[illegible] – [illegible] with Life Science Applications
The Substitution Rule
(Section 7.1)

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Section 7.1: The Substitution Rule

The substitution rule is the chain rule in integral form.
We therefore begin by recalling the chain rule.

Suppose that we wish to differentiate

$$f(x) = (6x^2 + 3)^3.$$

This is clearly a situation in which we need to use the chain rule.

We set $u = 6x^2 + 3$ so that $f(u) = u^3$.
The chain rule, using Leibniz notation, tells us that

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = 3u^2 \cdot (6 \cdot 2x) = 3(6x^2 + 3)^2(12x).$$

Reversing these steps and integrating along the way, we get

$$\int 3(6x^2+3)^2(12x)\,dx = \int 3u^2\,du = u^3 + C = (6x^2+3)^3 + C.$$

In the first step, we substituted u for $6x^2+3$ and used $du = 12x\,dx$.

This substitution simplified the integrand.

At the end, we substitute back $6x^2+3$ for u to get the final answer in terms of x.

We summarize this discussion, by stating the following **general principle**:

Substitution Rule for Indefinite Integrals

If $u = g(x)$, then $$\int f[\underbrace{g(x)}_{u}]\,\underbrace{g'(x)\,dx}_{du} = \int f(u)\,du.$$

Example 1

Evaluate the indefinite integral $\int \cos x \sin x \, dx$

- by using the substitution $u = \cos x$;
- by using the substitution $u = \sin x$;
- by using the trigonometric identity $\sin(2x) = 2 \sin x \cos x$.

Compare your answers.

Consider the indefinite integral $\int \cos x \cdot \sin x \, dx$.

(I) If we set $u = \cos x$ then $\frac{du}{dx} = -\sin x$ so that

$\sin x \cdot dx = -du$

Thus: $\int \underbrace{\cos x}_{u} \cdot \underbrace{\sin x \, dx}_{-du} = \int -u \, du = -\int u \, du = -\frac{1}{2} u^2 + c$

$= -\frac{1}{2} (\cos x)^2 + C_1 = -\frac{1}{2} \cos^2 x + C_1$

(II) If we set $u = \sin x$ then $\frac{du}{dx} = \cos x$ so that

$\cos x \, dx = du$

Thus: $\int \underbrace{\cos x \, dx}_{du} \underbrace{\sin x}_{u} = \int u \, du = \frac{1}{2} u^2 + C$

$= \frac{1}{2} (\sin x)^2 + C_2 = \frac{1}{2} \sin^2 x + C_2$

Notice that the 2 answers are consistent as we know that there is the trigonometric identity

$\cos^2 x + \sin^2 x = 1$. Thus

$$-\frac{1}{2}\cos^2 x + C_1 = -\frac{1}{2}(1-\sin^2 x) + C_1 = \frac{1}{2}\sin^2 x + \underbrace{C_1 - \frac{1}{2}}_{C_2}$$

first solution

(III) Recall the trig. identity $\boxed{\cos x \sin x = \frac{1}{2}\sin(2x)}$. Thus

$$\int \cos x \sin x \, dx = \int \frac{1}{2}\sin(2x)\, dx .$$ Set $u = 2x$ so that $du = 2\,dx$

$$= \int \frac{1}{2}\sin(u)\cdot\frac{1}{2}du = \int \frac{1}{4}\sin(u)\,du = \frac{1}{4}\int \sin(u)\,du$$

$$= \frac{1}{4}(-\cos(u)) + C_3 = -\frac{1}{4}\cos(2x) + C_3$$

This answer is consistent with the others: $\boxed{\cos(2x) = \cos^2 x - \sin^2 x}$

Example 2

Evaluate the indefinite integral $\int (2x+1)e^{x^2+x}\,dx$.

$$\int (2x+1)\, e^{x^2+x}\, dx$$

Set $u = x^2 + x$ so that $\frac{du}{dx} = 2x+1$;

hence $(2x+1)\, dx = du$.

Thus $\int (2x+1)\, e^{x^2+x} \cdot dx = \int e^u \, du =$

$$= e^u + C = \boxed{e^{x^2+x} + C}$$

substitute back

Substitution Rule for Definite Integrals

Part II of the FTC says that when we evaluate a definite integral, we must find an antiderivative of the integrand and then evaluate the antiderivative at the limits of integration.

When we use the substitution $u = g(x)$ to find an antiderivative of an integrand, the antiderivative will be given in terms of u at first.

To complete the calculation, we can proceed in either of two ways:

(1) we can leave the antiderivative in terms of u and change the limits of integration according to $u = g(x)$;

(2) we can substitute $g(x)$ for u in the antiderivative and then evaluate the antiderivative at the limits of integration in terms of x.

Substitution Rule for Definite Integrals

The first method (1) is the more common one, and we summarize the procedure as follows:

Substitution Rule for Definite Integrals

If $u = g(x)$, then $$\int_a^b f[\underbrace{g(x)}_{u}]\,\underbrace{g'(x)\,dx}_{du} = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 3 (Online Homework # 6)

Evaluate the definite integral $\displaystyle\int_1^{e^5} \frac{dx}{x(1+\ln x)}$.

$$\int_1^{e^5} \frac{dx}{x(1+\ln x)} = \text{rewrite it as} = \int_1^{e^5} \frac{1}{(1+\ln x)} \cdot \frac{1}{x}\, dx$$

and notice that if we set $u = 1 + \ln x$ then $\frac{du}{dx} = \frac{1}{x}$.

Thus the integral becomes

$$\int \underbrace{\frac{1}{(1+\ln x)}}_{\frac{1}{u}} \cdot \underbrace{\frac{1}{x}\, dx}_{du} = \int \frac{1}{u}\, du.$$

We also need to change the limits of integration!

$x = 1 \rightsquigarrow u = 1 + \ln x \rightsquigarrow u = 1 + \ln(1) = 1$

$x = e^5 \rightsquigarrow u = 1 + \ln x \rightsquigarrow u = 1 + \ln(e^5) = 1 + 5 = 6$

$$\therefore \int_1^6 \frac{1}{u} \cdot du = \ln|u| \Big]_1^6 = \ln(6) - \underbrace{\ln(1)}_{=0} = \boxed{\ln(6)}$$

≈ 1.79176

Example 4 (Online Homework # 8)

Consider the indefinite integral $\int \frac{3}{3+e^x}\,dx$.

- The most appropriate substitution to simplify this integral is $u = f(x)$ where $f(x) =$ ________.
  We then have $dx = g(u)du$ where $g(u) =$ ________.
  (**Hint:** you need to back substitute for x in terms of u for this part.)
- After substituting into the original integral we obtain $\int h(u)\,du$ where $h(u) =$ ________.
- To evaluate this integral rewrite the numerator as $3 = u - (u-3)$. Simplify, then integrate, thus obtaining $\int h(u)\,du = H(u)$ where $H(u) =$ ________ $+ C$.

Example 4, cont.ed (Online Homework # 8)

- After substituting back for u we obtain our final answer
$$\int \frac{3}{3+e^x}\,dx = \underline{\hspace{4cm}} + C.$$

(1) Consider $\int \frac{3}{3+e^x}\,dx$ we try the following substitution

$u = \underline{3+e^x}$. (This is our $f(x)$.) Thus $\frac{du}{dx} = e^x$ or

$du = e^x\,dx$. Notice that $e^x = u-3$ so that

↳ becomes $du = (u-3)\,dx \rightsquigarrow$ $\boxed{dx = \frac{1}{u-3}\,du}$

Thus $g(u) = \frac{1}{u-3}$.

(2) After substituting we obtain

$$\int \frac{3}{3+e^x}\,dx = \int \frac{3}{u}\cdot\frac{1}{(u-3)}\cdot du = \int \underbrace{\frac{3}{u(u-3)}}_{h(u)}\,du$$

(3) We are suggested to rewrite the numerator as

$$3 = u-(u-3)$$

Therefore $\displaystyle\int \frac{3}{u(u-3)}\,du = \int \frac{u-(u-3)}{u(u-3)}\,du =$

$$= \int \left[\frac{u}{u(u-3)} - \frac{u-3}{u(u-3)}\right] du = \int \left[\frac{1}{u-3} - \frac{1}{u}\right] du$$

$$= \ln|u-3| - \ln|u| + C = \underbrace{\ln\left|\frac{u-3}{u}\right|}_{H(u)} + C$$

(4) after substituting back $u = 3+e^x$, we obtain

$$\int \frac{3}{3+e^x}\,dx = \boxed{\ln\left|\frac{e^x}{3+e^x}\right| + C}$$

or $\quad x - \ln(3+e^x) + C$

Example 5 (Online Homework # 9)

Consider the definite integral $\int_0^1 x^2\sqrt{5x+6}\,dx$.

- Then the most appropriate substitution to simplify this integral is $u =$ ________. Then $dx = f(x)du$ where $f(x) =$ ________.
- After making the substitution and simplifying we obtain the integral
$$\int_a^b g(u)\,du$$
where $g(u) =$ ________, $a =$ ____ and $b =$ ____.
- This definite integral has value $=$ ________.

(1) Consider $\int_0^1 x^2\sqrt{5x+6}\,dx$. The most appropriate substitution seems to be $u = 5x+6$ so that

$\frac{du}{dx} = 5$ or $\boxed{dx = \frac{1}{5}du}$. For the substitution notice that we also need to write x in terms of u!

$u = 5x+6 \rightsquigarrow x = \frac{u-6}{5}$. Thus

$$\int_0^1 x^2\sqrt{5x+6}\,dx = \int_6^{11} \left(\frac{u-6}{5}\right)^2 \sqrt{u}\cdot\left(\frac{1}{5}du\right)$$

$u = 5x+6$

$dx = \frac{1}{5}du$

$x = \frac{u-6}{5}$

$x = 1 \rightsquigarrow u = 5x+6 \rightsquigarrow u = 11$

$x = 0 \rightsquigarrow u = 5x+6 \rightsquigarrow u = 6$

for the limits of integration

(2) So, after making the substitutions we obtain:

$$\int_6^{11} \left(\frac{u-6}{5}\right)^2 \cdot \sqrt{u} \cdot \left(\frac{1}{5}du\right) = \int_6^{11} \frac{1}{125}(u-6)^2 \cdot \sqrt{u} \cdot du$$

$$= \int_6^{11} \frac{1}{125}(u^2 - 12u + 36)\sqrt{u} \cdot du = \int_6^{11} \underbrace{\frac{1}{125}\left(u^{5/2} - 12u^{3/2} + 36u^{1/2}\right)}_{g(u)} du$$

multiply out

(3) $$= \underbrace{\frac{1}{125}\left(\frac{2}{7}u^{7/2} - \frac{24}{5}u^{5/2} + 36 \cdot \frac{2}{3}u^{3/2}\right)}_{\text{antiderivative}} \Bigg]_6^{11}$$

$$\cong \frac{1}{125} \cdot (210.5583 - 80.6232) \cong \frac{129.9351}{125}$$

$$\cong \boxed{1.03948}$$

Example 6 (similar to Example 5)

Consider the definite integral $\int_1^2 x^5\sqrt{x^3+2}\,dx$.

- Then the most appropriate substitution to simplify this integral is $u =$ ________. Then $dx = f(x)du$ where $f(x) =$ ________.
- After making the substitution and simplifying we obtain the integral

$$\int_a^b g(u)\,du$$

where $g(u) =$ ________, $a =$ ____ and $b =$ ____.

- This definite integral has value = ________.

This example $\int_1^2 x^5\sqrt{x^3+2}\,dx$ is similar to Example 5

Set $\boxed{u = x^3+2}$; so that $\frac{du}{dx} = 3x^2$ or $\boxed{x^2\,dx = \frac{1}{3}\,du}$

Thus we can rewrite the integral as :

$$\int_1^2 x^5\sqrt{x^3+2}\,dx = \int_1^2 \underbrace{x^3}_{u-2}\cdot\underbrace{\sqrt{x^3+2}}_{\sqrt{u}}\cdot\underbrace{x^2\,dx}_{\frac{1}{3}du}$$

Thus $= \boxed{\int_3^{10} \frac{1}{3}(u-2)\sqrt{u}\,du}$ as $x=1 \rightsquigarrow u=3$, $x=2 \rightsquigarrow u=10$

$$= \int_3^{10}\left(\frac{1}{3}u^{3/2} - \frac{2}{3}u^{1/2}\right)du = \frac{1}{3}\cdot\frac{2}{5}u^{5/2} - \frac{2}{3}\cdot\frac{2}{3}u^{3/2}\Big]_3^{10}$$

$$\cong 28.109 - (-0.231) \cong \underline{28.34}$$

Example 7 (Online Homework # 11)

Consider the indefinite integral $\int \frac{1}{3x + 7\sqrt{x}}\, dx$.

- Then the most appropriate substitution to simplify this integral is $u =$ ________. Then $dx = f(x)du$ where $f(x) =$ ________.
- After making the substitution and simplifying we obtain the integral

$$\int g(u)\, du$$

  where $g(u) =$ ________.
- This last integral is: $=$ ________ $+ C$.
  (Leave out constant of integration from your answer.
- After substituting back for u we obtain the following final form of the answer: $=$ ____________ $+ C$.

Consider the integral $\int \frac{1}{3x+7\sqrt{x}}\,dx$.

The idea to solve this integral is to notice that $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$. How does this help? Rewrite the integral as follows:

$$\int \frac{1}{3x+7\sqrt{x}}\,dx = \int \frac{1}{\sqrt{x}\,(3\sqrt{x}+7)}\,dx$$

Now set $\boxed{u = 3\sqrt{x}+7}$, so that $\frac{du}{dx} = 3 \cdot \frac{1}{2\sqrt{x}}$.

Thus $\boxed{dx = \frac{2}{3}\sqrt{x}\,du}$. After making the substitution we obtain: $\int \underbrace{\frac{2}{3} \cdot \frac{1}{u}}_{g(u)} \cdot du$.

$$= \frac{2}{3}\ln|u| + C = \boxed{\frac{2}{3}\ln|3\sqrt{x}+7| + C}$$