MA 137 — Calculus 1 with Life Science Applications
The Definite Integral
(Section 6.1)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

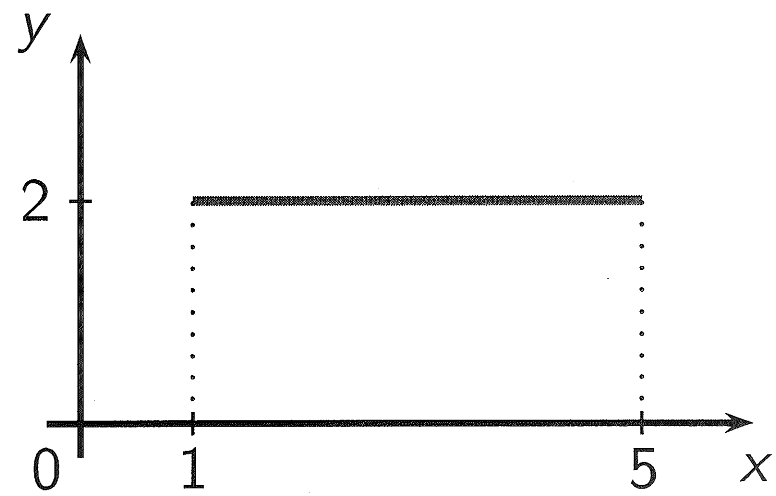
November 30, 2016

The Area Problem

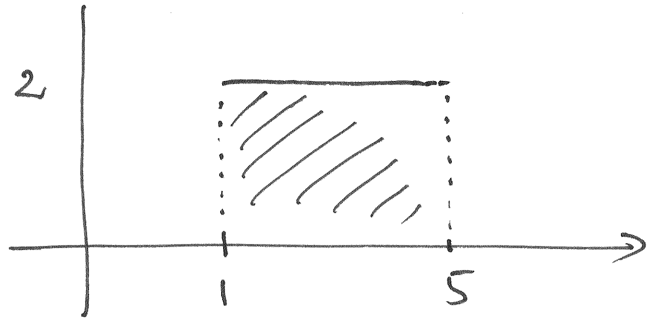
- We start with the area and distance problems and use them to formulate the idea of a definite integral, which is the basic concept of integral calculus. We discover that in trying to find the area under a curve or the distance traveled by a car, we end up with the same special type of limit.
- We will then see that there is a connection between integral calculus and differential calculus: The Fundamental Theorem of Calculus relates the integral to the derivative,
- Why would a biologist be interested in calculating an area? A botanist might want to know the area of a leaf and compare it with the leafs area at other stages of its development. An ecologist might want to know the area of a lake and compare it with the area in previous years. An oncologist might want to know the area of a tumor and compare it with the areas at prior times to see how quickly it is growing. But there are also indirect ways in which areas are of interest.

Example 1: (Easy Area Problem)

Find the area of the region in the xy-plane bounded above by the graph of the function $f(x) = 2$, below by the x-axis, on the left by the line $x = 1$, and on the right by the line $x = 5$.



$f(x) = 2$ defined on $[1, 5]$

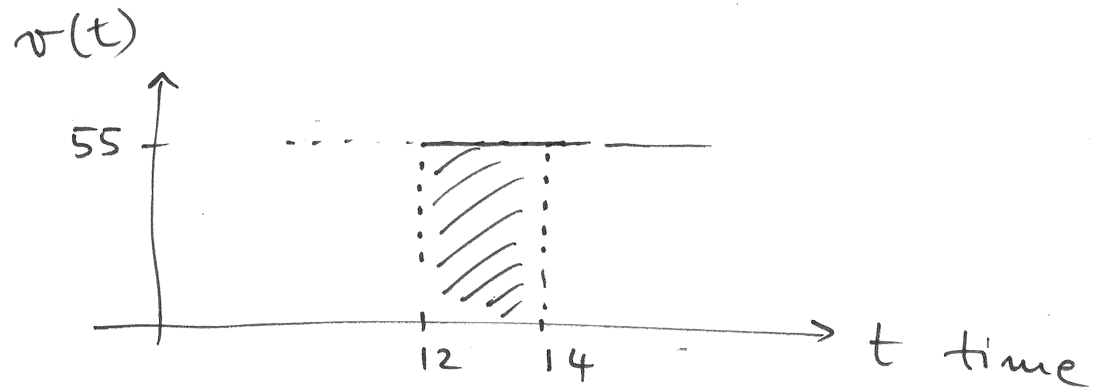


Area of the shaded region $= 2 \cdot \underbrace{(5-1)}_{\text{length of interval}} = 8$

Example 2: (Easy distance traveled problem)

Suppose a car is traveling due east at a constant velocity of 55 miles per hour. How far does the car travel between noon and 2:00 pm?

$v(t) = 55$ mph on the interval $[12, 14]$



distance traveled = area of the shaded region

$$= 55 \cdot \underbrace{(14-12)}_{2 \text{ hours}} = \underline{\underline{110 \text{ miles}}}$$

General Philosophy

By means of the integral, problems similar to the previous ones can be solved when the ingredients of the problem are no longer constant but rather changing or variable.

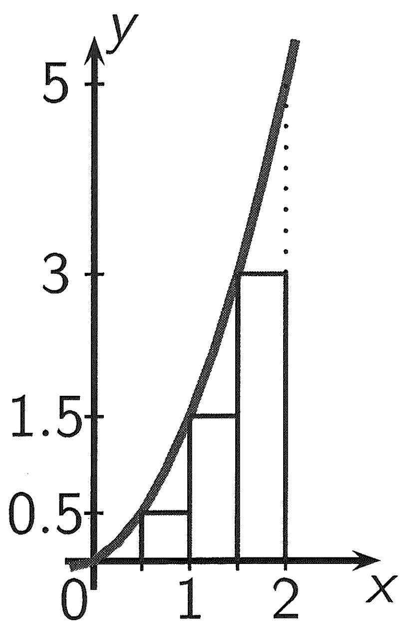
- We first learn how to *estimate* a solution to these more complex problems. The key idea is to notice that the value of the function does not vary very much over a small interval, and so it is approximatively constant over a small interval;
- We will then be able to solve these problems *exactly*;
- Finally, in Section 6.2, we will be able to solve them both exactly and *easily*.

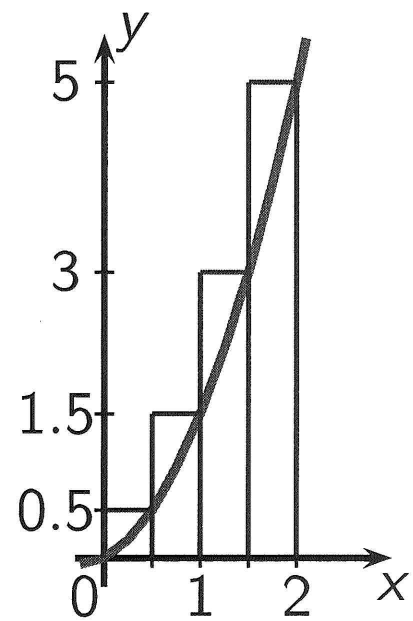
Example 3:

Estimate the area under the graph of $y = x^2 + \frac{1}{2}x$ for x between 0 and 2 in two different ways:

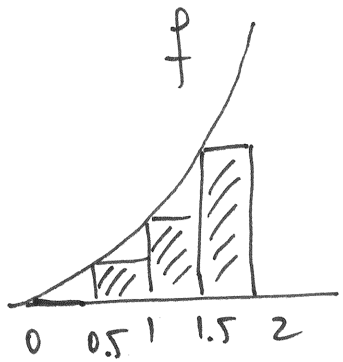
(a) Subdivide the interval $[0, 2]$ into four equal subintervals and use the left endpoint of each subinterval as "sample point".

(b) Subdivide the interval $[0, 2]$ into four equal subintervals and use the right endpoint of each subinterval as "sample point".





(a)



area of those 4 rectangles

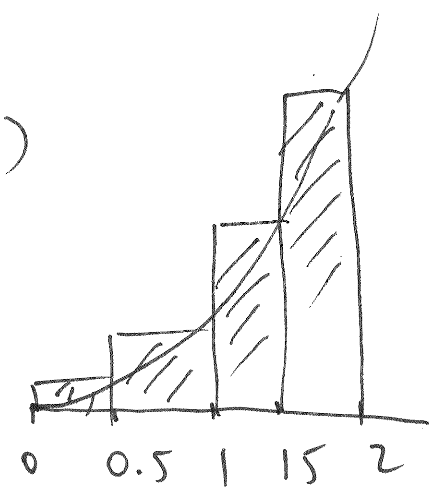
$$= 0.5 \cdot \underbrace{f(0)}_{\text{height}} + 0.5 \cdot \underbrace{f(0.5)}_{\text{height}} + 0.5 \cdot \underbrace{f(1)}_{\text{height}}$$

$$+ 0.5 \cdot \underbrace{f(1.5)}_{\text{height}}$$

$$= 0.5 \left[0 + 0.5 + 1.5 + 3 \right]$$

$$= 0.5 \, (5) = \underline{\underline{2.5}}$$

(b)



area of those 4 rectangles

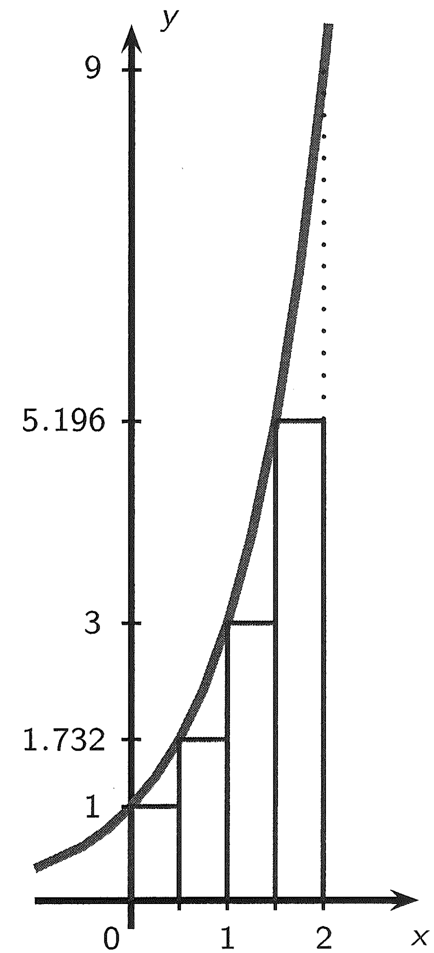
$$= 0.5 \cdot f(0.5) + 0.5 \cdot f(1) + 0.5 \cdot f(1.5) + 0.5 \cdot f(2)$$

$$= 0.5 \left[0.5 + 1.5 + 3 + 5 \right]$$

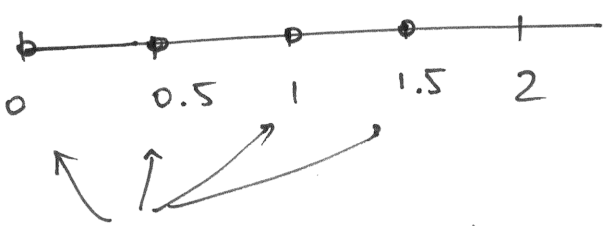
$$= 0.5 \left[10 \right] = \underline{\underline{5}}$$

Example 4:

Estimate the area under the graph of $y = 3^x$ for x between 0 and 2. Use a partition that consists of four equal subintervals of $[0, 2]$ and use the left endpoint of each subinterval as a sample point.



$f(x) = 3^x \qquad [0, 2]$



left end points

Area of the four rectangles using the <u>left</u>-endpoint

$$= \underbrace{0.5 \cdot f(0)}_{1^{st}\text{ rectangle}} + \underbrace{0.5 \cdot f(0.5)}_{2^{nd}\text{ rect.}} + \underbrace{0.5 \cdot f(1)}_{3^{rd}\text{ rect.}} + \underbrace{0.5 \cdot f(1.5)}_{4^{th}\text{ rect}}$$

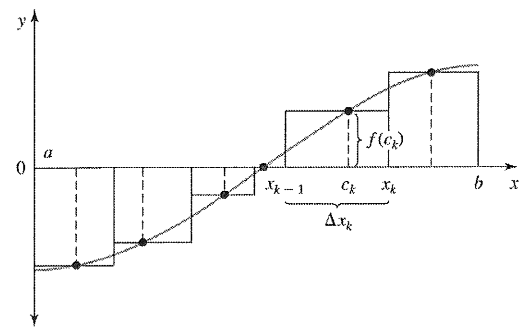
$$= 0.5 \cdot 1 + 0.5 \cdot 1.732 + 0.5 \cdot 3 + 0.5 \cdot 5.196$$

$$= 0.5\,(1 + 1.732 + 3 + 5.196) = \underline{\underline{5.464}}$$

Note:

- In the previous two examples we systematically chose the value of the function at one of the endpoints of each subinterval.
- However, since the guiding idea is that we are assuming that the values of the function over a small subinterval do not change by very much, we could take the value of the function at any point of the subinterval as a good sample or representative value of the function.

  We could also have chosen small subintervals of different lengths.



- However, we are trying to establish a systematic, general procedure.
- We can only expect the previous answers to be approximations of the correct answers. This is because the values of the function do change on each subinterval, even though they do not change by much.

Getting better estimates:

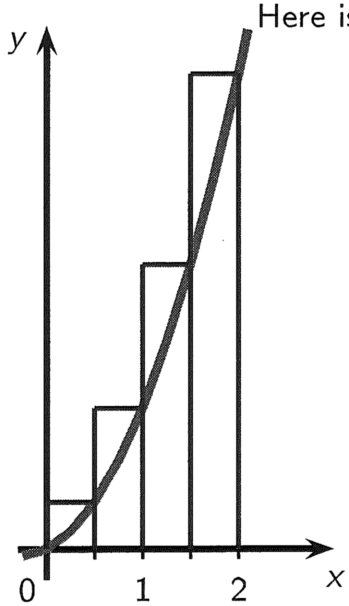
If we replace the subintervals we used by "smaller" subintervals we can reasonably expect the values of the function to vary by much less on each thinner subinterval. Thus, we can reasonably expect that the area of each thinner vertical strip under the graph of the function to be more accurately approximated by the area of these thinner rectangles. Then if we add up the areas of all these thinner rectangles, we should get a much more accurate estimate for the area of the original region.

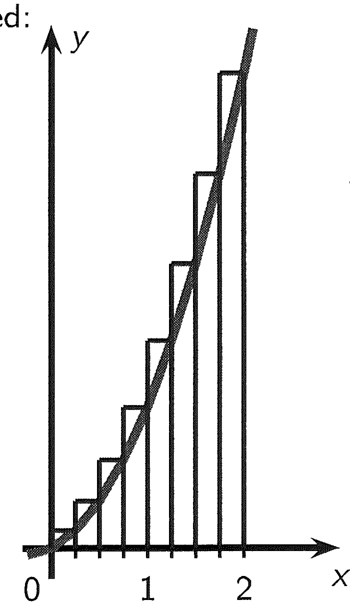
Here is Example 3(b), revisited:

$y = x^2 + \frac{1}{2}x$ on $[0, 2]$

$n = 4$ equal subintervals

Area ≈ 5





$y = x^2 + \frac{1}{2}x$ on $[0, 2]$

$n = 8$ equal subintervals

Area ≈ 4.3125

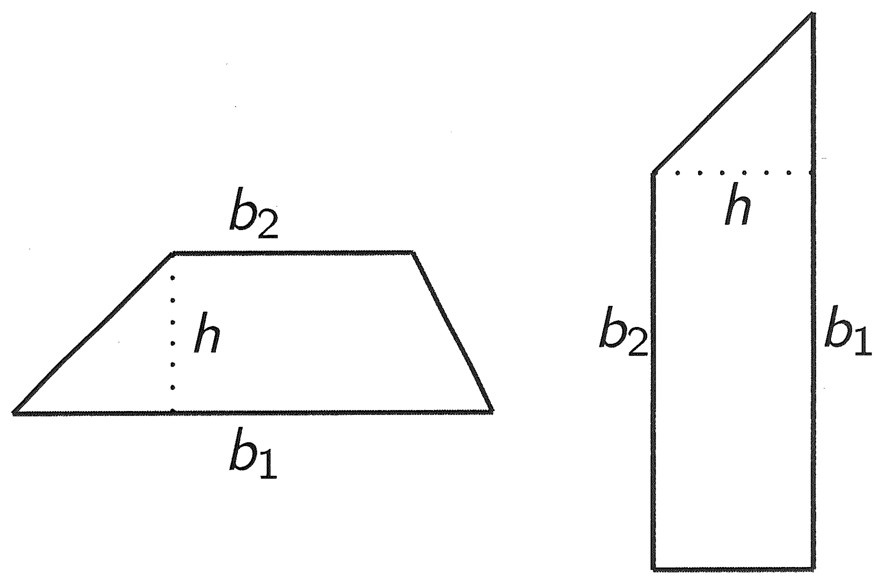
We will see later that the exact value of the area under consideration in Example 3 is $11/3 \approx 3.66$.

Trapezoids versus rectangles

We could use trapezoids instead of rectangles to obtain better estimates, even though the calculations get a little bit more complicated. (This will occur in Example 5.)

We recall that the area of a trapezoid is

$$\text{Area of a trapezoid} = \frac{(b_1 + b_2) \cdot h}{2}.$$

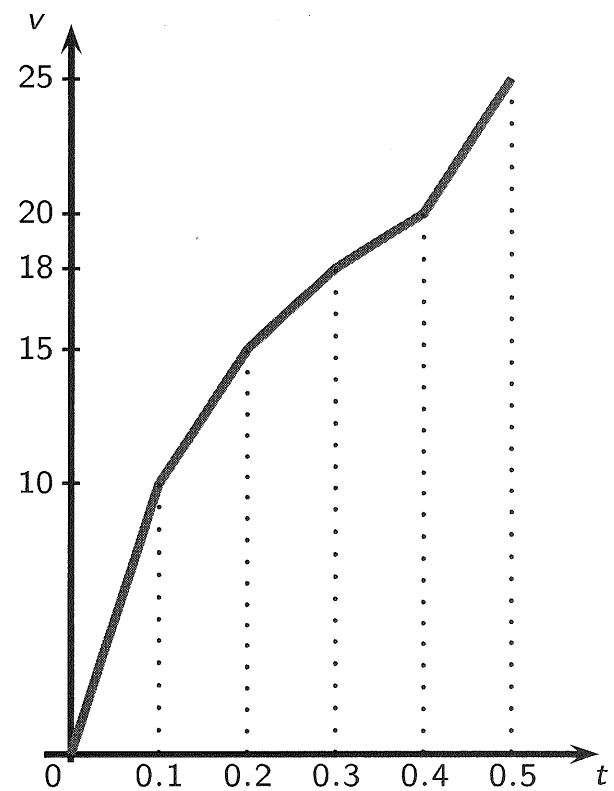


Example 5:

A train travels in a straight westward direction along a track. The velocity of the train varies, but it is measured at regular time intervals of 1/10 hour. The measurements for the first half hour are

time	0	0.1	0.2	0.3	0.4	0.5
velocity	0	10	15	18	20	25

The total distance traveled by the train is equal to the area underneath the graph of the velocity function and lying above the t-axis. Compute the total distance traveled by the train during the first half hour by assuming the velocity is a linear function of t on the subintervals. The velocity in the table is given in miles per hour.



the area under the graph consists of the areas of 5 trapezoids :

$$\text{Area} = 0.1\cdot\left(\frac{0+10}{2}\right) + 0.1\cdot\left(\frac{10+15}{2}\right) + 0.1\cdot\left(\frac{15+18}{2}\right)$$

$$+\ 0.1\cdot\left(\frac{18+20}{2}\right) + 0.1\cdot\left(\frac{20+25}{2}\right)$$

$$= 0.1\cdot 5 + 0.1\cdot 12.5 + 0.1\cdot 16.5 + 0.1\cdot 19 + 0.1\cdot 22.5$$

$$= 0.1\cdot\left[5 + 12.5 + 16.5 + 19 + 22.5\right]$$

$$= 0.1\,(75.5)$$

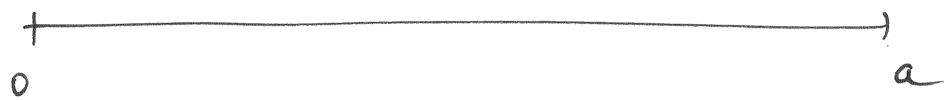
$$= \underline{7.55 \text{ miles}}$$

Example 6: (Neuhauser, Example # 1, p. 277/8)

We will try to find the area of the region below the parabola $f(x) = x^2$ and above the x-axis between 0 and a.

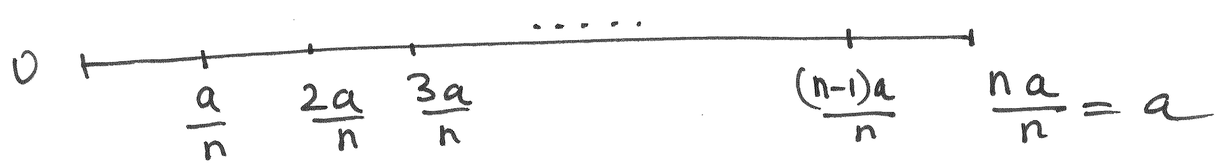
To do this, we divide the interval $[0, a]$ into n subintervals of equal length and approximate the area of interest by a sum of the areas of rectangles, the widths of whose bases are equal to the lengths of the subintervals and whose heights are the values of the function at the left endpoints of these subintervals.

On page 278, Neuhauser uses left-end points. We will use right end points instead:



n equal subintervals of length $\frac{a-0}{n} = \frac{a}{n}$

hence the end points are:



Area of the n-rectangles is:

$$\frac{a}{n} \cdot f\left(\frac{a}{n}\right) + \frac{a}{n} \cdot f\left(\frac{2a}{n}\right) + \frac{a}{n} \cdot f\left(\frac{3a}{n}\right) + \cdots + \frac{a}{n} \cdot f\left(\frac{na}{n}\right)$$

$$= \frac{a}{n} \cdot \left(\frac{a}{n}\right)^2 + \frac{a}{n} \cdot \left(\frac{2a}{n}\right)^2 + \frac{a}{n} \cdot \left(\frac{3a}{n}\right)^2 + \cdots + \frac{a}{n} \cdot \left(\frac{(n-1)a}{n}\right)^2 + \frac{a}{n}\left(\frac{na}{n}\right)^2$$

$$= \frac{a^3}{n^3} + \frac{a^3 \cdot 2^2}{n^3} + \frac{a^3 \cdot 3^2}{n^3} + \dots + \frac{a^3}{n^3} \cdot (n-1)^2 + \frac{a^3}{n^3} \cdot n^2$$

$$= \frac{a^3}{n^3} \left[1 + \underbrace{2^2 + 3^2 + \dots + (n-1)^2}_{} + n^2 \right]$$

$$= \frac{a^3}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] = \frac{a^3}{n^3} \left[\frac{2n^3 + 3n^2 + n}{6} \right]$$

$$= \frac{a^3 (2n^3 + 3n^2 + n)}{6n^3}$$

Now, if n goes to infinity this should be exactly the area under the parabola $y = x^2$

$$\lim_{n \to \infty} \frac{a^3 (2n^3 + 3n^2 + n)}{6n^3} = \lim_{n \to \infty} \frac{a^3 (2n^3 + 3n^2 + n) / n^3}{6n^3 / n^3}$$

$$= \lim_{n \to \infty} \frac{a^3}{6} \left[2 + 3\frac{1}{n} + \frac{1}{n^2} \right] = \boxed{\frac{a^3}{3}}$$

MA 137 — Calculus 1 with Life Science Applications
The Definite Integral
(Section 6.1)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

December 2, 2016

Sigma (Σ) Notation

In approximating areas we have encountered sums with many terms. A convenient way of writing such sums uses the Greek letter Σ (which corresponds to our capital S) and is called *sigma notation*. More precisely, if $a_1, a_2, \ldots, a_n$ are real numbers we denote the sum

$$a_1 + a_2 + \cdots + a_n$$

by using the notation

$$\sum_{k=1}^{n} a_k.$$

The integer k is called an *index* or *counter* and takes on the values $1, 2, \ldots, n$. For example,

$$\sum_{k=1}^{6} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 1 + 4 + 9 + 16 + 25 + 36 = 91$$

whereas

$$\sum_{k=3}^{6} k^2 = 3^2 + 4^2 + 5^2 + 6^2 = 9 + 16 + 25 + 36 = 86.$$

Summation Rules

The rules and formulas given next allow us to compute fairly easily Riemann sums where the number n of subintervals is rather large. We can also get compact and manageable expressions for the sum so that we can readily investigate what happens as n approaches infinity.

[sr₁] $$\sum_{k=1}^{n} c = n\,c$$

[sr₂] $$\sum_{k=1}^{n} (c\,a_k) = c \sum_{k=1}^{n} a_k$$

[sr₃] $$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

Note: The summations rules are nothing but the usual rules of arithmetic rewritten in the Σ notation.
For example, **[sr₂]** is nothing but the distributive law of arithmetic

$$c\,a_1 + c\,a_2 + \cdots + c\,a_n = c\,(a_1 + a_2 + \cdots + a_n);$$

[sr₃] is nothing but the commutative law of addition

$$(a_1 \pm b_1) + \cdots + (a_n \pm b_n) = (a_1 + \cdots + a_n) \pm (b_1 + \cdots + b_n).$$

Formulas [Neuhauser, Example #3 (p. 279); Problem # 31 (p. 291)]

[sf$_1$] $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$ **[sf$_2$]** $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof: In the case of **[sf$_1$]**, let S denote the sum of the integers $1, 2, 3, \ldots, n$. Let us write this sum S twice: we first list the terms in the sum in increasing order whereas we list them in decreasing order the second time:

$$\begin{array}{ccccccccc} S & = & 1 & + & 2 & + & \cdots & + & n \\ S & = & n & + & n-1 & + & \cdots & + & 1 \end{array}$$

If we now add the terms along the vertical columns, we obtain

$$2S = \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n \text{ times}} = n(n+1).$$

This gives our desired formula, once we divide both sides of the above equality by 2.

In the case of **[sf$_2$]**, let S denote the sum of the integers $1^2, 2^2, 3^2, \ldots, n^2$. The *trick* is to consider the sum $\sum_{k=1}^{n}[(k+1)^3 - k^3]$. On the one hand, this new sum collapses to

$$(2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \cdots + (n^3 - (n-1)^3) + ((n+1)^3 - n^3) = (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

On the other hand, using our summation rules together with **[sf$_1$]** gives us

$$\sum_{k=1}^{n}[(k+1)^3 - k^3] = \sum_{k=1}^{n}[3k^2 + 3k + 1] = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 3S + 3\,\frac{n(n+1)}{2} + n$$

Equating the right hand sides of the above identities gives us: $3S + 3\,\dfrac{n(n+1)}{2} + n = n^3 + 3n^2 + 3n.$

If we solve for S and properly factor the terms, we obtain our desired expression.

More Formulas

The next formulas can be verified in a sequential order using the same type of trick used in the proof for **[sf$_2$]**. The proofs get increasingly more tedious.

[sf$_3$] $$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

[sf$_4$] $$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Example 1: (Online Homework, HW 23, # 15)

Find the numerical value of the sums below:

- $\displaystyle\sum_{j=3}^{7}(4j-1)$

- $\displaystyle\sum_{i=3}^{5}(i^2-i)$

$$\sum_{j=3}^{7} (4j-1) = [4\cdot \underline{3}-1]+[4\cdot \underline{4}-1]+[4\cdot \underline{5}-1]+[4\cdot \underline{6}-1]+[4\cdot \underline{7}-1]$$

$$= 11 + 15 + 19 + 23 + 27 = 95$$

$$\sum_{i=3}^{5} (i^2-i) = [3^2-3] + [4^2-4] + [5^2-5]$$

$$= 6 + 12 + 20 = 38$$

Example 2:

Find the numerical value of the sums below:

- $\displaystyle\sum_{j=3}^{n}(4j-1)$

- $\displaystyle\sum_{i=3}^{n}(i^2-i)$

$$
\begin{aligned}
\sum_{j=3}^{n} (4j-1) &= \sum_{j=1}^{n} (4j-1) - \sum_{j=1}^{2} (4j-1) \\
&= \sum_{j=1}^{n} (4j-1) - \Big[\underbrace{(4\cdot 1-1) + (4\cdot 2-1)}_{=10}\Big] \\
&= 4\sum_{j=1}^{n} j - \sum_{j=1}^{n} 1 - 10 \\
&= 4\,\frac{n(n+1)}{2} - n - 10 \\
&= 2n(n+1) - n - 10 \\
&= 2n^2 + 2n - n - 10 \\
&= \boxed{2n^2 + n - 10}
\end{aligned}
$$

$$\sum_{i=3}^{n}(i^2-i) = \sum_{i=1}^{n}(i^2-i) - \sum_{i=1}^{2}(i^2-i)$$

$$= \sum_{i=1}^{n}(i^2-i) - [\underbrace{(1^2-1)+(2^2-2)}_{=0+2\;=2}]$$

$$= \left(\sum_{i=1}^{n} i^2\right) - \left(\sum_{i=1}^{n} i\right) - 2$$

use the rules

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} - 2$$

simplify

$$= \frac{2n^3+3n^2+n - 3n(n+1) - 12}{6}$$

$$= \frac{2n^3+\cancel{3n^2}+n-\cancel{3n^2}-3n-12}{6} = \frac{2n^3-2n-12}{6}$$

$$= \boxed{\frac{n^3-n-6}{3}}$$

Back to the Area Problem: Partitions

The idea we have used so far is to "to partition" or subdivide the given interval $[a, b]$ into smaller subintervals on each of which the variable x, and thus the function $f(x)$, does not change much.

Definition of a Partition

A *partition* of an interval $[a, b]$ is a set of points $\{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$, listed increasingly, on the x-axis with $a = x_0$ and $x_n = b$. That is:

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

These points subdivide the interval $[a, b]$ into n *subintervals*

$$[a, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \ldots,\ [x_{n-1}, b].$$

The k-th subinterval is thus of the form $[x_{k-1}, x_k]$ and it has *length*

$$\Delta x_k = x_k - x_{k-1}.$$

Assumption

Set $\|P\| = \max\limits_{1 \le i \le n} \{\Delta x_i\}$. We assume that our partition P is such that $\|P\| \to 0$ as $n \to \infty$. In other words, we assume that the length of the longest (and, hence, of all) subinterval(s) tend(s) to zero whenever the number of subintervals in P becomes very large.

The Definite Integral

Let $f(x)$ be a function defined on an interval $[a, b]$.

- Partition the interval $[a, b]$ in n subintervals of lengths $\Delta x_1, \ldots, \Delta x_n$, respectively.
- For $k = 1, \ldots, n$ pick a representative point c_k in the corresponding k-th subinterval.

The **definite integral of** f **from** a **to** b is defined as

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

and it is denoted by $\displaystyle\int_a^b f(x)\,dx.$

The sum $\displaystyle\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$ is called a *Riemann sum* in honor of the German mathematician Bernhard Riemann (1826-1866), who developed the above ideas in full generality. The symbol $\int$ is called the *integral sign*. It is an elongated capital S, of the kind used in the 1600s and 1700s. The letter S stands for the summation performed in computing a Riemann sum. The numbers a and b are called the *lower and upper limits of integration*, respectively. The function $f(x)$ is called the *integrand* and the symbol dx is called the *differential* of x. You can think of the dx as representing what happens to the term Δx in the limit, as the size Δx of the subintervals gets closer and closer to 0.

- The role of x in a definite integral is the one of a *dummy variable*. In fact $\displaystyle\int_a^b x^2\,dx$ and $\displaystyle\int_a^b t^2\,dt$ have the same meaning. They represent the same number.
- We recall that a limit does not necessarily exist. However:

Theorem

If f is continuous on $[a, b]$ then $\displaystyle\int_a^b f(x)\,dx$ exists.

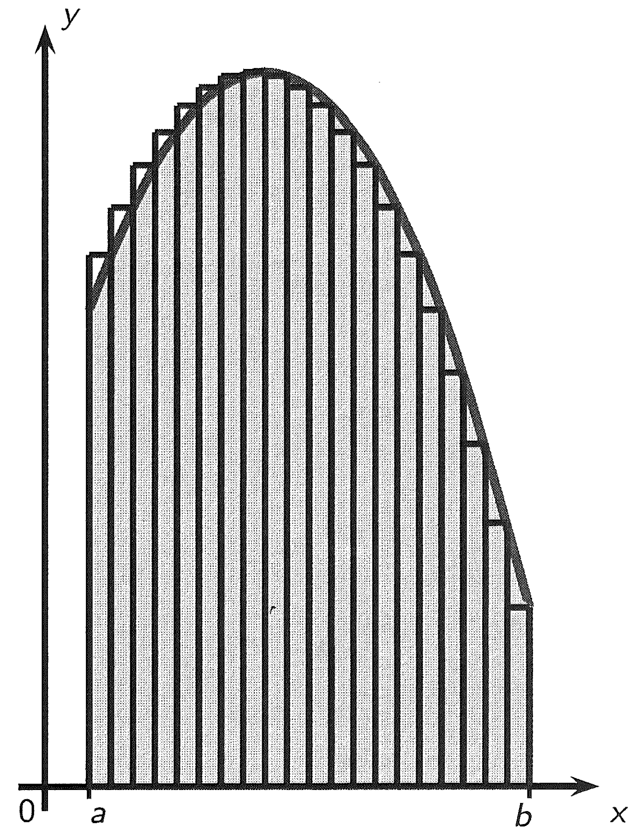
- As we observed earlier, it is computationally easier to partition the interval $[a, b]$ into n subintervals of equal length. Therefore each subinterval has length $\Delta x = \dfrac{b-a}{n}$ (we drop the index k as it is no longer necessary). In this case, there is a simple formula for the points of the partition, namely:

$$x_0 = a+0\cdot\Delta x = a,\ \ x_1 = a+\Delta x,\ \dots\ x_k = a+k\cdot\Delta x,\ \dots,\ x_n = a+n\cdot\Delta x = b$$

or, more concisely, $\quad x_k = a + k\cdot\dfrac{b-a}{n} \quad$ for $\quad k = 0, 1, 2, \dots, n.$

Definite Integrals and Areas

We stress the fact that if the function f takes on only positive values then the definite integral is nothing but the area of the region below the graph of f, lying above the x-axis, and bounded by the vertical lines $x = a$ and $x = b$.

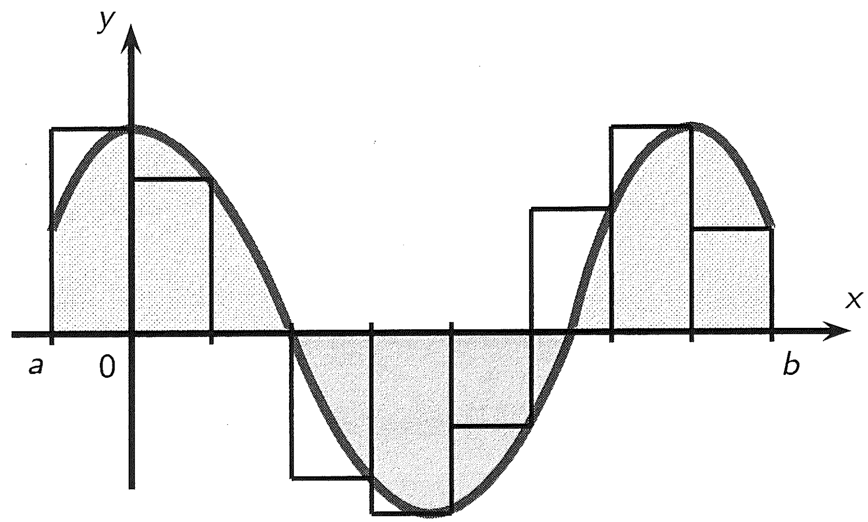


Distance traveled by an object:

If the positive valued function under consideration is the velocity $v(t)$ of an object at time t, then the area underneath the graph of the velocity function and lying above the t-axis represents the total distance traveled by the object from $t = a$ to $t = b$.

What if the Function Takes on Negative Values?

If f happens to take on both positive and negative values, then the Riemann sum is the sum of the areas of rectangles that lie above the x-axis and the negatives of the areas of rectangles that lie below the x-axis. Passing to the limit, we obtain that, in general, a definite integral can be interpreted as a difference of areas:



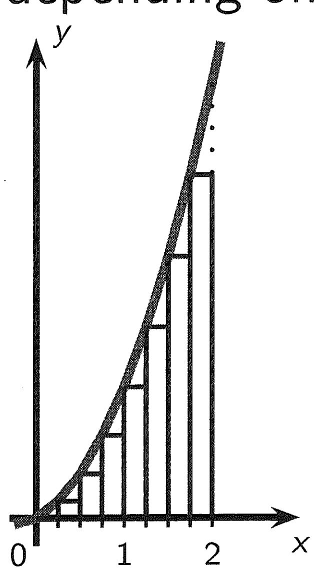
$$\int_a^b f(x)\,dx = [\text{area of the region(s) lying above the } x\text{-axis}] - [\text{area of the region(s) lying below the } x\text{-axis}]$$

Right Versus Left Endpoint Estimates

Observe that x_k, the right endpoint of the k-th subinterval, is also the left endpoint of the $(k+1)$-th subinterval. Thus the Riemann sum estimate for the definite integral of a function f defined over an interval $[a,b]$ can be written in either of the following two forms

$$\sum_{k=0}^{n-1} f(x_k)\cdot\Delta x_{k+1} \qquad\qquad \sum_{k=1}^{n} f(x_k)\cdot\Delta x_k$$

depending on whether we use left or right endpoints, respectively.

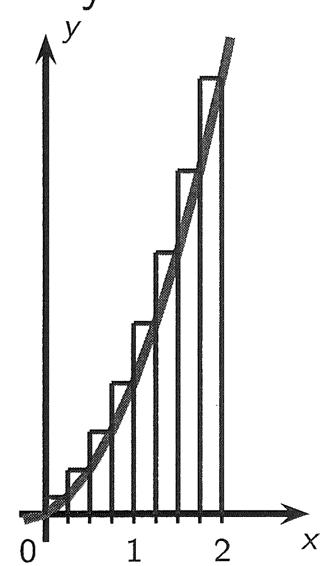


Left endpoint
Riemann sum estimate

If we are dealing with a regular partition, the above sums become

$$\sum_{k=0}^{n-1} f(a+k\cdot\Delta x)\cdot\Delta x \qquad\qquad \sum_{k=1}^{n} f(a+k\cdot\Delta x)\cdot\Delta x$$

with $\Delta x = \dfrac{b-a}{n}$ and $x_k = a+k\cdot\Delta x$ for $k=0,1,\ldots,n$.



Right endpoint
Riemann sum estimate

Example 3: (Online Homework, HW 23, # 11)

Express the limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}$$

From page 13, the formula for a Riemann sum using the right endpoints is:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\cdot \underbrace{\frac{b-a}{n}}_{\Delta x}\right)\cdot \underbrace{\frac{b-a}{n}}_{\Delta x}$$

Hence in our case:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(5 + \frac{2i}{n}\right)^{10} \cdot \frac{2}{n}$$

Says that this is $\displaystyle\int_5^7 \underbrace{x^{10}}_{f(x)} \cdot dx$

Example 4: (Online Homework, HW 23, # 12)

Express the limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\sqrt{1+\frac{4i}{n}}$$

We can interpret $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\sqrt{1+\frac{4}{n}i}$

as $\int_0^4 \underbrace{\sqrt{1+x}}_{f(x)}\,dx$ (this is the type of answer that WeBWork seeks)

(or $\int_1^5 \sqrt{x}\,dx$ ... which are actually equivalent)

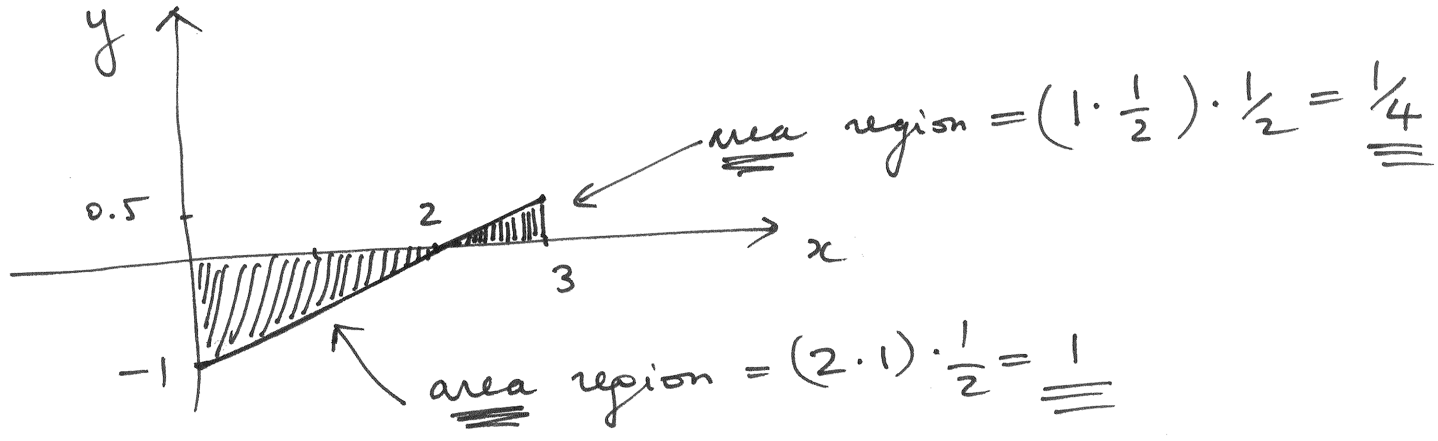
there is just an horizontal shift!

Example 5: (Online Homework, HW 23, # 7)

Evaluate the following integral by interpreting it in terms of areas:

$$\int_0^3 \left(\frac{1}{2}x - 1\right) dx$$

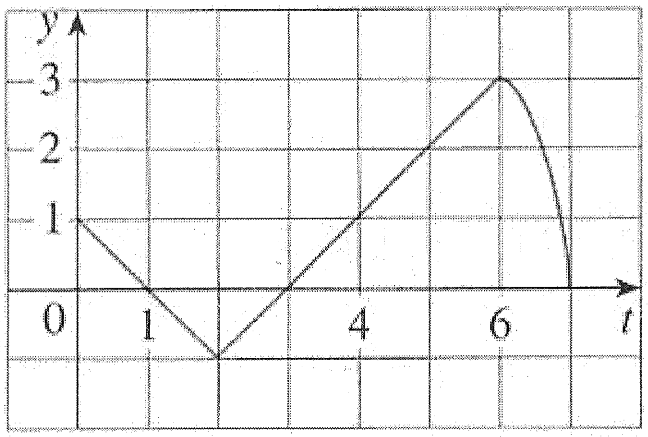
Let's graph the function $y = \frac{1}{2}x - 1$ on $[0, 3]$



$$\int_0^3 \left(\frac{1}{2}x - 1\right) dx = \underbrace{\boxed{-1}}_{\text{"signed area" as it is below the x-axis}} + \frac{1}{4} = \boxed{-\frac{3}{4}}$$

Example 6: (Online Homework, HW 23, # 10)

Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown below.

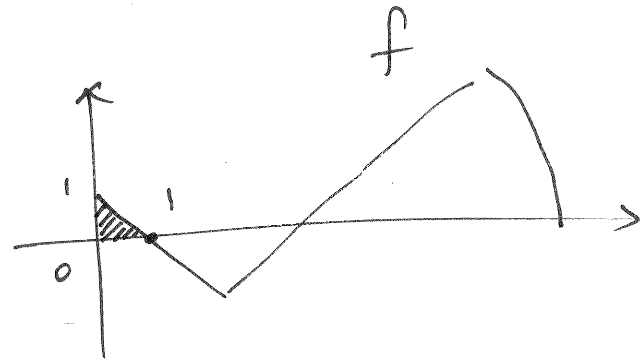


- Evaluate $g(x)$ for $x = 0, 1, 2, 3, 4, 5,$ and 6.
- Estimate $g(7)$.
- At what value of x does g attain its maximum?
- At what value of x does g attain its minimum?

$$g(x) = \int_0^x f(t)\,dt$$

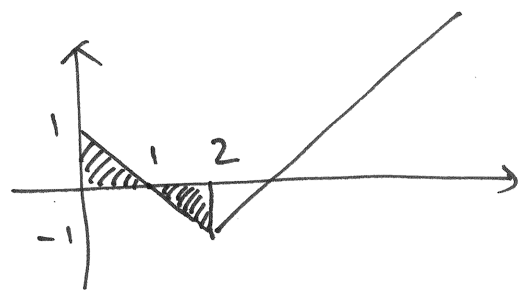
$$g(0) = \int_0^0 f(t)\,dt = 0$$

$$g(1) = \int_0^1 f(t)\,dt$$



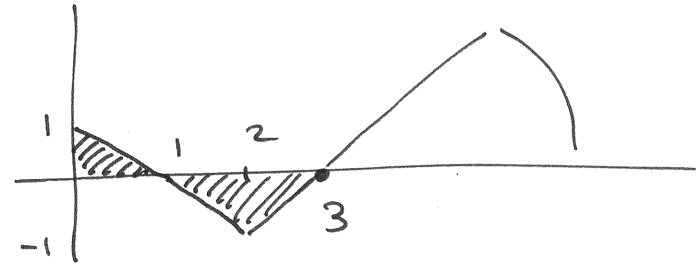
$$= \frac{1 \cdot 1}{2} = \boxed{\frac{1}{2}}$$

$$g(2) = \int_0^2 f(t)\,dt$$



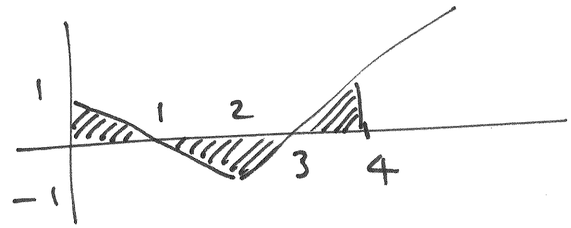
$$= \frac{1}{2} - \frac{1}{2} = \boxed{0}$$

$$g(3) = \int_0^3 f(t)\,dt = \frac{1}{2} - 1 = \boxed{-\tfrac{1}{2}}$$



$$g(4) = \int_0^4 f(t)\,dt = \boxed{0}$$

$$= \frac{1}{2} - 1 + \frac{1}{2} = 0$$



$g(5) = \boxed{3/2}$ and $g(6) = \boxed{4}$ check

$g(7) \cong 6.2$ (estimate)

Max of g occurs at $\boxed{x=7}$; minimum at $\boxed{x=3}$

MA 137 — Calculus 1 with Life Science Applications
The Definite Integral
(Section 6.1)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

December 5, 2016

Some Properties of Definite Integrals

1. $$\int_a^a f(x)\,dx = 0$$

2. $$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$$

3. $$\int_a^b (f(x) \pm g(x))\,dx = \left(\int_a^b f(x)\,dx\right) \pm \left(\int_a^b g(x)\,dx\right)$$

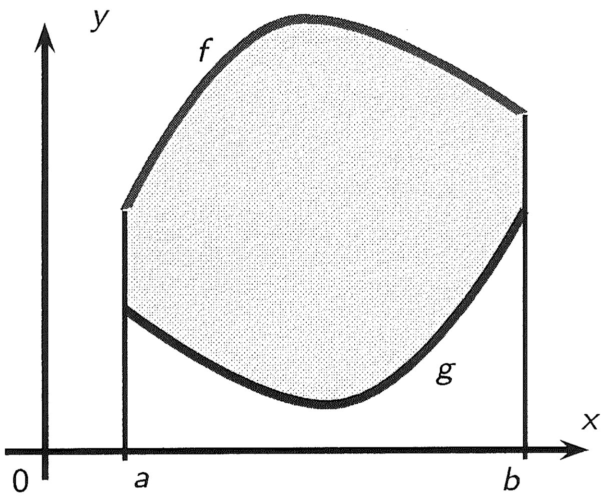
4. $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

5. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

6. If $m \le f(x) \le M$ on $[a, b]$ then
$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Geometric Illustration of Some of the Properties

Property **3.** says that if f and g are two positive valued functions with



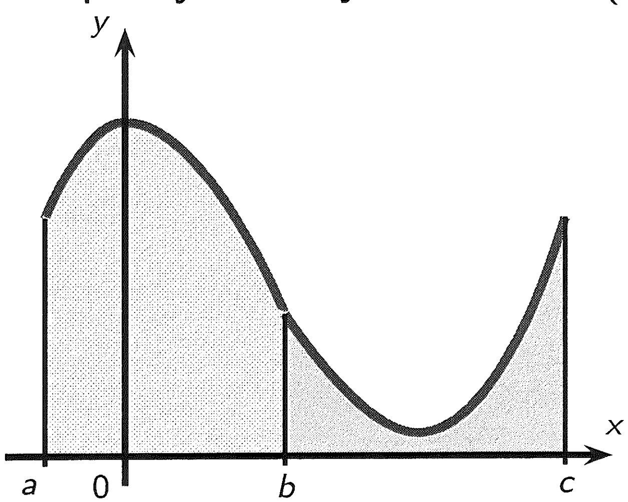
f greater than g, then

$$\int_a^b (f(x) - g(x))\, dx$$

gives the area between the graphs of f and g

$$\int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

Property **4.** says that if $f(x)$ is a positive valued function then the area

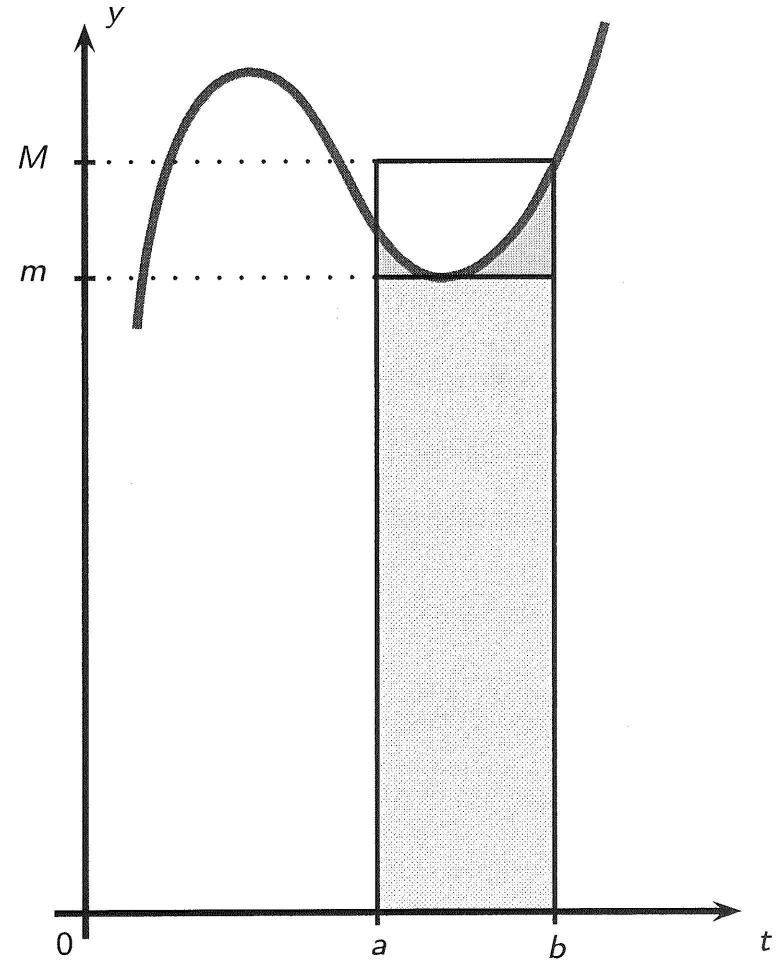


underneath the graph of $f(x)$ between a and b plus the area underneath the graph of $f(x)$ between b and c <u>equals</u> the area underneath the graph of $f(x)$ between a and c.

Property **5.** follows from Properties **4.** and **1.** by letting $c = a$.

$$0 = \int_a^a f(x)\,dx = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx.$$

Property **6.** is illustrated in the picture below.



Example 1: (Online Homework, HW23, # 8)

The sum

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

can be written as a single definite integral of the form

$$\int_{a}^{b} f(x)\,dx$$

for appropriate a and b. Determine these values.

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx \underbrace{- \int_{-2}^{-1} f(x)\,dx}_{+\int_{-1}^{-2} f(x)\,dx}$$

because of property 5.

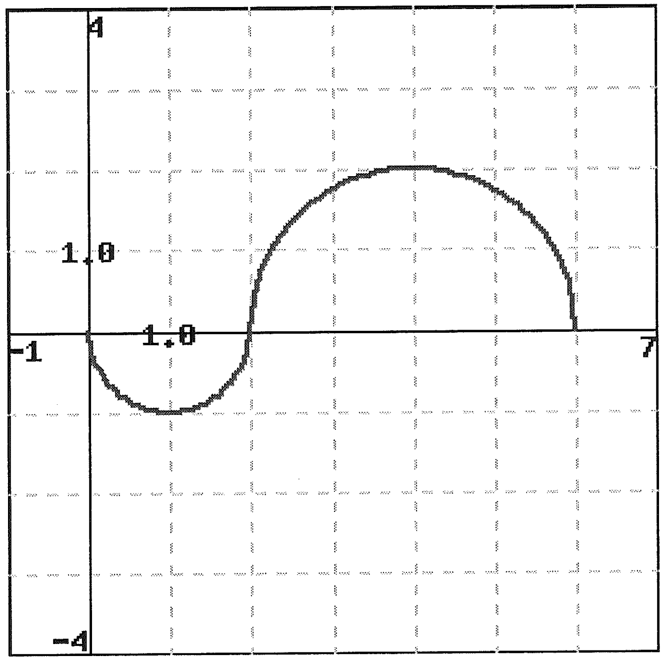
$$= +\int_{-1}^{-2} f(x)\,dx + \int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx$$

$$= \int_{-1}^{5} f(x)\,dx$$

because of property 4.

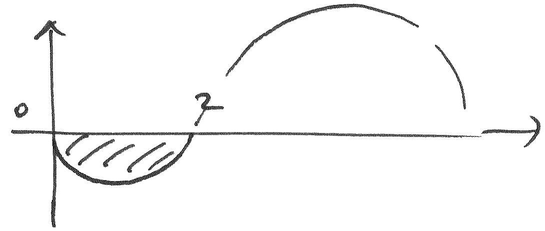
Example 2: (Online Homework, HW23, # 5)

Evaluate the integrals for $f(x)$ shown in the figure below. The two parts of the graph are semicircles.



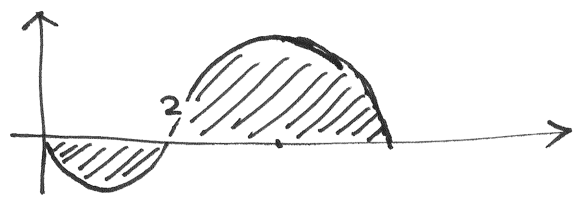
$$\int_0^2 f(x)\,dx \qquad \int_0^6 f(x)\,dx \qquad \int_1^4 f(x)\,dx \qquad \int_1^6 |f(x)|\,dx.$$

$$\int_0^2 f(x)\,dx =$$



$$= -\frac{\pi(1)^2}{2} = \boxed{-\pi/2} \cong -1.5708$$

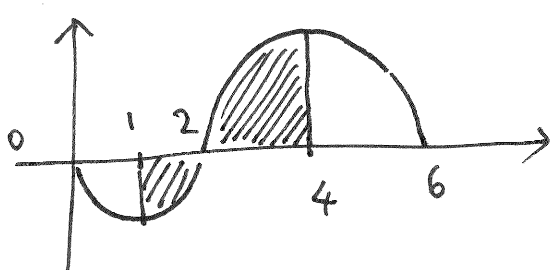
$$\int_0^6 f(x)\,dx =$$



$$= -\frac{\pi(1)^2}{2} + \frac{\pi(2)^2}{2} = -\pi/2 + 2\pi = \boxed{\frac{3}{2}\pi}$$

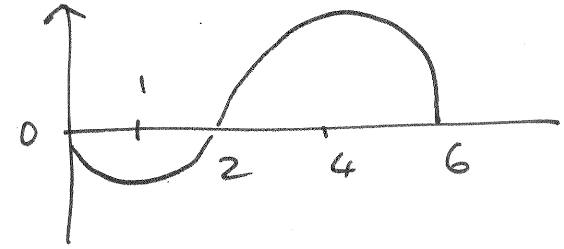
$$\cong 4.7124$$

$$\int_1^4 f(x)\,dx =$$

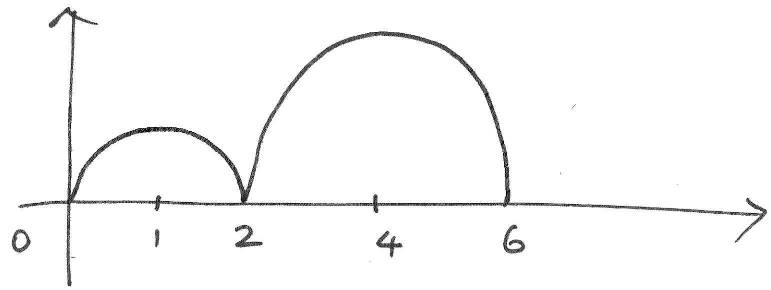


$$= -\frac{\pi(1)^2}{4} + \frac{\pi(2)^2}{4} = -\frac{\pi}{4} + \pi = \boxed{\frac{3}{4}\pi} \cong 2.3562$$

If f has the graph

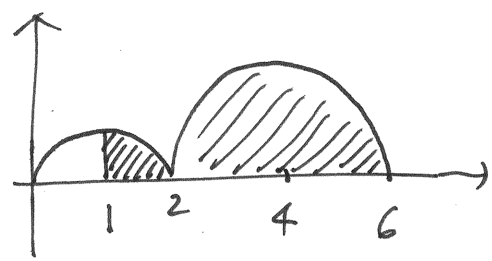


then $|f|$ has the following graph



Hence

$$\int_1^6 |f(x)|\,dx =$$



$$= + \frac{\pi(1)^2}{4} + \frac{\pi(2)^2}{2} = \frac{\pi}{4} + 2\pi = \boxed{\frac{9}{4}\pi}$$

$$\cong 7.0686$$

Example 3: (Neuhauser, Problem # 61, p. 293)

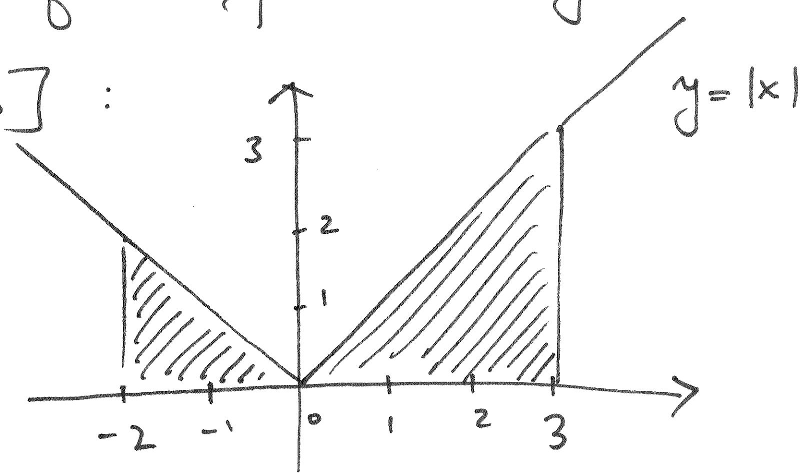
Use an area formula from geometry to find the value of the integral below

$$\int_{-2}^{3} |x| \, dx$$

by interpreting it as the (signed) area under the graph of an appropriately chosen function.

$$\int_{-2}^{3} |x|\,dx$$

Let's look at the graph of the function $y = |x|$ over the interval $[-2, 3]$:



Hence $\int_{-2}^{3} |x|\,dx$ gives the area of the 2 shaded regions (which are triangles) :

$$\int_{-2}^{3} |x|\,dx = \frac{2 \cdot 2}{2} + \frac{3 \cdot 3}{2} = \frac{13}{2} = \underline{\underline{6.5}}$$

Example 4: (Neuhauser, Problem # 65, p. 293)

Use an area formula from geometry to find the value of the integral below

$$\int_{-2}^{2} \left(\sqrt{4-x^2}-2\right) dx$$

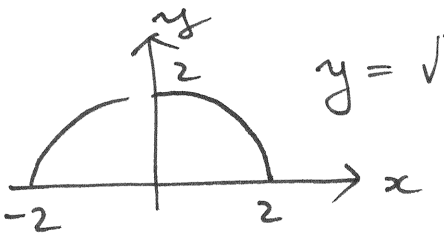
by interpreting it as the (signed) area under the graph of an appropriately chosen function.

$$\int_{-2}^{2}\left(\sqrt{4-x^2}-2\right)dx$$

Let's graph the function

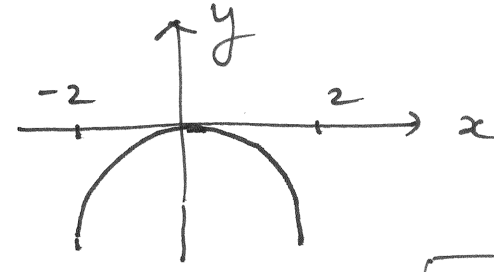
$$y=\sqrt{4-x^2}-2 \quad \text{on } [-2,2]$$

Notice that $y=\sqrt{4-x^2}$ by itself is the graph of the upper half of the circle of radius 2 centered at the origin:



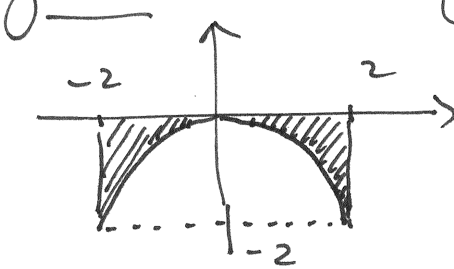
$y=\sqrt{4-x^2}$

hence



$y=\sqrt{4-x^2}-2$

$$\int_{-2}^{2}\left(\sqrt{4-x^2}-2\right)dx = \text{"signed" area of the region}$$



$$\approx \boxed{-1.7168}$$

$$= -\left[\text{area rectangle} - \text{area semicircle}\right] = -\left[4\cdot 2 - \frac{\pi\, 2^2}{2}\right]$$

Example 5: (Neuhauser, Problem # 68(c),(f), p. 293)

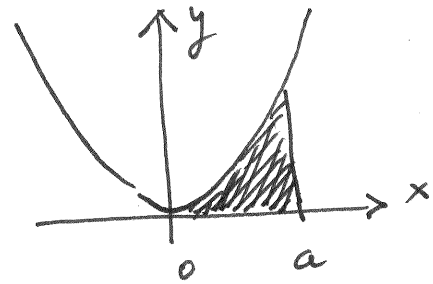
Given that

$$\int_0^a x^2\,dx = \frac{1}{3}a^3$$

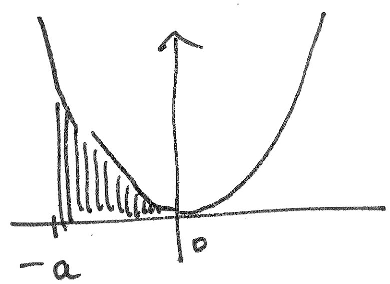
evaluate the following

$$\int_{-1}^3 \frac{1}{3}x^2\,dx \qquad \int_2^4 (x-2)^2\,dx.$$

$$\int_0^a x^2\,dx = \frac{1}{3}a^3 = \text{area of the shaded region}$$



By symmetry of the function we also have that

$$\int_{-a}^{0} x^2\,dx =$$ 

$$= \frac{1}{3}a^3$$

positive sign

Hence :

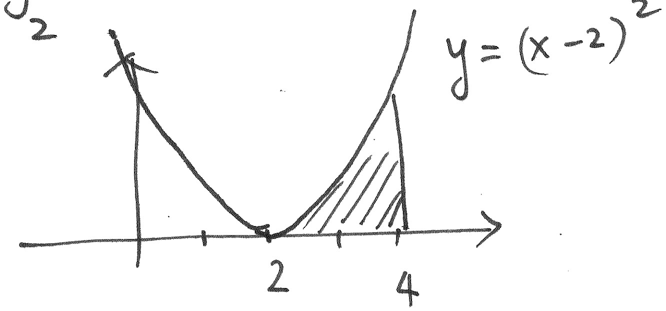
$$\int_{-1}^{3} \frac{1}{3}x^2\,dx = \frac{1}{3}\int_{-1}^{3} x^2\,dx = \frac{1}{3}\left[\int_{-1}^{0} x^2\,dx + \int_0^3 x^2\,dx\right]$$

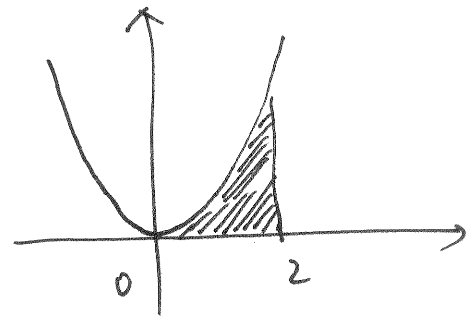
$$= \frac{1}{3}\left[\frac{1}{3}\cdot 1^2 + \frac{1}{3}\cdot 3^2\right] = \frac{1}{3}\left[\frac{1}{3}+3\right] = \boxed{\frac{10}{9}}$$

the graph of $y=(x-2)^2$ is obtained from the graph of $y=x^2$ by shifting it of 2 units to the right;

hence:

$$\int_2^4 (x-2)^2\,dx \quad \text{must be the same as} \quad \int_0^2 x^2\,dx$$





Hence $\displaystyle\int_2^4 (x-2)^2\,dx = \int_0^2 x^2\,dx = \frac{1}{3}2^3 = \boxed{\frac{8}{3}}$

MA 137 — Calculus 1 with Life Science Applications
The Fundamental Theorem of Calculus
(Section 6.2)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

December 7 & 9, 2016

An Example

- The easiest procedure for computing definite integrals is not by computing a limit of a Riemann sum, but by relating integrals to (anti)derivatives.
- This relationship is so important in Calculus that the theorem that describes it is called

  the Fundamental Theorem of Calculus.
- We introduce the theorem by first analyzing a simple example.

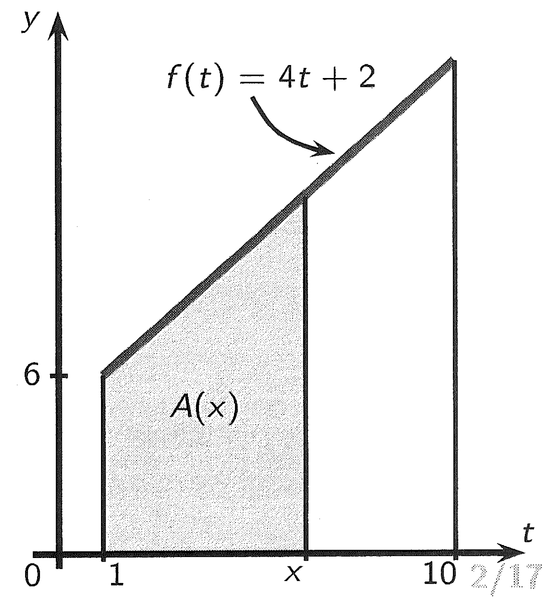
Example 1

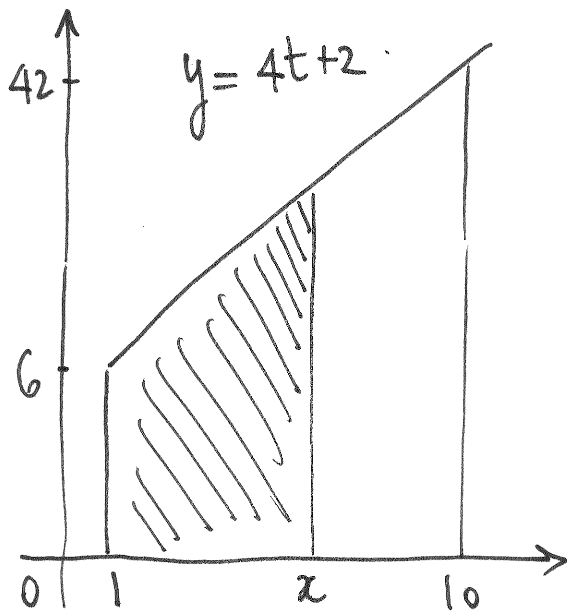
Find a formula for $A(x) = \int_1^x (4t+2)\,dt$,

where $1 \leq x \leq 10$.

Find the values $A(5)$, $A(10)$, and $A(1)$.

What is the derivative of $A(x)$ with respect to x?





$$A(x) = \int_1^x (4t+2)\,dt = \text{area of the shaded trapezoid}$$

Bases : 6 and $4x+2$

height : $x-1$

Hence $A(x) = \dfrac{[6 + (4x+2)]\cdot(x-1)}{2}$

$$\therefore\ A(x) = \int_1^x (4t+2)\,dt = \frac{(4x+8)(x-1)}{2} = (2x+4)(x-1)$$

$$= 2x^2 + 2x - 4$$

$$A(1) = 2\cdot 1^2 + 2\cdot 1 - 4 = \underline{\underline{0}}$$

$$A(5) = 2\cdot 5^2 + 2\cdot 5 - 4 = \underline{\underline{56}}$$

$$A(10) = 2\cdot 10^2 + 2\cdot 10 - 4 = \underline{\underline{216}}$$

$$\boxed{\frac{d}{dx}A(x) = \frac{d}{dx}(2x^2+2x-4) = \underline{\underline{4x+2}}}$$

The Main Idea of the FTC

Suppose that for *any* function $f(t)$ it were true that the area function

$$A(x) = \int_a^x f(t)\,dt$$

satisfies

$$A(a) = \int_a^a f(t)\,dt = 0 \qquad\qquad A'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt \right) = f(x).$$

Moreover, suppose that $B(x)$ is any function such that: $B'(x) = f(x) = A'(x)$. By a consequence of the MVT we know that there is a constant value c such that $B(x) = A(x) + c$.

All these facts put together help us easily evaluate $\displaystyle\int_a^b f(t)\,dt$.

Indeed

$$\begin{aligned} \underbrace{\int_a^b f(t)\,dt}_{} &= A(b) = A(b) - 0 \\ &= \underbrace{A(b) - A(a)}_{} = [A(b)+c] - [A(a)+c] \\ &= \underbrace{B(b) - B(a)}_{} \end{aligned}$$

The FTC

The previous 'speculations' are actually true for any continuous function on the interval over which we are integrating. These results are stated in the following theorem, which is divided into two parts:

Theorem (The Fundamental Theorem of Calculus)

PART I: *Let $f(t)$ be a continuous function on the interval $[a, b]$. Then the function $A(x)$, defined by the formula*

$$A(x) = \int_a^x f(t)\,dt$$

for all x in the interval $[a, b]$, is an antiderivative of $f(x)$, that is

$$A'(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt \right) = f(x)$$

for all x in the interval $[a, b]$.

PART II: *Let $F(x)$ be any antiderivative of $f(x)$ on $[a, b]$, so that $F'(x) = f(x)$ for all x in the interval $[a, b]$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Special Notation for Part II:

Part II of the FTC tells us that evaluating a definite integral is a two-step process:

- find *any* antiderivative $F(x)$ of the function $f(x)$; and then
- compute the difference $F(b) - F(a)$.

A notation has been devised to separate the two steps of this process: $F(x)\Big|_a^b$ stands for the difference $F(b) - F(a)$. Thus

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

About the Proof of the FTC:

We already gave an explanation of why the second part of the Fundamental Theorem of Calculus follows from the first one.

To prove the first part we need to use the definition of the derivative.

Proof of Part I

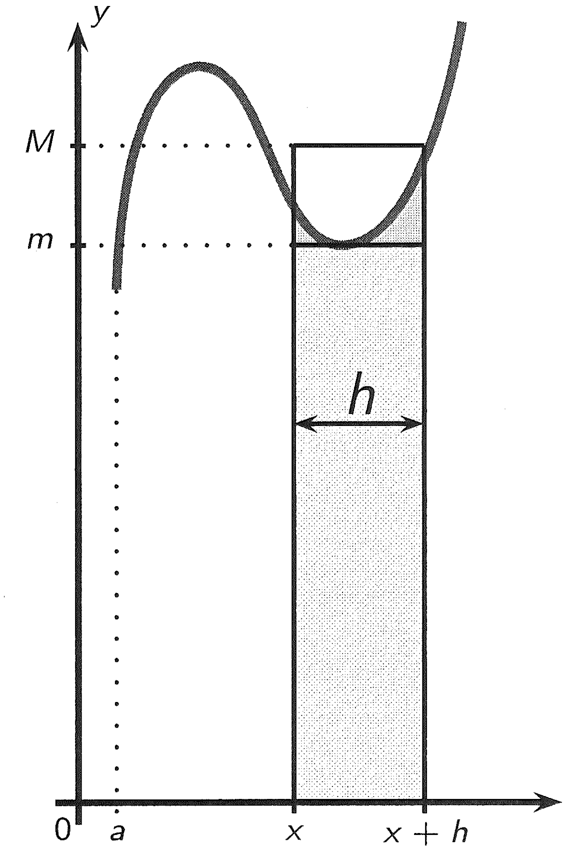
We must show that

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x).$$

For convenience, let us assume that f is a positive valued function. Given that $A(x)$ is defined by $\displaystyle\int_a^x f(t)\,dt$, the numerator of the above difference quotient is

$$A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt.$$

Using properties **4.** and **5.** of definite integrals, the above difference equals $\displaystyle\int_x^{x+h} f(t)\,dt$.



As the function f is continuous over the interval $[x, x+h]$, the Extreme Value Theorem says that there are values c_1 and c_2 in $[x, x+h]$ where f attains the minimum and maximum values, say m and M, respectively.

Thus $\quad m \leq f(t) \leq M \quad$ on $[x, x+h]$. As the length of the interval $[x, x+h]$ is h, by property **6.** of definite integrals we have that

$$f(c_1)h = mh \leq \int_x^{x+h} f(t)\,dt \leq Mh = f(c_2)h$$

or, equivalently,

$$f(c_1) \leq \frac{\displaystyle\int_x^{x+h} f(t)\,dt}{h} \leq f(c_2).$$

Finally, as f is continuous we have that $\lim\limits_{h\to 0} f(c_1) = f(x) = \lim\limits_{h\to 0} f(c_2)$.

This concludes the proof.

Example 2: (Online Homework HW24, # 2)

Suppose
$$f(x) = \int_0^x \frac{t^2 - 16}{2 + \cos^2(t)}\, dt$$
For what value(s) of x does $f(x)$ have a local maximum?

$$f(x) = \int_0^x \frac{t^2 - 16}{2 + \cos^2(t)}\, dt$$

By the FTC part I :

$$f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \int_0^x \frac{t^2-16}{2+\cos^2(t)}\, dt = \frac{x^2-16}{2+\cos^2(x)}$$

Hence $f'(x) = 0 \iff x^2 - 16 = 0 \iff x = \pm 4$

Now you can use test points to determine the <u>sign</u> <u>of $f'(x)$</u> :

$+ + + +$ at -4 $- - - - - - -$ at 4 $+ + + +$

f inc. ↗ f dec. ↘ f inc. ↗

$\therefore$ f has a local max at $\boxed{x = -4}$

Example 3: (Online Homework HW24, # 6)

Find a function f and a number a such that

$$2 + \int_a^x \frac{f(t)}{t^7}\, dt = 4x^{-3}$$

$$2 + \int_a^x \frac{f(t)}{t^7}\,dt = 4x^{-3}$$

Notice that if we plug in $\boxed{x=a}$ in the above equation we get :

$$2 + \underbrace{\int_a^a \frac{f(t)}{t^7}\,dt}_{=0} = 4a^{-3} \qquad \therefore \quad 2 = \frac{4}{a^3}$$

as the interval of integration has length 0

$$\text{or} \quad a^3 = \frac{4}{2}$$

$$\therefore \quad \boxed{a = \sqrt[3]{2}}$$

Hence we obtain so far :

$$\boxed{2 + \int_{\sqrt[3]{2}}^{x} \frac{f(t)}{t^7}\,dt = 4x^{-3}}$$

In order to find f, let's take the derivative of both sides with respect to x, and let's apply the FTC Part I :

$$\frac{d}{dx}\left[2 + \int_{\sqrt[3]{2}}^{x} \frac{f(t)}{t^7}\,dt\right] = \frac{d}{dx}\,4x^{-3}$$

$$\Longrightarrow \quad \frac{f(x)}{x^7} = 4(-3)\cdot x^{-4}$$

Hence $\quad f(x) = -12\,x^{-4}\cdot x^{7}$

$$\therefore \quad \boxed{f(x) = -12x^3}$$

Leibniz's Rule

Combining the chain rule and the FTC (Part I), we can differentiate integrals with respect to x when the upper and/or lower limits of integration are function of x.

We summarize these facts into the following result:

Leibniz's Rule

If $g(x)$ and $h(x)$ are differentiable functions and $f(u)$ is continuous for u between $g(x)$ and $h(x)$, then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(u)\,du = f[h(x)]h'(x) - f[g(x)]g'(x)$$

Example 4: (Neuhauser, Example # 4, p. 296)

Compute

$$\frac{d}{dx}\int_{\sin x}^{1} u^2\,du$$

$$\frac{d}{dx}\int_{\sin x}^{1} u^2\,du = \frac{d}{dx}\left[-\int_{1}^{\boxed{\sin x}} u^2\,du\right] =$$

Set $\boxed{w = \sin x}$ then by the chain rule we have

$$= -\left[\frac{d}{dw}\int_{1}^{w} u^2\,du\right]\cdot\left[\frac{dw}{dx}\right] = -w^2\cdot\frac{dw}{dx}$$

$$= -(\sin x)^2\cdot\cos x = \boxed{-\sin^2 x\cdot\cos x}$$

this confirms the Leibniz's Rule we described earlier.

Example 5: (Online Homework HW24, # 5)

Find the derivative of the following function

$$F(x) = \int_{x^4}^{x^6} (2t-1)^3 \, dt$$

using the Fundamental Theorem of Calculus.

$$F(x) = \int_{x^4}^{x^6} (2t-1)^3\,dt = \int_{x^4}^{0} (2t-1)^3\,dt + \int_{0}^{x^6} (2t-1)^3\,dt$$

for example

$$= \int_{0}^{x^6} (2t-1)^3\,dt - \int_{0}^{x^4} (2t-1)^3\,dt$$

now apply FTC Part I together with the chain rule

$$F'(x) = \frac{d}{dx}\int_{0}^{x^6} (2t-1)^3\,dt - \frac{d}{dx}\int_{0}^{x^4} (2t-1)^3\,dt$$

$$= (2x^6-1)^3 \cdot \underbrace{6x^5}_{} - (2x^4-1)^3 \cdot \underbrace{4x^3}_{}$$

which confirms the Leibniz's Rule we described earlier

Example 6: (Online Homework HW24, # 7)

Evaluate the definite integral

$$\int_4^7 \left(\frac{d}{dt} \sqrt{3 + 3t^4} \right) dt$$

using the Fundamental Theorem of Calculus.

Notice that

$$\int \left(\frac{d}{dt}\sqrt{3+3t^4}\right) dt = \sqrt{3+3t^4} + C$$

as the process of antidifferentiation is the inverse of the process of differentiation.

Hence :

$$\int_4^7 \left(\frac{d}{dt}\sqrt{3+3t^4}\right) dt = \sqrt{3+3t^4}\Big]_4^7$$

we choose the constant to be $\underline{C=0}$

$$= \sqrt{3+3\cdot 7^4} - \sqrt{3+3\cdot 4^4} = \sqrt{7206} - \sqrt{771}$$

$$= 84.8882 - 27.7669 \cong 57.1213$$

Example 7: (Online Homework HW24, # 12)

Evaluate the definite integral

$$\int_1^4 \frac{x^2+5}{x}\,dx$$

$$\int_1^4 \frac{x^2+5}{x}\,dx$$ We use FTC Part 2

We first need an antiderivative of $\frac{x^2+5}{x}$:

$$\int \frac{x^2+5}{x}\,dx = \int \left(x + \frac{5}{x}\right) dx = \int x\,dx + 5\int \frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2 + 5\ln|x| + C$$

Now we have :

$$\int_1^4 \frac{x^2+5}{x}\,dx = \frac{1}{2}x^2 + 5\ln|x| + C\Big]_1^4 =$$

$$= \left[\frac{1}{2}4^2 + 5\ln(4) + \cancel{C}\right] - \left[\frac{1}{2}1^2 + 5\ln(1) + \cancel{C}\right]$$

$$= 8 + 5\ln 4 - \frac{1}{2} = \boxed{15/2 + 5\ln 4} \cong 14.4315$$

Example 8: (Online Homework HW24, # 14)

Evaluate the definite integral

$$\int_0^1 (x^2 + 8 - 2e^{-2x})\, dx$$

$$\int (x^2 + 8 - 2e^{-2x})\, dx =$$

$$\int x^2\, dx + 8\int 1 \cdot dx + \int -2e^{-2x}\, dx$$

$$= \frac{1}{3}x^3 + 8x + e^{-2x} + C$$

<u>Hence</u> :

$$\int_0^1 (x^2 + 8 - 2e^{-2x})\, dx = \left. \frac{1}{3}x^3 + 8x + e^{-2x} \right]_0^1$$

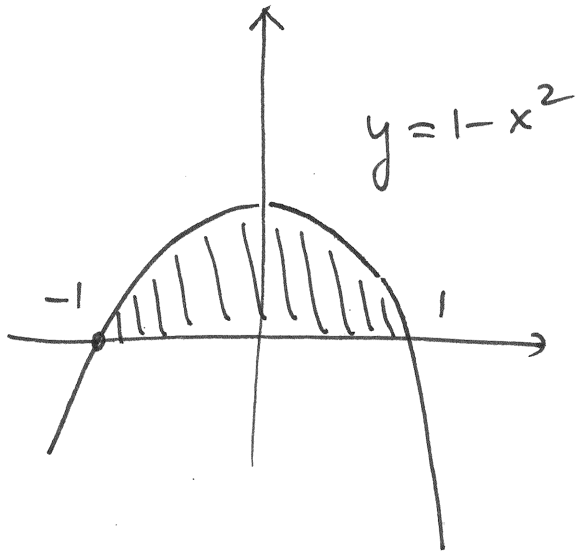
we need just one antiderivative
we choose the one with $C = 0$

$$= \left[\frac{1}{3}\cdot 1^3 + 8\cdot 1 + e^{-2\cdot 1}\right] - \left[\frac{1}{3}\cdot 0^3 + 8\cdot 0 + e^{-2\cdot 0}\right]$$

$$= \frac{1}{3} + 8 + e^{-2} - 1 = \frac{22}{3} + e^{-2} \cong 7.4687$$

Example 9: (Online Homework HW24, # 15)

Find the area bounded by the function $y = 1 - x^2$ and the x-axis.



Notice that the intersections with the x-axis are given by

$$0 = 1 - x^2$$

$$\iff \quad x = 1 .$$

Hence we need to compute $\displaystyle\int_{-1}^{1} (1-x^2)\,dx$

Observe that by the symmetry of the function ($\because f(x)$ is $\underline{\underline{\text{even}}}$) then

$$\int_{-1}^{1} (1-x^2)\,dx = 2\int_{0}^{1} (1-x^2)\,dx = 2\left(x - \frac{1}{3}x^3\right)\Big]_0^1$$

$$= \left[2\left(1 - \frac{1}{3}\right)\right] - [0] = \boxed{4/3}$$