MA 138 – Calculus 2 with Life Science Applications
Rational Functions and Partial Fractions
(Section 7.3)

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Example 1 (Online Homework # 7)

Evaluate the indefinite integral: $\displaystyle\int \frac{3}{(x+a)(x+b)}\,dx.$

We need to consider 2 cases:

$\boxed{a = b}$ the integral becomes $\displaystyle\int \frac{3}{(x+a)^2}\,dx$

so by setting $\boxed{u = x + a}$; $\dfrac{du}{dx} = 1$ so $\boxed{du = dx}$

Thus $\displaystyle = \int \frac{3}{u^2}\,du = \int 3u^{-2}\,du = -3u^{-1} + C$

$$= -\frac{3}{u} + C = \boxed{-\frac{3}{x+a} + C}$$

$\boxed{a \neq b}$ We need to write $\dfrac{3}{(x+a)(x+b)}$ as $\dfrac{A}{x+a} + \dfrac{B}{x+b}$

for some A, B constants.

$$\frac{3}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} = \frac{A(x+b) + B(x+a)}{(x+a)(x+b)}$$

Thus the numerators must be the same:

$$3 = A(x+b) + B(x+a)$$

$$= (A+B)\,x + (Ab + Ba)$$

$\therefore$ $A+B = 0$ and $Ab + Ba = 3$

$A = -B$ $\qquad$ $-Bb + Ba = 3$

$\therefore$ $\boxed{B = \dfrac{3}{a-b}}$ and $\boxed{A = -B = \dfrac{-3}{a-b}}$

Thus $$\frac{3}{(x-a)(x+b)} = \frac{-3/a-b}{x+a} + \frac{3/a-b}{x+b}$$

$$= \frac{3}{a-b}\left[\frac{1}{x+b} - \frac{1}{x+a}\right]$$

$$\therefore \int \frac{3}{(x+a)(x+b)}\,dx = \frac{3}{a-b}\left[\int \frac{1}{x+b}\,dx - \int \frac{1}{x+a}\,dx\right]$$

$$\therefore \int \frac{3}{(x+a)(x+b)}\,dx = \frac{3}{a-b}\Big[\ln|x+b| - \ln|x+a|\Big] + C$$

$$\boxed{= \frac{3}{a-b}\cdot \ln\left|\frac{x+b}{x+a}\right| + C}$$

$$\left[\; = \frac{3}{b-a}\cdot \ln\left|\frac{x+a}{x+b}\right| + C \;\right]$$

Example 2

Consider the rational function

$$f(x) = \frac{4x^2 - x - 1}{(x+1)^2(x-3)}$$

which has a repeated factor at the denominator.

Try to find constants A and B such that

$$\frac{4x^2 - x - 1}{(x+1)^2(x-3)} = \frac{A}{(x+1)^2} + \frac{B}{(x-3)}.$$

Let's try to find A and B such that

$$\frac{4x^2 - x - 1}{(x+1)^2(x-3)} = \frac{A}{(x+1)^2} + \frac{B}{x-3}$$

$$= \frac{A(x-3) + B(x+1)^2}{(x+1)^2(x-3)}$$

$$= \frac{Ax - 3A + Bx^2 + 2Bx + B}{(x+1)^2(x-3)}$$

$$= \frac{Bx^2 + (A+2B)x + (B-3A)}{(x+1)^2(x-3)}$$

Thus: $\underline{B=4}$, $\underline{A+2B=-1}$, $\boxed{\underline{B-3A=-1}}$

$A + 8 = -1$ $\therefore$ $\boxed{A=-9}$

But: $\underline{\underline{4-3(-9) = -1}}$ not

IMPOSSIBLE!!!

Example 2 (again)

The previous calculation didn't work.

Try now to find constants A, B and C such that

$$\frac{4x^2 - x - 1}{(x+1)^2(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x-3)}.$$

Then evaluate the definite integral

$$\int \frac{4x^2 - x - 1}{(x+1)^2(x-3)}\, dx.$$

$$\frac{4x^2-x-1}{(x+1)^2(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{x-3}$$

$$= \frac{A(x+1)(x-3) + B(x-3) + C(x+1)^2}{(x+1)^2(x-3)}$$

The denominators are the same :

$$4x^2-x-1 = A(x+1)(x-3) + B(x-3) + C(x+1)^2 \qquad (*)$$

If we evaluate $(*)$ for $x=-1$ we obtain

$$4(-1)^2-(-1)-1 = A \cdot 0 + B(-4) + C \cdot 0$$

$$\therefore \quad -4B = 4 \qquad \therefore \quad \boxed{B=-1}$$

If we evaluate $(*)$ at $x=3$ we obtain

$$4(3)^2-3-1 = A \cdot 0 + B \cdot 0 + 16C$$

$$\therefore \quad 16C = 32 \qquad \therefore \quad \boxed{C=2}$$

Thus the (Heaviside) cover-up method doesn't produce directly A ! Well substitute in (*) for example $x=0$ and get :

$$4 \cdot 0^2 - 0 - 1 = A \cdot (1)(-3) + (-1)(0-3) + (2)(0+1)^2$$

So: $-1 = -3A + 3 + 2 \longleftrightarrow 3A = 6 \therefore \boxed{A=2}$

Bottom line :

$$\int \frac{4x^2 - x - 1}{(x+1)^2(x-3)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{-1}{(x+1)^2}\,dx + \int \frac{2}{x-3}\,dx$$

$$= 2\ln|x+1| + \frac{1}{x+1} + 2\ln|x-3| + C$$

$$= \ln\left[(x+1)^2(x-3)^2\right] + \frac{1}{x+1} + C$$

using the properties of logarithms

Partial Fraction Decomposition

(repeated linear factors)

Case of Repeated Linear Factors

$Q(x)$ is a product of m distinct linear factors to various powers. $Q(x)$ is thus of the form

$$Q(x) = a(x - \alpha_1)^{n_1}(x - \alpha_2)^{n_2} \cdots (x - \alpha_m)^{n_m}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the m distinct roots of $Q(x)$ and $n_1, n_2, \ldots, n_m$ are positive integers such that $n_1 + n_2 + \cdots + n_m = \deg Q(x)$.
The rational function can then be written as

$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\sum_{i=1}^{m} \frac{A_{i,1}}{x - \alpha_i} + \frac{A_{i,2}}{(x - \alpha_i)^2} + \cdots + \frac{A_{i,n_i}}{(x - \alpha_i)^{n_i}}\right].$$

Example 3 (Online Homework # 5)

Evaluate the integral

$$\int \frac{-10x^2}{(x+1)^3}\,dx.$$

We need to find A, B, and C such that

$$\frac{-10x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

$$= \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3}$$

Thus: $\quad -10x^2 = A(x+1)^2 + B(x+1) + C \qquad (*)$

Evaluate $(*)$ at $x=-1$. We get "C" !

$$-10(-1)^2 = A \cdot 0 + B \cdot 0 + C \qquad \therefore \boxed{C=-10}$$

If we plug-in in $(*)$ $x=0$ and $x=1$ for example we get:

$x=0$: $\quad 0 = A(1)^2 + B(1) - 10$

$x=1$: $\quad -10 = A(2)^2 + B(2) - 10$

So: $\begin{cases} A+B=10 \\ 4A+2B=0 \end{cases} \rightsquigarrow \begin{cases} B=-2A \\ A+(-2A)=10 \end{cases}$

$\therefore$ $\boxed{A=-10}$ and $\boxed{B=20}$

Thus: $$\frac{-10x^2}{(x+1)^3} = \frac{-10}{x+1} + \frac{20}{(x+1)^2} + \frac{-10}{(x+1)^3}$$

and $$\int \frac{-10x^2}{(x+1)^3}\,dx = -10\int \frac{1}{x+1}\,dx + 20\int \frac{1}{(x+1)^2} - 10\int \frac{1}{(x+1)^3}\,dx$$

$$= -10\ln|x+1| + 20\left[-1\cdot(x+1)^{-1}\right] - 10\left[-\frac{1}{2}(x+1)^{-2}\right] + C$$

$$\boxed{= -10\ln|x+1| + \frac{-20}{x+1} + \frac{5}{(x+1)^2} + C}$$

$$\left[= -10\ln|x+1| + \frac{-20x-15}{(x+1)^2} + C \right]$$

$$= \int \frac{-10x^2}{(x+1)^3}\,dx$$ you can also $u = x+1$ $du = dx$

$$\Longrightarrow \quad x = u-1$$

$$= \int \frac{-10(u-1)^2}{u^3}\,du = \int \frac{-10(u^2-2u+1)}{u^3}\,du$$

$$= \int \left[-10\frac{u^2}{u^3} + 20\frac{u}{u^3} - 10\frac{1}{u^3}\right] du =$$

$$= \int \left[-10\frac{1}{u} + 20\frac{1}{u^2} - 10\frac{1}{u^3}\right] du =$$

$$= -10\ln|u| - 20\frac{1}{u} + 5\frac{1}{u^2} + C$$

$$= \boxed{-10\ln|x+1| - 20\frac{1}{x+1} + 5\frac{1}{(x+1)^2} + C}$$

Example 4

Evaluate the integral

$$\int \frac{1}{x^2(x-1)^2}\,dx.$$

We want to write $\dfrac{1}{x^2(x-1)^2}$ as the following sum of fractions

$$\frac{1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

for some constants A, B, C, and D. Give common denom-

$$= \frac{Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2}{x^2(x-1)^2}$$

$$= \frac{A(x^3-2x^2+x) + B(x^2-2x+1) + C(x^3-x^2) + Dx^2}{x^2(x-1)^2}$$

$$= \frac{x^3[A+C] + x^2[-2A+B-C+D] + x[A-2B] + B}{x^2(x-1)^2}$$

Equate the coefficients of the denominators:

$A+C=0$; $-2A+B-C+D=0$; $A-2B=0$; $B=1$

Thus $B = 1$; $A = 2B = 2$; $C = -A = -2$.

Finally from $-2A + B - C + D = 0$ we get

$-4 + 1 - (-2) + D = 0$ so $D = 1$

Hence :

$$\frac{1}{x^2(x-1)^2} = \frac{2}{x} + \frac{1}{x^2} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}$$ using this, we get

$$\int \frac{1}{x^2(x-1)^2}\,dx = \boxed{2\ln|x| - \frac{1}{x} - 2\ln|x-1| - \frac{1}{(x-1)} + C}$$

$$\left[= \ln\left[\frac{x^2}{(x-1)^2}\right] + \frac{1-2x}{x(x-1)} + C \right]$$

after simplifying and using the properties of logarithms

Example 5 (Online Homework #11)

If $f(x)$ is a quadratic function such that $f(0) = 1$ and $\displaystyle\int \frac{f(x)}{x^2(x+1)^3}\,dx$ is a rational function, find the value of $f'(0)$.

$f(x)$ is a quadratic function; so $f(x) = ax^2 + bx + c$.

We know that $f(0) = 1$ so $f(0) = a \cdot 0^2 + b \cdot 0 + c = \boxed{c = 1}$

Thus $f(x) = ax^2 + bx + 1$. We are asked to compute $f'(0)$. Now $f'(x) = 2ax + b$. Hence

we are asked to compute $f'(0) = 2a \cdot 0 + b = \boxed{b}$

We are asking that

$$\int \frac{f(x)}{x^2(x+1)^3}\,dx = \int \frac{ax^2+bx+1}{x^2(x+1)^3}\,dx$$ is a rational function.

Think about the partial fraction decomposition

$$\frac{ax^2+bx+1}{x^2(x+1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

If A and C were to be $\neq 0$ those 2 fractions would give rise to logarithmic terms. Thus we need a partial fraction decomposition of the form:

$$\frac{ax^2+bx+1}{x^2(x+1)^3} = \frac{B}{x^2} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

$$= \frac{B(x+1)^3 + Dx^2(x+1) + Ex^2}{x^2(x+1)^3}$$

$$= \frac{B(x^3+3x^2+3x+1) + D(x^3+x^2) + Ex^2}{x^2(x+1)^3}$$

$$= \frac{x^3[B+D] + x^2[3B+D+E] + 3Bx + B}{x^2(x+1)^3}$$

Thus the coefficients of the numerators must be the same:

$B + D = 0 \qquad 3B + D + E = a \qquad 3B = b \qquad B = 1$

Thus $B = 1$ so $\boxed{b = 3B = 3}$

Note that $D = -B = -1$ so that

$$E = a - 3B - D = a - 3 - (-1) = a - 2$$

Hence

$$\frac{ax^2 + 3x + 1}{x^2(x+1)^3} = \frac{1}{x^2} - \frac{1}{(x+1)^2} + \frac{a-2}{(x+1)^3}$$

Partial Fraction Decomposition
(irreducible quadratic factors)

Irreducible quadratic factors in the denominator of a proper rational functions are dealt with in the partial-fraction decomposition as follows:

Case of Irreducible Quadratic Factors

If the irreducible quadratic factor $ax^2 + bx + c$ is contained n times in the factorization of the denominator of a proper rational function, then the partial-fraction decomposition contains terms of the form

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}.$$

Example 6 (Example 6, Section 7.3, page 348)

Write the partial faction decomposition of

$$f(x) = \frac{2x^3 - x^2 + 2x - 2}{(x^2+1)(x^2+2)}.$$

$$\frac{2x^3 - x^2 + 2x - 2}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

for some $A, B, C,$ and D.

$$= \frac{(Ax+B)(x^2+2) + (Cx+D)(x^2+1)}{(x^2+1)(x^2+2)}$$

$$= \frac{Ax^3 + 2Ax + Bx^2 + 2B + Cx^3 + Cx + Dx^2 + D}{(x^2+1)(x^2+2)}$$

$$= \frac{x^3[A+C] + x^2[B+D] + x[2A+C] + 2B+D}{(x^2+1)(x^2+2)}$$

So that, if we equate the coefficients of the numerators

$\boxed{A+C=2}$ $\boxed{B+D=-1}$ $\boxed{2A+C=2}$ $\boxed{2B+D=-2}$

So we need to solve

$$\begin{cases} A+C=2 \\ 2A+C=2 \end{cases} \qquad \begin{cases} B+D=-1 \\ 2B+D=-2 \end{cases}$$

These 2 systems give

$A=0 \quad C=2 \qquad B=-1 \quad D=0$

Hence

$$\boxed{\frac{2x^3-x^2+2x-2}{(x^2+1)(x^2+2)} = \frac{-1}{x^2+1} + \frac{2x}{x^2+2}}$$

$$\left[\int \frac{2x^3-x^2+2x-2}{(x^2+1)(x^2+2)} = \ln(x^2+2) - \tan^{-1}x + C\right]$$