c$, where c is a constant.

The Indefinite Integral

Notation

The *indefinite integral* of $f(x)$, denoted by

$$\int f(x)\,dx$$

represents the *general* antiderivative of $f(x)$.

For example, $\int 3x^2\,dx = x^3 + c$, where c is any constant.

Rules for Indefinite Integrals

A. $\int k\,f(x)\,dx = k \int f(x)\,dx$ $\quad k$ any constant

B. $\int [f(x) \pm g(x)]\,dx = \left[\int f(x)\,dx\right] \pm \left[\int g(x)\,dx\right]$

Basic Indefinite Integrals

The formulas below can be verified by differentiating the righthand side of each expression. The quantities a and c below denote (nonzero) constants.

1. $$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c \qquad n \neq -1$$

2. $$\int \frac{1}{x} \, dx = \ln |x| + c$$

3. $$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + c$$

4. $$\int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) + c$$

5. $$\int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + c$$

Warning: We do not have simple derivative rules for products and quotients, so we should not expect simple integral rules for products and quotients.

Example 1: (Online Homework HW22, # 2)

Find the antiderivative F of $f(x) = 5x^4 - 2x^5$ that satisfies $F(0) = -10$.

In other words:

$$F(x) = \int (5x^4 - 2x^5)\,dx =$$

$$= 5\int x^4\,dx - 2\int x^5\,dx$$

$$= 5\cdot\left[\frac{1}{5}x^5\right] - 2\left[\frac{1}{6}x^6\right] + C$$

$$= \underline{x^5 - \frac{1}{3}x^6 + C}$$

describes all antiderivatives of $f(x)$.

We want the one such that $F(0) = -10$. Thus

$$-10 = F(0) = 0^5 - \frac{1}{3}0^6 + C \quad \therefore \boxed{C = -10}$$

$$\therefore \boxed{F(x) = x^5 - \frac{1}{3}x^6 - 10}$$

Example 2:

Evaluate the indefinite integral $\int (t^3 + 3t^2 + 4t + 9)\, dt$.

$$\int (t^3 + 3t^2 + 4t + 9)\, dt$$

$$= \int t^3\, dt + 3\int t^2\, dt + 4\int t\, dt + 9\int 1 \cdot dt$$

$$= \frac{1}{4} t^4 + 3 \cdot \left(\frac{1}{3} t^3\right) + 4\left(\frac{1}{2} t^2\right) + 9 \cdot t + C$$

$$= \frac{1}{4} t^4 + t^3 + 2t^2 + 9t + C$$

Example 3: (Online Homework HW22, # 5)

Evaluate the indefinite integral $\int x(10 - x^4)\, dx$.

$$\int \underbrace{x \cdot (10 - x^4)} \, dx$$

there are no rules for the antiderivative of a product

$$= \int (10x - x^5) \, dx = 10 \int x \, dx - \int x^5 \, dx$$

$$= 10\left(\frac{1}{2}x^2\right) - \left(\frac{1}{6}x^6\right) + C$$

$$= 5x^2 - \frac{1}{6}x^6 + C$$

Example 4: (Online Homework HW22, # 7)

Evaluate the indefinite integral $\displaystyle\int \frac{9u^4 + 7\sqrt{u}}{u^2}\, du.$

$$\int \frac{9u^4 + 7\sqrt{u}}{u^2}\, du$$

there are no rules for the antiderivative of a quotient ....

$$= \int \left(\frac{9u^4}{u^2} + \frac{7\sqrt{u}}{u^2} \right) du = \int \left(9u^2 + 7u^{\frac{1}{2}-2} \right) du$$

$$= \int \left(9u^2 + 7u^{-3/2} \right) du = 9\int u^2\, du + 7\int u^{-3/2}\, du$$

$$= 9\left(\frac{1}{3}u^3\right) + 7\left(\frac{1}{-\frac{3}{2}+1}u^{-3/2+1}\right) + C =$$

$$= 3u^3 + 7\left(\frac{1}{-1/2}\, u^{-1/2}\right) + C = 3u^3 - 14\frac{1}{\sqrt{u}} + C$$

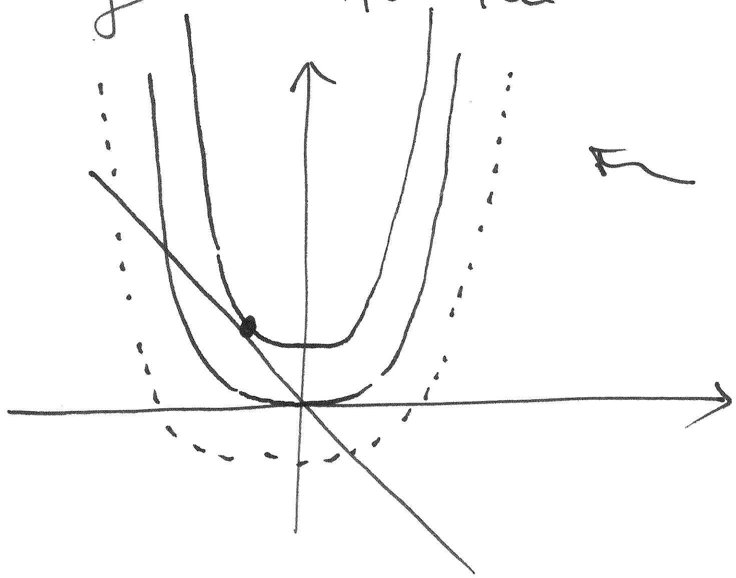
$$= \boxed{\frac{3u^3\sqrt{u} - 14}{\sqrt{u}} + C}$$

Example 4: (Online Homework HW22, # 10)

Find a function f such that $f'(x) = 4x^3$ and the line $x + y = 0$ is tangent to the graph of f.

$$f'(x) = 4x^3 \implies f(x) = x^4 + C$$

We also know that "somewhere" this function is tangent to the line $y = -x$.



← all vertical translates of $\underline{\underline{y = x^4}}$

$\underline{\underline{\text{I.e.}}}$ at some point of $f(x) = x^4 + C$ the tangent line has slope -1.

So $f'(x) = 4x^3 = -1 \iff x = -\sqrt[3]{1/4} \cong -0.62996$

What is the value of f at $x = -0.62996$?

It must be the value of the tangent line at that point!

Hence $f(-0.62996) = 0.62996$

since the tangent line is $y = -x$.

Hence $f(x) = x^4 + C$ is such that

$$0.62996 = (-0.62996)^4 + C$$

$$\therefore \quad C = 0.47247$$

Hence : $\boxed{f(x) = x^4 + 0.47247}$

Example 5: (Online Homework HW22, # 13)

Find f if $f'''(x) = \sin(x)$, $f(0) = 8$, $f'(0) = 4$, and $f''(0) = -10$.

$f'''(x) = \sin x$, $f(0) = 8$, $f'(0) = 4$, $f''(0) = -10$

<u>Now</u> : $f'''(x) = \sin x \Longrightarrow f''(x) = -\cos x + C_1$

Since $f''(0) = -10$ we have $-10 = -\underbrace{\cos(0)}_{1} + C_1$

$\therefore \quad C_1 = -10 + 1 = -9$

So $f''(x) = -\cos x - 9$. Thus

$$f'(x) = -\sin x - 9x + C_2$$

Since $f'(0) = 4$ we have $4 = f'(0) = -\underbrace{\sin(0)}_{0} + C_2$

$\therefore \quad C_2 = 4$. Hence

$$f'(x) = -\sin x - 9x + 4$$

Since $f'(x) = -\sin x - 9x + 4$

we have that

$$f(x) = \cos x - \frac{9}{2}x^2 + 4x + C_3$$

Since $f(0) = 8$ we have

$$8 = f(0) = \underbrace{\cos(0)}_{1} + C_3$$

$$\therefore \quad C_3 = 8 - 1 = 7$$

Finally $\boxed{f(x) = \cos x - \frac{9}{2}x^2 + 4x + 7}$

Solving Simple Differential Equations

In this course, we have repeatedly encountered differential equations ($\equiv$ DEs). Occasionally, we showed that a certain function would solve a given differential equation.

What we learned so far translates into solving DEs of the form

$$\frac{dy}{dx} = f(x).$$

That is, the rate of change of y with respect to x depends only on x. We now know that if we can find one such function y such that $y' = f(x)$, then there is a whole family of functions with this property, all related by vertical translations.

If we want to pick out one of these functions, we need to specify an initial condition — a point (x_0, y_0) on the graph of the function. Such a function is called a solution of the **initial-value problem**

$$\frac{dy}{dx} = f(x) \qquad \text{with} \quad y = y_0 \quad \text{when} \quad x = x_0.$$

Example 6: (Neuhauser, Example 5, p. 270)

Solve the initial-value problem $\dfrac{dy}{dx} = -2x^2 + 3$ with $y_0 = 10$ when $x_0 = 3$.

$$\frac{dy}{dx} = -2x^2 + 3 \iff y = -\frac{2}{3}x^3 + 3x + C$$

Now when $x_0 = 3$ $y_0 = 10$. So

$$10 = -\frac{2}{3}(3)^3 + 3\cdot(3) + C$$

$$\iff$$

$$10 = -2\cdot 9 + 9 + C$$

$$10 = -9 + C$$

$$\therefore \quad C = 19$$

$$\therefore \quad \boxed{y = -\frac{2}{3}x^3 + 3x + 19}$$

Note that we could think of

$$\frac{dy}{dx} = -2x^2 + 3 \qquad \text{as}$$

$$dy = (-2x^2+3)\,dx$$

Now take the indefinite integral of both sides

$$\int 1 \cdot dy = \int (-2x^2+3)\,dx$$

$$y + c_1 = -\tfrac{2}{3}x^3 + 3x + C_2$$

hence

$$y = -\frac{2}{3}x^3 + 3x + \underbrace{(c_2 - c_1)}_{C \text{ constant}}$$

Example 7:

What about finding the solution of the initial-value problem
$\dfrac{dy}{dx} = r\,y$ with $y(0) = y_0$ and r a constant? How can we do it?

We have already solved this differential equation while studying Section 5.1.

Now we give a nicer proof:

$$\frac{dy}{dx} = ry$$

$$\Longleftrightarrow$$

$$\underbrace{\frac{1}{y} \cdot \frac{dy}{dx}}_{} = r$$

What is this? By the chain rule

$$\frac{d}{dx}(\ln y) = r$$

If the derivative of a function is a constant then the antiderivative is

$$\frac{d}{dx}\left[\ln y\right] = r \iff \ln y = rx + C$$

Hence, by taking the exponential of both sides,

$$e^{\ln y} = e^{rx + C} \iff y = e^{rx} \cdot e^{C}$$

When $x = 0$ we have $y(0) = y_0$ so

$$y_0 = y(0) = \underbrace{e^{r \cdot 0}}_{1} \cdot e^{C} \quad \therefore \quad e^{C} = y_0$$

Hence the solution is :

$$\boxed{y(x) = y_0 \, e^{rx}}$$

As we did in Example 6, we could think of

$$\frac{dy}{dx} = ry$$ as

$$\frac{1}{y}\,dy = r\,dx$$

That is we separated the variables ; now we take the indefinite integral of both sides

$$\int \frac{1}{y}\,dy = \int r\,dx$$

hence $$\boxed{\ln y = rx + C}$$

now proceed as before ....