

MA 138 – Calculus 2 with Life Science Applications  
**Improper Integrals**  
(Section 7.4)

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# Improper Integrals

We discuss definite integrals of two types with the following characteristics:

- (1) **one or both limits of integration are infinite**; that is, the integration interval is unbounded. For example

$$\int_1^{\infty} e^{-x} dx \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx;$$

(These integrals are very important in Probability and Statistics!)

- (2) **the integrand becomes infinite at one or more points of the interval of integration**. For example

$$\int_{-1}^1 \frac{1}{x^2} dx \quad \text{or} \quad \int_0^1 \frac{1}{2\sqrt{x}} dx.$$

We call such integrals **improper integrals**.

## Type 2: Unbounded Integrand

What if the integrand becomes infinite at one or both endpoints of the interval of integration?

- If  $f$  is continuous on  $(a, b]$  and  $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ , we define

$$\int_a^b f(x) dx := \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$

provided that this limit exists.

- If  $f$  is continuous on  $[a, b)$  and  $\lim_{x \rightarrow b^-} f(x) = \pm\infty$ , we define

$$\int_a^b f(x) dx := \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$

provided that this limit exists.

If the limit does not exist, we say that the integral diverges.

**Example 1** (Problem #12, Section 7.4, page 362)

Determine whether the improper integral

$$\int_1^e \frac{1}{x\sqrt{\ln x}} dx$$

is convergent. If the integral is convergent, compute its value.

$$= \int_1^e \frac{1}{x \sqrt{\ln x}} dx$$

use the substitution

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$= \int_0^1 \frac{1}{\sqrt{u}} du$$

notice that the integrand is not defined when  $u=0$  !!

$$= \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{\sqrt{u}} du = \lim_{a \rightarrow 0^+} \left[ 2\sqrt{u} \right]_a^1 =$$

$$= \lim_{a \rightarrow 0^+} \left[ 2\sqrt{1} - 2\sqrt{a} \right] = 2 - 2 \cdot \underbrace{\lim_{a \rightarrow 0^+} \sqrt{a}}_{=0}$$

$$= \boxed{2}$$

**Example 2** (Problem #26, Section 7.4, page 362)

Determine whether the improper integral

$$\int_1^e \frac{1}{x \ln x} dx$$

is convergent. If the integral is convergent, compute its value.

$$= \int_1^e \frac{1}{x \cdot \ln x} dx$$

use the substitution  $u = \ln x$

$$du = \frac{1}{x} dx \quad \text{Thus}$$

$$= \int_0^1 \frac{1}{u} du$$

notice that the integrand  $\frac{1}{u}$  is not defined when  $u=0$ .

$$= \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{u} du = \lim_{a \rightarrow 0^+} \left[ \ln|u| \right]_a^1 =$$

$$= \lim_{a \rightarrow 0^+} \left[ \underbrace{\ln(1)}_0 - \ln(a) \right] =$$

$$= - \lim_{a \rightarrow 0^+} \ln(a) = -(-\infty) = \underline{\underline{+\infty}}$$

diverges



### Example 3 (Online Homework #7)

Determine whether the improper integral

$$\int_0^9 \frac{4}{(x-6)^2} dx$$

is convergent. If the integral is convergent, compute its value.

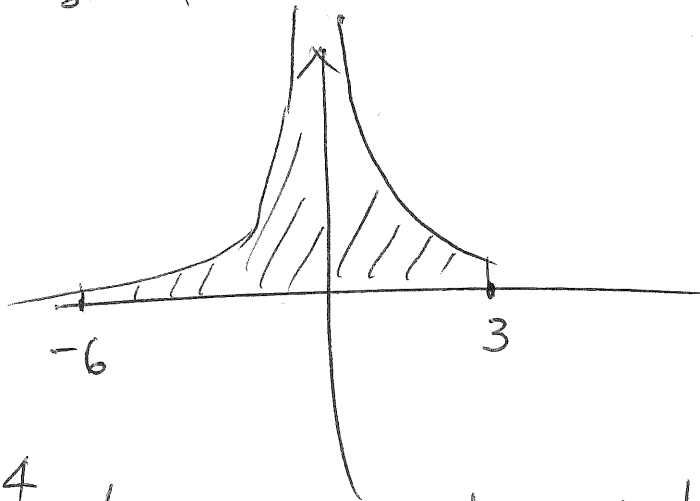


$$= \int_0^9 \frac{4}{(x-6)^2} dx$$

$$= \int_{-6}^3 \frac{4}{u^2} du$$

$$= \int_{-6}^0 \frac{4}{u^2} du + \int_0^3 \frac{4}{u^2} du$$

use the substitution  $u = x - 6$   
so that  $du = dx$



the integrand  $\frac{1}{u^2}$   
is not defined at u=0

Note that both integrals are "to  $\infty$ ". So the whole integral does not exist (or diverges).

(e.g.)  $\int_0^3 \frac{4}{u^2} du = \lim_{a \rightarrow 0^+} \int_a^3 \frac{4}{u^2} du = \lim_{a \rightarrow 0^+} \left[ -\frac{4}{u} \right]_a^3 =$

$$= \lim_{a \rightarrow 0^+} \left( -\frac{4}{3} - \left( -\frac{4}{a} \right) \right)$$

$$= \lim_{a \rightarrow 0^+} \left[ \frac{4}{a} - \frac{4}{3} \right] = +\infty - \frac{4}{3} = \underline{\underline{+\infty}}$$

Similarly:

$$\int_{-6}^0 \frac{4}{u^2} du = \lim_{b \rightarrow 0^-} \int_{-6}^b \frac{4}{u^2} du = \lim_{b \rightarrow 0^-} \left[ -\frac{4}{u} \right]_{-6}^b$$

$$= \lim_{b \rightarrow 0^-} \left[ -\frac{4}{b} - \left( -\frac{4}{(-6)} \right) \right] = \lim_{b \rightarrow 0^-} \left[ -\frac{4}{b} - \frac{2}{3} \right]$$

$$= - \left[ \lim_{b \rightarrow 0^-} \frac{4}{b} \right] - \frac{2}{3} = -(-\infty) - \frac{2}{3} = \underline{\underline{+\infty}}$$

## Example 4 (Problem #34, Section 7.4, page 363)

Let  $p$  be a positive real number. Show that

$$\int_0^1 \frac{1}{x^p} dx = \begin{cases} \frac{1}{1-p} & \text{for } 0 < p < 1 \\ \infty & \text{for } p \geq 1. \end{cases}$$

**E.g.:**  $\int_0^1 \frac{1}{x} dx$  and  $\int_0^1 \frac{1}{x^2} dx$  both diverge (as  $p = 1, 2$ , respectively).

**E.g.:**  $\int_0^1 \frac{1}{\sqrt{x}} dx = 2$  and  $\int_0^1 \frac{1}{\sqrt[3]{x}} dx = \frac{3}{2}$  (as  $p = 1/2, 1/3$ , respectively).

$$\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \int_a^1 x^{-p} dx$$

for  $p \neq 1$

$$= \lim_{a \rightarrow 0^+} \left[ \frac{1}{1-p} x^{1-p} \right]_a^1 = \lim_{a \rightarrow 0^+} \left[ \frac{1}{1-p} - \frac{1}{1-p} a^{1-p} \right]$$

$$= \frac{1}{1-p} - \frac{1}{1-p} \lim_{a \rightarrow 0^+} \left[ a^{1-p} \right]$$

diverges

$\infty$   
for  $p > 1$

converges

0 for  
 $0 < p < 1$

for  $p = 1$

$$\lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x} dx = \lim_{a \rightarrow 0^+} \left[ \ln|x| \right]_a^1 = \lim_{a \rightarrow 0^+} \left[ 0 - \ln(a) \right]$$

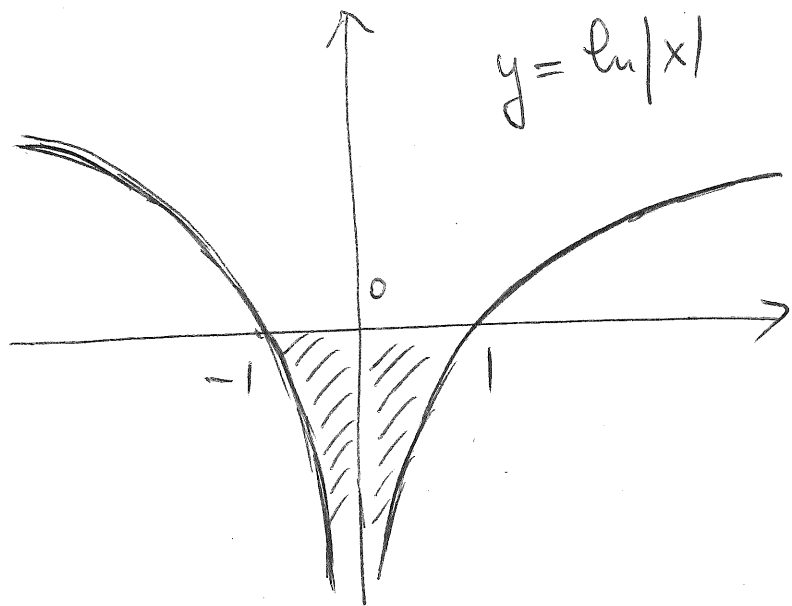
$$= 0 - (-\infty) = \underline{\underline{+\infty}} \quad \text{diverges}$$

**Example 5** (Problem #15, Section 7.4, page 362)

Determine whether the improper integral

$$\int_{-1}^1 \ln |x| dx.$$

is convergent. If the integral is convergent, compute its value.



Notice that

$$\int_{-1}^1 \ln|x| dx = \int_{-1}^0 \ln|x| dx + \int_0^1 \ln|x| dx$$


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If both integrals exist, by symmetry, then

$$\int_{-1}^1 \ln|x| dx = 2 \int_0^1 \ln x dx$$

$$= \boxed{2 \lim_{a \rightarrow 0^+} \int_a^1 \ln x dx}$$

Recall that (integration by parts):

$$\begin{aligned} \int \ln x dx &= x \cdot \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx \\ &= x \ln x - x + C = \underline{x(\ln x - 1) + C} \end{aligned}$$

Thus :

$$\int_{-1}^1 \ln|x| dx = 2 \lim_{a \rightarrow 0^+} \int_a^1 \ln x dx$$

$$= 2 \lim_{a \rightarrow 0^+} \left[ x(\ln x - 1) \right]_a^1 =$$

$$= 2 \lim_{a \rightarrow 0^+} \left\{ [1 \cdot (\ln(1) - 1)] - [a(\ln a - 1)] \right\}$$

$$= 2 \lim_{a \rightarrow 0^+} \left\{ -1 - \underbrace{a \cdot \ln(a)}_{\rightarrow 0} + \underbrace{a}_{\rightarrow 0} \right\}$$

$$= \boxed{-2} //$$

Note :  $\lim_{a \rightarrow 0^+} a \cdot \ln(a) = 0 \cdot (-\infty) = \lim_{a \rightarrow 0^+} \frac{\ln(a)}{1/a} \stackrel{\text{L'Hôpital}}{=} \lim_{a \rightarrow 0^+} \frac{1/a}{-1/a^2}$

$$= \lim_{a \rightarrow 0^+} (-a) = 0$$