MA 138 – Calculus 2 with Life Science Applications
Rational Functions and Partial Fractions
(Section 7.3)

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Example 1

Evaluate the following indefinite integrals

- $$\int \frac{5}{(3x+2)^4}\,dx;$$
- $$\int \frac{2x-2}{(x^2-2x+5)^3}\,dx.$$

(1) $\int \frac{5}{(3x+2)^4}\,dx$ set $\boxed{u = 3x+2}$ so that $\frac{du}{dx} = 3$

$\boxed{dx = \frac{1}{3}\,du}$

hence if we substitute back

$$= \int \frac{5}{u^4}\cdot\frac{1}{3}\,du = \int \frac{5}{3}\,u^{-4}\,du = \frac{5}{3}\cdot\frac{1}{-3}\,u^{-3} + C$$

$$= -\frac{5}{9}\cdot\frac{1}{u^3} + C = \boxed{-\frac{5}{9}\cdot\frac{1}{(3x+2)^3} + C}$$

(2) $\int \frac{2x-2}{(x^2-2x+5)^3}\,dx$ set $\boxed{u = x^2 - 2x + 5}$ so

$\frac{du}{dx} = 2x - 2$ $\boxed{(2x-2)\,dx = du}$

hence

$$= \int \frac{du}{u^3} = \int u^{-3}\,du = -\frac{1}{2}\,u^{-2} + C = -\frac{1}{2u^2} + C$$

$$= \boxed{-\frac{1}{2(x^2-2x+5)^2} + C}$$

Section 7.3: Rational Functions and Partial Fractions

- A rational function f is the quotient of two polynomials. That is,
$$f(x) = \frac{P(x)}{Q(x)}$$
where $P(x)$ and $Q(x)$ are polynomials.
- To integrate such a function, we write $f(x)$ as a sum of a polynomial and simpler rational functions (=partial-fraction decomposition).
- These simpler rational functions, which can be integrated with the methods we have learned, are of the form
$$\frac{A}{(ax+b)^n} \quad \text{or} \quad \frac{Bx+C}{(ax^2+bx+c)^n}$$
where $A, B, C, a, b,$ and c are constants and n is a positive integer.
- In this form, the quadratic polynomial $ax^2 + bx + c$ can no longer be factored into a product of two linear functions with real coefficients.

Proper Rational Functions

- The rational function $f(x) = P(x)/Q(x)$ is said to be **proper** if the degree of the polynomial in the numerator, $P(x)$, is strictly less than the degree of the polynomial in the denominator, $Q(x)$,

$$f(x) = \frac{P(x)}{Q(x)} \quad \textbf{proper} \quad \Longleftrightarrow \quad \deg P(x) < \deg Q(x).$$

- Which of the following three rational functions

$$f_1(x) = \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x + 5} \qquad f_2(x) = \frac{x}{x+2} \qquad f_3(x) = \frac{2x-3}{x^2+x}$$

is proper? Only $f_3(x)$ is proper.

- The first step in the partial-fraction decomposition procedure is to use the long division algorithm to write $f(x)$ as a sum of a polynomial and a **proper** rational function.

Algebra Review

Dividing polynomials is much like the familiar process of dividing numbers. This process is the *long division algorithm for polynomials*.

Long Division Algorithm

If $A(x)$ and $B(x)$ are polynomials, with $B(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree strictly less than the degree of $B(x)$, such that

$$A(x) = Q(x) \cdot B(x) + R(x)$$

The polynomials $A(x)$ and $B(x)$ are called the **dividend** and **divisor**, respectively; $Q(x)$ is the **quotient** and $R(x)$ is the **remainder**.

Example 2

Divide the polynomial

$$A(x) = 2x^2 - x - 4 \quad \text{by} \quad B(x) = x - 3.$$

$$
\begin{array}{r|rrrrr}
 & 2x & & & & \\ \hline
x - 3 & 2x^2 & - & x & - & 4 \\
 & 2x^2 & - & 6x & & \\ \hline
 & & + & 5x & - & 4
\end{array}
$$

(Complete the above table and check your work!)

$$
\begin{array}{r|l}
 & 2x \quad + 5 \\
\hline
x - 3 & 2x^2 - x - 4 \\
 & 2x^2 - 6x \\
\hline
 & \quad 5x - 4 \\
 & \quad 5x - 15 \\
\hline
 & \qquad 11
\end{array}
$$

subtract

subtract

11 ← remainder

This means that

$$\boxed{2x^2 - x - 4 = (x-3)\cdot(2x+5) + 11}$$

check: $(2x^2 + 5x - 6x - 15) + 11$

$= 2x^2 - x - 4$ ✓

- **Synthetic division** is a quick method of dividing polynomials; it can be used when the divisor is of the form $x - c$, where c is a number. In synthetic division we write only the essential part of the long division table.
- In synthetic division we abbreviate the polynomial $A(x)$ by writing only its coefficients.
  Moreover, instead of $B(x) = x - c$, we simply write 'c.'
  Writing c instead of $-c$ allows us to add instead of subtract!

Example 2 (revisited): Divide

$$A(x) = 2x^2 - x - 4 \text{ by } B(x) = x - 3.$$

3	2	−1	−4
		6	15
	2	5	11

We obtain $Q(x) = 2x + 5$ and $R(x) = 11$. That is,

$$2x^2 - x - 4 = (2x + 5)(x - 3) + 11.$$

Example 3 (Online Homework # 3)

Use the Long Division Algorithm to write $f(x)$ as a sum of a polynomial and a proper rational function

$$f(x) = \frac{x^3}{x^2 + 4x + 3}.$$

$$x - 4$$

$$x^2 + 4x + 3 \,\overline{\big)\, x^3}$$

$$x^3 + 4x^2 + 3x$$ subtract

$$-4x^2 - 3x$$

$$-4x^2 - 16x - 12$$ subtract

$$13x + 12$$ ← remainder

This means that

$$\boxed{x^3 = (x-4)(x^2+4x+3) + 13x + 12}$$ 1m

Thus $$\frac{x^3}{x^2+4x+3} = \frac{(x-4)(x^2+4x+3) + 13x + 12}{x^2+4x+3}$$

Now split the fraction on the right hand side as the sum of 2 fractions:

$$\frac{x^3}{x^2+4x+3} = \frac{(x-4)(x^2+4x+3)}{x^2+4x+3} + \frac{13x+12}{x^2+4x+3}$$

simplify

$$= x-4 + \frac{13x+12}{x^2+4x+3}$$

this is a <u>proper</u> rational fraction

Partial Fraction Decomposition (linear factors)

Case of Distinct Linear Factors

$Q(x)$ is a product of m distinct linear factors. $Q(x)$ is thus of the form

$$Q(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_m)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the m distinct roots of $Q(x)$.
The rational function can then be written as

$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \cdots + \frac{A_m}{x-\alpha_m}\right]$$

We will see in the next examples how the constants $A_1, A_2, \ldots, A_m$ are determined.

Example 3 (cont.d)

Evaluate the indefinite integral: $$\int \frac{x^3}{x^2+4x+3}\,dx.$$

Note: from the calculations carried out in the first part of the example, we know that our problem reduces to

$$\int (x-4)\,dx + \int \frac{13x+12}{(x+3)(x+1)}\,dx.$$

We have shown before that

$$\frac{x^3}{x^2+4x+3} = x-4 + \frac{13x+12}{x^2+4x+3}$$

Thus :

$$\int \frac{x^3}{x^2+4x+3}\,dx = \underbrace{\int (x-4)\,dx}_{\text{easy}} + \underbrace{\int \frac{13x+12}{x^2+4x+3}\,dx}_{\text{let's look at this}}$$

easy

$$= \frac{1}{2}x^2 - 4x + C$$

let's look at this

<u>Notice</u> :

$$\frac{13x+12}{x^2+4x+3} = \frac{13x+12}{(x+1)(x+3)} = \frac{A}{x+3} + \frac{B}{x+1}$$

↑ <u>want</u> for some constants A and B

$$\frac{13x+12}{(x+1)(x+3)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1) + B(x+3)}{(x+3)(x+1)}$$

give common denominator

$$= \frac{Ax + A + Bx + 3B}{(x+3(x+1)} = \frac{(A+B)x + (A+3B)}{(x+1)(x+3)}$$

This mean that

$$13x + 12 = (A+B)x + (A+3B) \quad \underline{\underline{\text{so}}}$$

$$\begin{cases} A+B = 13 \\ A+3B = 12 \end{cases} \longleftrightarrow \begin{cases} A = 13 - B \\ A = 12 - 3B \end{cases}$$

so $\quad 13 - B = 12 - 3B \longleftrightarrow 2B = -1 \longrightarrow \boxed{B = -1/2}$

and $\quad A = 13 - B = 13 - (-1/2) = \frac{27}{2}$

This means that $\boxed{\dfrac{13x+12}{x^2+4x+3} = \dfrac{27}{2}\cdot\dfrac{1}{x+3} - \dfrac{1}{2}\cdot\dfrac{1}{x+1}}$

Thus

$$\int \frac{13x+12}{x^2+4x+3}\,dx = \frac{27}{2}\int \frac{1}{x+3}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{27}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + C$$

Thus:

$$\int \frac{x^3}{x^2+4x+3} = \underbrace{\frac{1}{2}x^2 - 4x}\ + \underbrace{\frac{27}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + C}$$

(Heaviside) cover-up method

We illustrate this method by using the previous example:

$$\frac{13x+12}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1)+B(x+3)}{(x+3)(x+1)}$$

$$\leftrightsquigarrow$$

$$\boxed{A(x+1)+B(x+3) = 13x+12} \qquad (*)$$

Set $x = -1$ in $(*)$. We obtain

$$A \cdot 0 + B \cdot (-1+3) = 13(-1)+12$$

$$\leftrightsquigarrow$$

$$B \cdot (2) = -1$$

$$\leftrightsquigarrow$$

$$B = -1/2$$

Set $x = -3$ in $(*)$. We obtain

$$A \cdot (-3+1) + 0 = 13(-3)+12$$

$$\leftrightsquigarrow$$

$$A \cdot (-2) = -27$$

$$\leftrightsquigarrow$$

$$A = 27/2$$

Example 4 (Online Homework # 8)

Find the integral: $\displaystyle\int_2^5 \frac{2}{x^2-1}\,dx.$

Consider the fraction $\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)}$.

We want to decompose it as:

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} = \text{give common denominator}$$

$$= \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}$$

Thus $2 = A(x+1) + B(x-1)$

evaluate at $\boxed{x=1}$: $2 = 2 \cdot A + B \cdot 0$ $\therefore \boxed{A=1}$

evaluate at $\boxed{x=-1}$: $2 = A \cdot (0) + B(-2)$ $\therefore \boxed{B=-1}$

Thus: $$\boxed{\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}}$$

Thus :
$$\int \frac{2}{x^2-1}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{1}{x+1}\,dx$$
$$= \ln|x-1| - \ln|x+1| + C$$
$$= \ln\left|\frac{x-1}{x+1}\right| + C$$

Finally :
$$\int_2^5 \frac{2}{x^2-1}\,dx = \ln\left|\frac{x-1}{x+1}\right|\Bigg]_2^5 =$$
$$= \ln\left(\tfrac{4}{6}\right) - \ln\left(\tfrac{1}{3}\right)$$
$$= \ln\left(\tfrac{4}{6}\cdot 3\right) = \underline{\underline{\ln(2)}} \cong \underline{0.6931}$$

Example 5 (Online Homework # 6)

Evaluate the indefinite integral: $\displaystyle\int \frac{1}{x(x+1)}\,dx.$

We need to decompose the fraction $\frac{1}{x(x+1)}$ as $\frac{A}{x} + \frac{B}{x+1}$. Thus:

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1) + B \cdot x}{x(x+1)}$$

Thus $1 = A(x+1) + Bx$

- evaluate at $\boxed{x=0}$

$1 = A \cdot 1 + B \cdot 0$

$\therefore \boxed{A=1}$

- evaluate at $\boxed{x=-1}$

$1 = A \cdot 0 + B(-1)$

$\therefore \boxed{B=-1}$

$(A+B)x + A = 1$

implies

$$\begin{cases} A+B = 0 \\ A = 1 \end{cases}$$

$\therefore \boxed{A=1}$ $B=-1$

No matter which method we choose, we obtain

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

Thus

$$\int \frac{1}{x(x+1)}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx$$

$$= \ln|x| - \ln|x+1| + C$$

$$= \ln\left|\frac{x}{x+1}\right| + C$$