MA 138 – Calculus 2 with Life Science Applications
Improper Integrals
(Section 7.4)

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Improper Integrals

We discuss definite integrals of two types with the following characteristics:

(1) **one or both limits of integration are infinite**; that is, the integration interval is unbounded. For example

$$\int_1^\infty e^{-x}\,dx \qquad \text{or} \qquad \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx;$$

(These integrals are very important in Probability and Statistics!)

(2) **the integrand becomes infinite at one or more points of the interval of integration**. For example

$$\int_{-1}^1 \frac{1}{x^2}\,dx \qquad \text{or} \qquad \int_0^1 \frac{1}{2\sqrt{x}}\,dx.$$

We call such integrals **improper integrals**.

Type 1: Unbounded Intervals

Let $f(x)$ be continuous on the interval $[a, \infty)$. If

$$\lim_{b \longrightarrow \infty} \int_a^b f(x)\, dx$$

exists and has a finite value, we say that the improper integral

$$\int_a^\infty f(x)\, dx$$

converges and define

$$\int_a^\infty f(x)\, dx := \lim_{b \longrightarrow \infty} \int_a^b f(x)\, dx$$

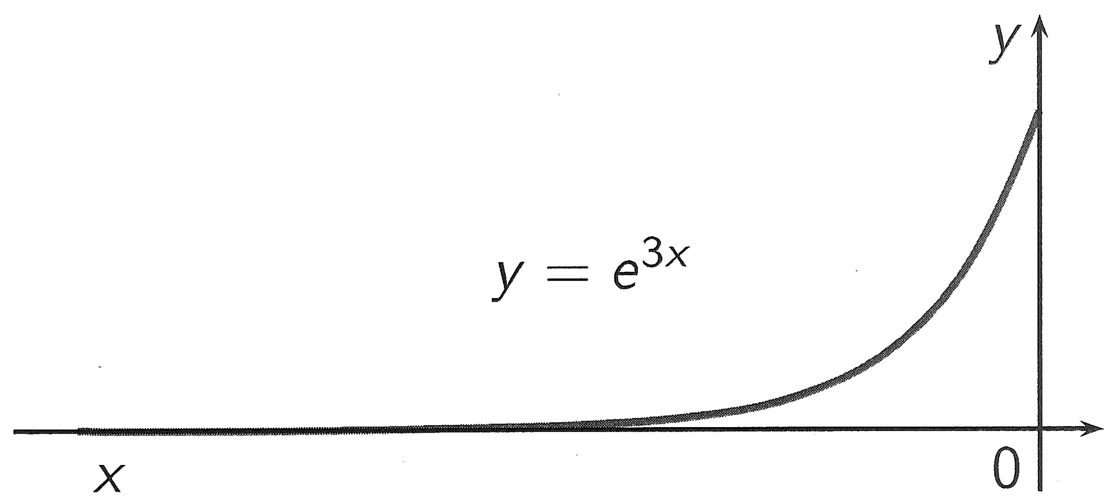
Otherwise, we say that the improper integral **diverges**.

Analogous definitions apply when the lower limit of integration is infinite.

Example 1 (Online Homework # 1)

Determine if the improper integral converges and, if so, evaluate it.

$$\int_{-\infty}^{0} e^{3x}\,dx.$$



(1) $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx = \lim_{a\to-\infty}\int_a^0 e^{3x}\,dx =$

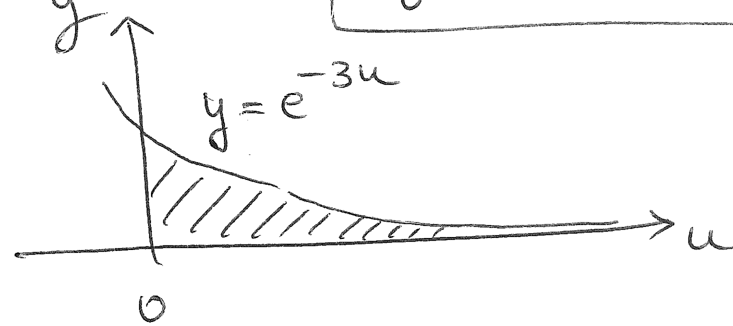
$$= \lim_{a\to-\infty}\left[\frac{1}{3}e^{3x}\right]_a^0 = \lim_{a\to-\infty}\left[\frac{1}{3}-\frac{1}{3}e^{3a}\right] =$$

$$= \tfrac{1}{3} - \tfrac{1}{3}\lim_{a\to-\infty} e^{3a} = \tfrac{1}{3} - \tfrac{1}{3}\cdot 0 = \boxed{\tfrac{1}{3}}$$

(2) Notice that by using the substitution

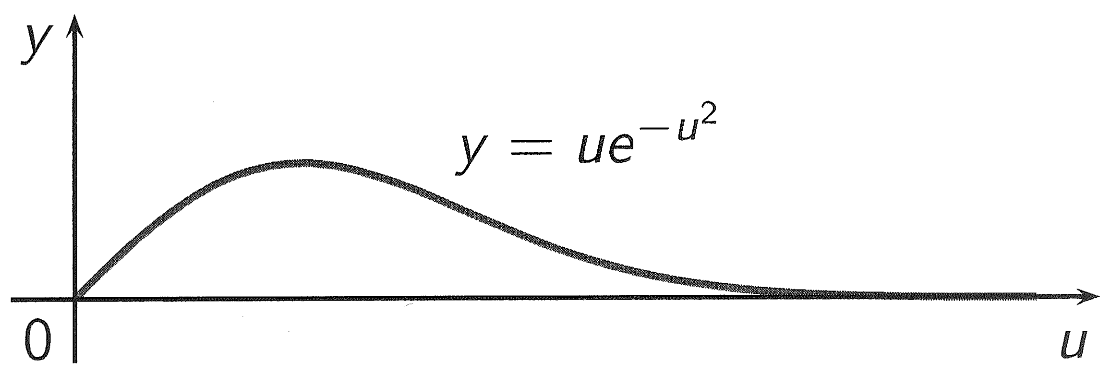
$\underline{u=-x}$; $\underline{du=-dx}$ we obtain

$$\int_{-\infty}^{0} e^{3x}\,dx = \int_{+\infty}^{0} e^{-3u}(-du) = \boxed{\int_0^{\infty} e^{-3u}\,du}$$



$\boxed{\textbf{Example 2}}$ (Problem # 8(a), Exam 1, Spring 2014)

Compute $\displaystyle\int_1^\infty ue^{-u^2}\,du.$



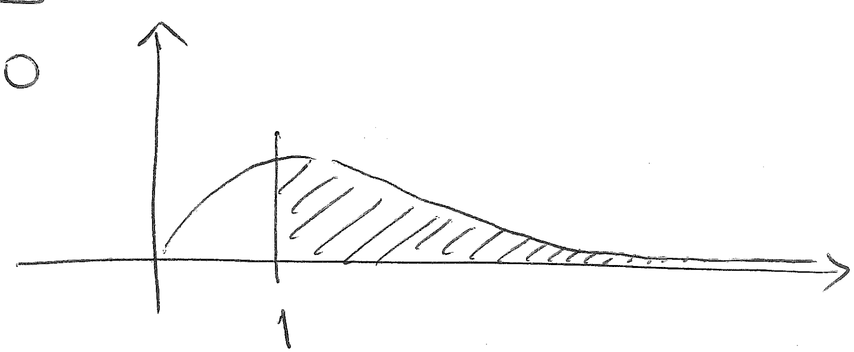
$$\int_1^{\infty} u e^{-u^2}\, du \quad ; \text{ substitute } \boxed{x = u^2}, \quad dx = 2u\, du$$

$$\boxed{\tfrac{1}{2}\, dx = u\, du}$$

$$= \int_1^{\infty} \frac{1}{2} e^{-x}\, dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{2} e^{-x}\, dx$$

$$= \lim_{b \to \infty} \left[-\frac{1}{2} e^{-x} \right]_1^{b} = \lim_{b \to \infty} \left[-\tfrac{1}{2} e^{-b} + \frac{1}{2} e^{-1} \right]$$

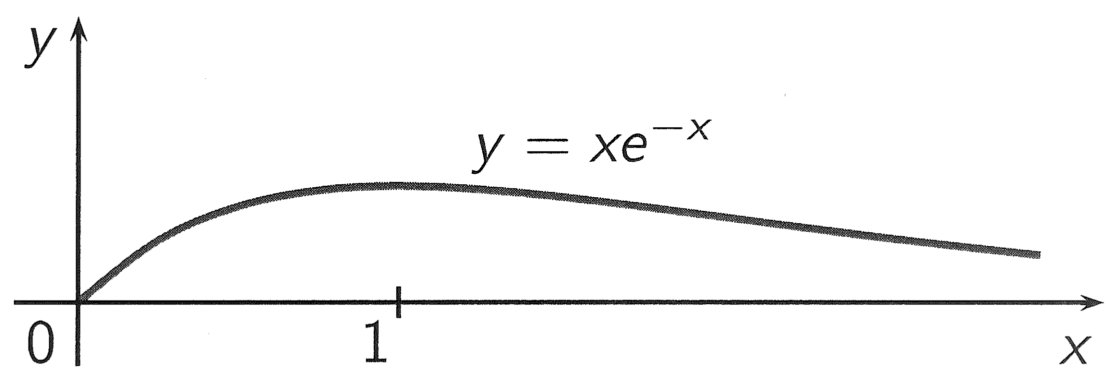
$$= \frac{1}{2e} - \tfrac{1}{2} \underbrace{\lim_{b \to +\infty} e^{-b}}_{\to 0} = \frac{1}{2e}$$



Example 3 (Problem #2, Section 7.4, page 362)

Determine if the improper integral converges and, if so, evaluate it.

$$\int_0^\infty x\, e^{-x}\, dx.$$



First, let's find the antiderivative of $x e^{-x}$!

$$\int x e^{-x} \, dx \underset{\text{by parts}}{=} x(-e^{-x}) - \int 1 \cdot (-e^{-x}) \, dx$$

$$= -x e^{-x} + \int e^{-x} \, dx = -x e^{-x} + (-e^{-x}) + C$$

$$= \boxed{-e^{-x}(x+1) + C}$$

Thus:

$$\int_0^{\infty} x e^{-x} \, dx = \lim_{b \to \infty} \int_0^b x e^{-x} \, dx =$$

$$= \lim_{b \to \infty} \left[-(x+1) e^{-x} \right]_0^b = \lim_{b \to \infty} \left\{ -(b+1) e^{-b} + 1 \right\}$$

[note that $-(0+1) e^{-0} = -1$]

Thus $\displaystyle\int_0^\infty x e^{-x} dx = 1 - \lim_{b\to\infty} (b+1)e^{-b}$

$$= 1 - \underbrace{\lim_{b\to\infty} \frac{b+1}{e^b}}_{=0}$$

$$= 1$$

Note: $\displaystyle\lim_{b\to\infty} \frac{b+1}{e^b} = \frac{\infty}{\infty}$ by L'Hôpital's rule we get

$$= \lim_{b\to\infty} \frac{1}{e^b} = \frac{1}{\infty} = 0$$

Example 4 (Problem #33, Section 7.4, page 362)

Let p be a positive real number. Show that

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1} & \text{for } p > 1 \\ \infty & \text{for } 0 < p \le 1. \end{cases}$$

E.g.: $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ and $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ both diverge (as $p = 1, 1/2$, respectively).

E.g.: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = 1$ and $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx = \frac{1}{2}$ (as $p = 2, 3$, respectively).

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx$$

$\boxed{\text{for } p \neq 1}$

$$= \lim_{b\to\infty} \frac{1}{-p+1}\, x^{-p+1}\Big]_1^b =$$

$$= \lim_{b\to\infty} \left[\frac{1}{1-p} \cdot b^{1-p} \underbrace{- \left(\frac{1}{1-p}\right)\cdot (1)^{1-p}}_{\frac{1}{p-1}} \right]$$

$$= \frac{1}{p-1} + \boxed{\frac{1}{1-p} \lim_{b\to\infty} b^{1-p}}$$

$\boxed{\infty \text{ if } 0<p<1}$ diverges

$\boxed{0 \text{ if } p>1}$ converges

for $\boxed{p=1}$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \Big[\ln|x| \Big]_1^b$$

$$= \lim_{b\to\infty} \Big(\ln b - \underbrace{\ln(1)}_{=0} \Big)$$

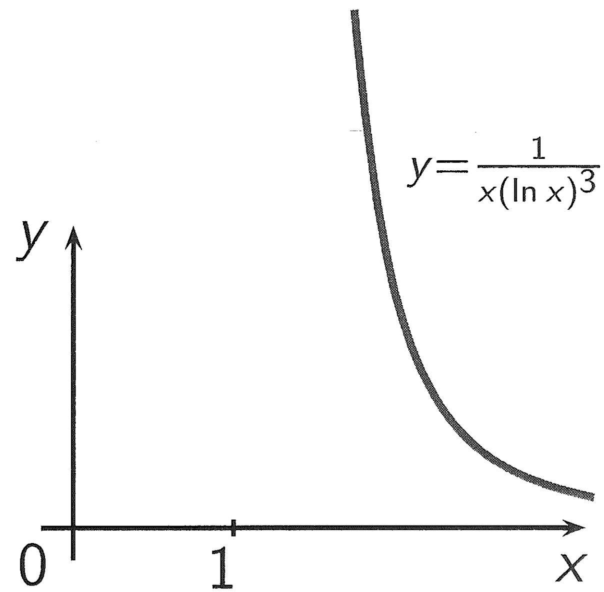
$$= \lim_{b\to\infty} \ln b = \underline{\underline{+\infty}}$$

diverges

Example 5 (Problem #7(a), Exam 1, Spring 2014)

Compute the following improper integral

$$\int_e^{\infty} \frac{1}{x(\ln x)^3}\, dx.$$



$$\int_e^\infty \frac{1}{x[\ln(x)]^3}\,dx$$

use the substitution $\boxed{u = \ln x}$

$\dfrac{du}{dx} = \dfrac{1}{x}$ so $du = \dfrac{1}{x}\,dx$

$$= \int_1^\infty \frac{1}{u^3}\,du = \lim_{b\to\infty} \int_1^b \frac{1}{u^3}\,du = \lim_{b\to\infty} \int_1^b u^{-3}\,du$$

$$= \lim_{b\to\infty} \left[\frac{1}{-3+1} u^{-3+1}\right]_1^b = \lim_{b\to\infty} \left[-\frac{1}{2} u^{-2}\right]_1^b =$$

$$= \lim_{b\to\infty} \left[-\frac{1}{2u^2}\right]_1^b = \lim_{b\to\infty} \left\{-\frac{1}{2b^2} - \left(-\frac{1}{2(1)^2}\right)\right\}$$

$$= \frac{1}{2} - \frac{1}{2} \cdot \underbrace{\lim_{b\to\infty} \frac{1}{b^2}}_{=0} = \boxed{1/2}$$

Remarks

- Assume that $f(x)$ is continuous on $(-\infty, \infty)$. Then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

where a is a real number. If **both** improper integrals on the right-hand side of the above equation are convergent, then the value of the improper integral on the left-hand side of the above equation is the sum of the two limiting values on the right-hand side.

- It is important to realize that the definition of $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ is **different** from that of

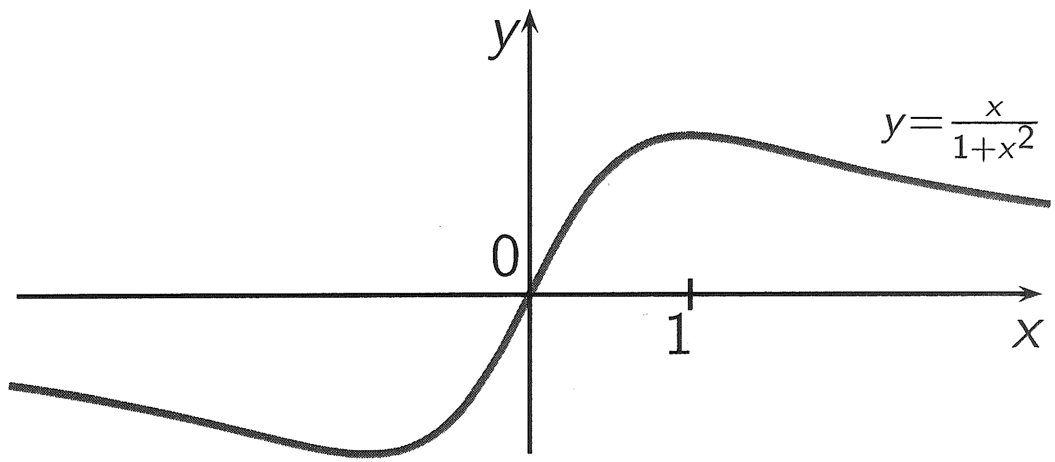
$$\lim_{b \longrightarrow \infty} \int_{-b}^{b} f(x)\,dx.$$

Example 6 (Example #5, Section 7.4, page 356)

Show that

$$\int_{-\infty}^{\infty} x\,dx \qquad \int_{-\infty}^{\infty} x^3\,dx \qquad \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$$

are all divergent integrals.



(1) Observe that $\displaystyle\int_{-\infty}^{+\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$

but both integrals diverge.

$$\int_{0}^{\infty} x\,dx = \lim_{b\to\infty} \int_{0}^{b} x\,dx = \lim_{b\to\infty} \left[\frac{1}{2}x^2\right]_0^b =$$

$$= \lim_{b\to\infty} \left[\tfrac{1}{2}b^2 - 0\right] = \underline{+\infty}$$

Similarly $\displaystyle\int_{-\infty}^{0} x\,dx = \lim_{a\to-\infty} \int_{a}^{0} x\,dx = \lim_{a\to-\infty}\left[-\tfrac{1}{2}a^2\right]$

$$= \underline{-\infty}$$

(2) As in (1) $\displaystyle\int_{-\infty}^{+\infty} x^3\,dx = \int_{-\infty}^{0} x^3\,dx + \int_{0}^{+\infty} x^3\,dx$

and both integrals diverge

$$\int_0^\infty x^3 dx = \lim_{b\to\infty} \int_0^b x^3 dx = \lim_{b\to\infty} \left[\frac{1}{4}x^4\right]_0^b = \lim_{b\to\infty}\left\{\frac{1}{4}b^4 - 0\right\}$$

$$= \lim_{b\to\infty} \frac{1}{4} b^4 = \underline{\underline{+\infty}}$$

$$\int_{-\infty}^0 x^3 dx = \lim_{a\to-\infty} \int_a^0 x^3 dx = \lim_{a\to-\infty}\left(-\frac{1}{4}a^4\right) = \underline{\underline{-\infty}}$$

(3) $$\int_{-\infty}^{+\infty} \frac{x}{1+x^2} dx = \int_{-\infty}^0 \frac{x}{1+x^2} dx + \int_0^{+\infty} \frac{x}{1+x^2} dx$$

Observe that $$\int_0^{+\infty} \frac{x}{1+x^2} dx = \int_1^{+\infty} \frac{1}{2}\cdot\frac{1}{u} du =$$

(use $u = 1+x^2$)

$$= \lim_{b\to\infty} \int_1^b \frac{1}{2}\cdot\frac{1}{u} du = \lim_{b\to\infty} \frac{1}{2} \ln|u| \Big]_1^b =$$

$$= \lim_{b\to\infty} \frac{1}{2}\ln|b| - \underbrace{\frac{1}{2}\ln|1|}_{=0} = \lim_{b\to\infty} \frac{1}{2}\ln b = \underline{\underline{+\infty}}$$

Similarly $\displaystyle\int_{-\infty}^{0} \frac{x}{1+x^2}\,dx = \underline{\underline{-\infty}}$.

Hence all the 3 integrals <u>diverge</u>!

They <u>are not</u> <u>equal to zero</u> !!!

Example 7 (Drug pharmacokinetics)

The plasma drug concentration of a new drug was modeled by the function

$$C(t) = 23\, t\, e^{-2\,t}$$

where t is measured in hours and C in μg/mL.

- What is the maximum drug concentration and when did it occur?
- Calculate $\int_0^\infty C(t)\, dt$ and explain its significance.

Remark: One could ask what the area under the curve represent in this biological context. For a drug to work, it needs to be "available" to interact with the target tissue (or target pathogen if the drug is an antibiotic). Availability can be increased by increasing the concentration or by increasing the time the drug lingers before it is cleared through metabolism. The area under the graph is a combined measure of these, and therefore is a composite measure of the overall "availability."

$$C(t) = 23 \cdot t \cdot e^{-2t}$$

(1) the maximum drug concentration occurs when $C'(t) = 0$ and sign C': $\underset{\nearrow}{+++}\ \underset{\searrow}{---}$

Now, $C'(t) = 23e^{-2t} + 23t \cdot (-2)e^{-2t}$

$$= 23e^{-2t}(1-2t) = 0$$

$$\iff 1 - 2t = 0 \iff t = 1/2$$

$$\boxed{C(1/2) = 4.23 \ \mu g/mL}$$

sign of $C'(t) = 23e^{-2t}(1-2t)$

$23e^{-2t}$:	$++++$	$++++$
$(1-2t)$:	$++++$	$----$
	$1/2$	
$23e^{-2t}(1-2t)$:	$+++$	$----$
	increas. $\nearrow$ $1/2$	$\searrow$ decreas.

(2) Let's first find the antiderivative of $C(t)$:

$$\int 23t\, e^{-2t}\, dt = 23t \cdot \left(-\tfrac{1}{2} e^{-2t}\right) - \int 23 \left(-\tfrac{1}{2} e^{-2t}\right) dt$$

$$= -\frac{23}{2} t\, e^{-2t} + \frac{23}{2} \int e^{-2t}\, dt$$

$$= -\frac{23}{2} t\, e^{-2t} + \frac{23}{2} \left(-\frac{1}{2} e^{-2t}\right) + C$$

$$\boxed{= -\frac{23}{2} e^{-2t} \left(t + \frac{1}{2}\right) + C}$$

Thus:

$$\int_0^\infty 23t\, e^{-2t}\, dt = \lim_{b \to \infty} \int_0^b 23t\, e^{-2t}\, dt$$

$$= \lim_{b \to \infty} \left[-\frac{23}{2} e^{-2t} \left(t + \frac{1}{2}\right) \right]_0^b$$

$$\therefore \int_0^\infty 23t\, e^{-2t}\, dt = \lim_{b \to \infty} \left[-\frac{23}{2} e^{-2b}\left(b+\tfrac{1}{2}\right) + \frac{23}{2} e^{0}\left(\tfrac{1}{2}\right) \right]$$

$$= \frac{23}{4} - \frac{23}{2} \underbrace{\lim_{b \to \infty} \frac{b+\frac{1}{2}}{e^{2b}}}_{=0 \text{ by l'Hôpital's rule}}$$

(as in Example 3

Thus $$\boxed{\int_0^\infty 23t\, e^{-2t}\, dt = \frac{23}{4} = 5.75 \ \mu g/mL \times \text{hours}}$$

This is the long term "availability" of a single drug use.