MA 138 – Calculus 2 with Life Science Applications
Solving Differential Equations
(Section 8.1)

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February 13, 2017

The Exponential Growth Model

A biological population with plenty of food, space to grow, and no threat from predators, tends to grow at a rate that is proportional to the population – that is, in each unit of time, a certain percentage of the individuals produce new individuals.

If reproduction takes place more or less continuously, then this growth rate is represented by

$$\frac{dN}{dt} = rN,$$

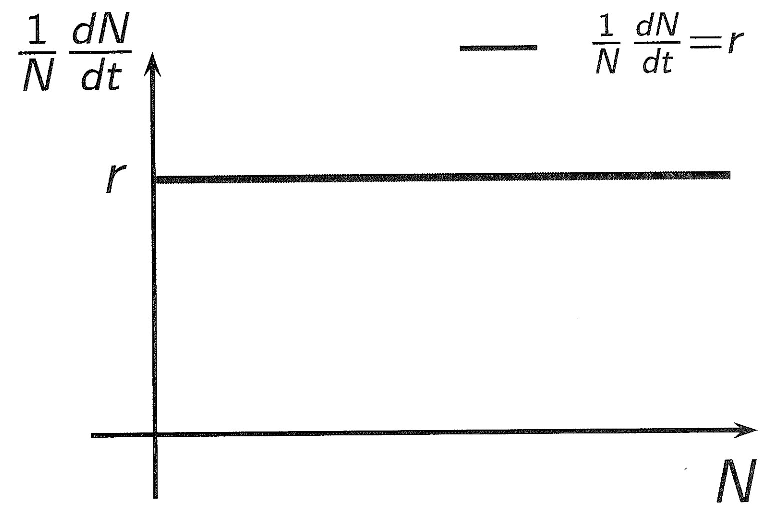
where $N = N(t)$ is the population as a function of time t and r is the growth rate. Assume also that N_0 is the population at time $t = 0$.

Note: $r =$ birth rate $-$ mortality rate.

Rewriting this differential equation as

$$\frac{1}{N}\frac{dN}{dt} = r$$

says that the per capita growth rate in the exponential model is a constant function of population size.



To obtain the solution to this differential equation we proceed as follows:

$$\frac{dN}{dt} = rN \rightsquigarrow \frac{1}{N}\,dN = r\,dt \rightsquigarrow \int \frac{1}{N}\,dN = \int r\,dt.$$

$$\rightsquigarrow \ln(N) = rt + C \rightsquigarrow N = Ae^{rt},$$

where C and $A = e^C$ are constants.

To determine the value of the constant A we now use the initial condition $N(0) = N_0$. We find that $A = N_0$. Thus $\boxed{N(t) = N_0 e^{rt}}$.

The Logistic Growth Model ($\equiv$ Verhulst Model)

- In short, unconstrained natural growth is exponential growth.
- However, we may account for the growth rate declining to 0 by including a factor $1 - N/K$ in the model, where K is a positive constant.
- The factor $1 - N/K$ is close to 1 (that is, has no effect) when N is much smaller than K, and is close to 0 when N is close to K.
- The resulting model,

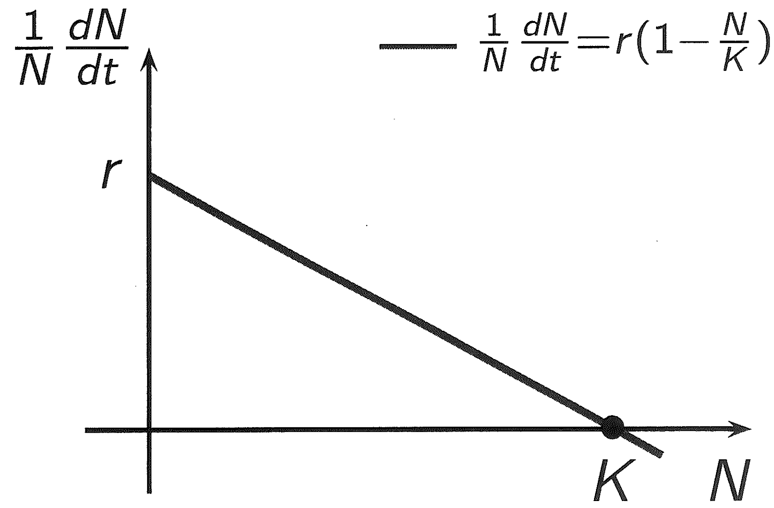
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \qquad \text{with} \qquad N(0) = N_0$$

is called the **logistic growth model** or the **Verhulst model**.

The word "logistic" has no particular meaning in this context, except that it is commonly accepted. The second name honors **Pierre François Verhulst** (1804–1849), a Belgian mathematician who studied this idea in the 19th century. Using data from the first five U.S. censuses, he made a prediction in 1840 of the U.S. population in 1940 – and was off by less than 1%.

Rewriting this differential equation as

$$\frac{1}{N}\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)$$



says that the per capita growth rate in the logistic equation is a linearly decreasing function of population size.

Note: r (=growth rate) and K (=carrying capacity) are positive constants.

To obtain the solution to this differential equation we proceed as follows:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \rightsquigarrow \frac{1}{N\left(1 - \frac{N}{K}\right)}\, dN = r\,dt \rightsquigarrow \frac{K}{N(K-N)}\, dN = r\,dt.$$

Next, we use the method of partial fractions, integration and a few manipulations, to obtain the general solution to this differential equation.

$$\frac{K}{N(K-N)}\,dN = r\,dt \quad \rightsquigarrow \quad \int\left(\frac{1}{N}+\frac{1}{K-N}\right)dN = \int r\,dt$$

$$\rightsquigarrow \quad \ln(N)-\ln(K-N) = rt + C \quad \rightsquigarrow \quad \frac{N(t)}{K-N(t)} = Ae^{rt},$$

where C and $A = e^C$ are constants.

To determine the value of the constant A we now use the initial condition $N(0) = N_0$. We find that $A = N_0/(K-N_0)$. Thus our solution looks like

$$\frac{N(t)}{K-N(t)} = \frac{N_0}{K-N_0}e^{rt} \quad \rightsquigarrow \quad \frac{K-N(t)}{N(t)} = \frac{K-N_0}{N_0e^{rt}}$$

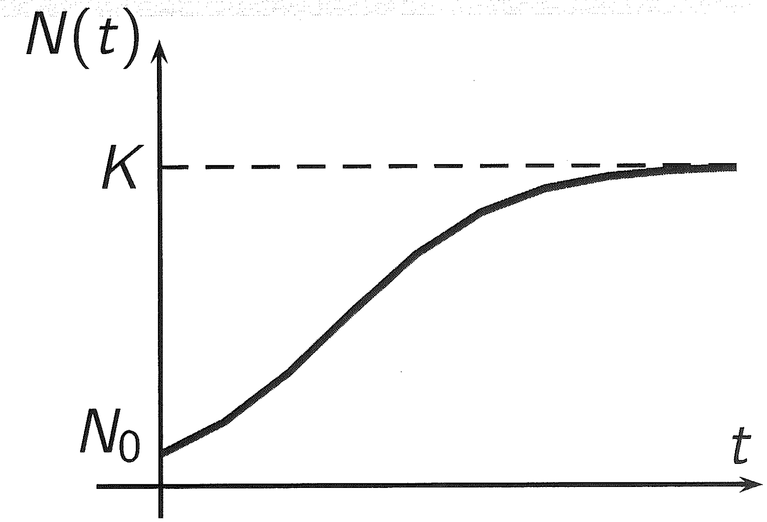
$$\rightsquigarrow \quad K-N(t) = N(t)\left(\frac{K}{N_0}-1\right)e^{-rt} \quad \rightsquigarrow \quad \boxed{N(t) = \frac{K}{1+\left(K/N_0-1\right)e^{-rt}}}$$

Observe that $\lim_{t\to\infty} N(t) = K$.

This justifies the fact that the constant K is dubbed **carrying capacity**.

Here is a typical graph of the logistic curve

$$N(t) = \frac{K}{1 + (K/N_0 - 1)e^{-rt}}.$$



Example 1 (Problem # 38, Section 8.1, p. 405)

Assume the size of a population, denoted $N(t)$, evolves according to the logistic equation. Find the intrinsic rate of growth r if the carrying capacity K is 100, $N(0) = 1$, and $N(1) = 20$.

$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}}$$

If $K = 100$, $N(0) = 1$, $N(1) = 20$

we have:

$$N(t) = \frac{100}{1 + \left(\frac{100}{1} - 1\right) e^{-rt}} = \frac{100}{1 + 99e^{-rt}}$$

after we used the first 2 conditions.

$N(1) = 20$ says

$$20 = \frac{100}{1 + 99e^{-r}} \implies 1 + 99e^{-r} = 5$$

$$\implies 99e^{-r} = 4 \quad \therefore \quad e^{-r} = 4/99$$

$$\Rightarrow \quad \ln e^{-r} = \ln(4/99)$$

$$\Rightarrow \quad -r = \ln(4/99)$$

$$\Rightarrow \quad r = -\ln(4/99) = \ln\left[\left(\frac{4}{99}\right)^{-1}\right]$$

$$= \ln(99/4) \simeq 3.208$$

this means that the growth rate is 320.8 %

Example 2 (Online Homework # 8)

Newton's Law of Cooling states that the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium. Thus, if an object is taken from an oven at 303°F and left to cool in a room at 76°F, its temperature T after t hours will satisfy the differential equation

$$\frac{dT}{dt} = k(T - 76)$$

where k is a positive constant.

If the temperature fell to 210°F in 0.8 hour(s), what will it be after 5 hour(s)?

$$\frac{dT}{dt} = k(T-76)$$

$T(0) = 303\ °F$; $T(0.8) = 210\ °F$

$\boxed{T(5) = ?}$

$$\frac{dT}{T-76} = k\,dt \implies \int \frac{dT}{T-76} = \int k\,dt$$

$$\ln(T-76) = kt + C$$

Take exponential

$$T - 76 = e^{kt} \cdot A$$

where $A = e^{C}$

$$T(0) = 303 \implies 303 - 76 = e^{0} \cdot A$$

$\therefore$ $A = 227$

$\therefore$ $\boxed{T(t) = 76 + 227\,e^{kt}}$

We now need to find k.

$$T(0.8) = 210 \implies 210 = 76 + 227\, e^{0.8k}$$

$$\therefore \quad e^{0.8k} = \frac{210-76}{227}$$ take $\ln$ of both sides

$$\implies \quad 0.8k = \ln\left(\frac{134}{227}\right)$$

$$\implies \quad k = \frac{1}{0.8} \ln\left(\frac{134}{227}\right) \simeq -0.65888$$

Thus

$$\boxed{T(t) = 76 + 227\, e^{-0.65888\, t}}$$

finally

$$T(5) = 76 + 227\, e^{-0.65888 \cdot 5}$$

$$\simeq 84.419 \,^\circ F$$

In Example 2 we just discovered that the constant k is ≈ -0.65888. Thus, it is customary to rewrite the DE as follows:

Newton's Law of Cooling

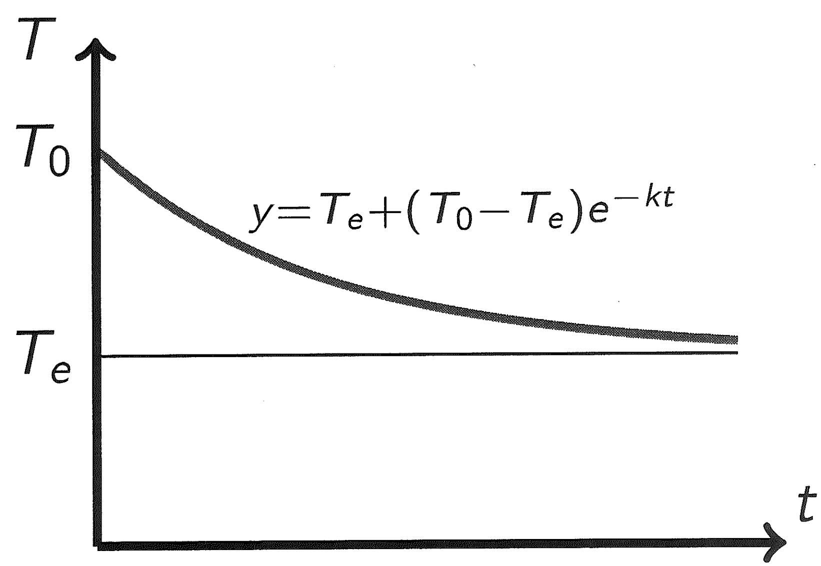
It states that the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium:

$$\frac{dT}{dt} = -k(T - T_e) \qquad T(0) = T_0,$$

where k is a positive constant.

We can easily show that the solution of this IVP is given by

$$\boxed{T(t) = T_e + (T_0 - T_e)e^{-kt}}.$$



Notice also that

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty} [T_e + (T_0 - T_e)e^{-kt}] = T_e.$$

Example 3 (Problem # 22, Section 8.1, p. 404)

Consider the differential equation below, where $L = L(t)$ is a function of t

$$\frac{dL}{dt} = k(34 - L) \qquad L(0) = 2.$$

- Solve the differential equation
- Determine k under the assumption that $L(4) = 10$.

$$\frac{dL}{dt} = k(34 - L) \qquad L(0) = 2$$

$$\frac{dL}{34-L} = k\,dt \implies \int \frac{-1}{34-L}\,dL = \int -k\,dt$$

$$\ln(34-L) = -kt + C$$

Take exponential of both sides

$$34 - L(t) = e^{-kt} \cdot A \qquad A = e^C$$

When $t=0$ $L(0)=2$. So

$$34-2 = e^0 \cdot A \qquad \therefore \quad A = 32$$

$$\therefore \quad \boxed{L(t) = 34 - 32e^{-kt}}$$

If $L(4) = 10$ we have that

$$10 = 34 - 32e^{-4k}$$

$\therefore \quad 32e^{-4k} = 34 - 10 \quad \therefore \quad e^{-4k} = \frac{24}{32}$

$\therefore \quad e^{-4k} = \frac{3}{4}$ Take ln of both sides

$-4k = \ln(3/4) \quad \therefore \quad k = -\frac{1}{4}\ln(3/4)$

$$= \boxed{\frac{1}{4}\ln(4/3)}$$

7.2% rate of growth ⟷ ≈ 0.0719

The Von Bertalanffy (Restricted) Growth Equation

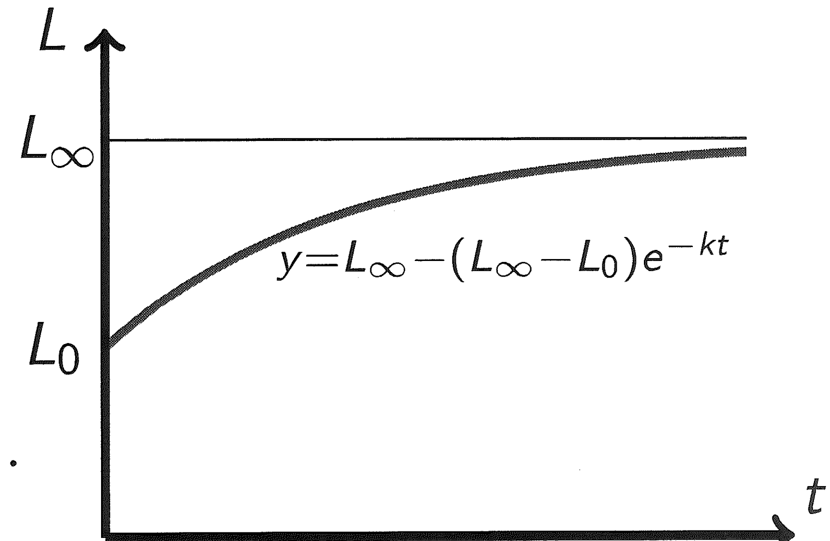
A commonly used DE for the growth, in length, of an individual fish is

$$\frac{dL}{dt} = k(L_\infty - L) \qquad L(0) = L_0,$$

where $L(t)$ is length at age t, L_∞ is the asymptotic length and k is a positive constant. The DE captures the idea that the rate of growth is proportional to the difference between asymptotic and current length.

We can easily show that the solution of this IVP is given by

$$\boxed{L(t) = L_\infty - (L_\infty - L_0)e^{-kt}}.$$



Notice also that

$$\lim_{t\to\infty} L(t) = \lim_{t\to\infty} [L_\infty - (L_\infty - L_0)e^{-kt}] = L_\infty.$$

Example 4 (Online Homework # 12)

Let $P(t)$ be the **performance level** of someone learning a skill as a function of the training time t. Its derivative represents the rate at which performance improves. If M is the maximum level of performance of which the learner is capable, then a model for learning is given by the differential equation $\frac{dP}{dt} = k(M - P)$ where k is a positive constant.

Two new workers, John and Bob, were hired for an assembly line.

John could process 12 units per minute after one hour and 15 units per minute after two hours. Bob could process 10 units per minute after one hour and 16 units per minute after two hours.

Using the above model and assuming that $P(0) = 0$, estimate the maximum number of units per minute that each worker is capable of processing.

$$\frac{dP}{dt} = k(M-P) \qquad P(0) = 0$$

$$\frac{dP}{M-P} = k\,dt \rightsquigarrow \frac{-1\,dP}{M-P} = -k\,dt$$

$$\rightsquigarrow \int \frac{-1}{M-P}\,dP = \int -k\,dt \rightsquigarrow \ln(M-P) = -kt + C$$

take exponential of both sides:

$$M - P = e^{-kt} \cdot \underbrace{e^{C}}_{A} \qquad \therefore \boxed{M - P(t) = A e^{-kt}}$$

Since $P(0) = 0$ we get: $M - 0 = A \cdot \underbrace{e^{0}}_{1}$

$\therefore A = M$ Thus

$$\boxed{P(t) = M - Me^{-kt} = \underline{M(1-e^{-kt})}}$$

For BOB : $P(1) = 10$ $P(2) = 16$

$$P_{BOB}(t) = M_{BOB}\left(1 - e^{-k_{BOB} t}\right)$$

Thus

$$\begin{cases} 10 = M_{BOB}\left(1 - e^{-k_{BOB}}\right) \\ 16 = M_{BOB}\left(1 - e^{-2k_{BOB}}\right) \end{cases}$$

$$\frac{10}{M_{BOB}} = 1 - e^{-k_{BOB}} \qquad \frac{16}{M_{BOB}} = 1 - \left(e^{-k_{BOB}}\right)^2$$

$$e^{-k_{BOB}} = 1 - \frac{10}{M_{BOB}} \qquad \left(e^{-k_{BOB}}\right)^2 = 1 - \frac{16}{M_{BOB}}$$

Compare and substitute $\left(1 - \frac{10}{M_{BOB}}\right)^2 = 1 - \frac{16}{M_{BOB}}$

$$\left(M_{BOB} - 10\right)^2 = \left(M_{BOB} - 16\right) M_{BOB}$$

$$\cancel{M_B^2} - 20M_B + 100 = \cancel{M_B^2} - 16M_B$$

$$\longrightarrow \quad 100 = 20M_B - 16M_B \longrightarrow 4M_B = 100$$

$$\boxed{M_B = 25}$$ <u>maximum level of performance of BOB.</u>

Similarly for <u>John</u>: $P(1) = 12 \qquad P(2) = 15$

$$P_{JOHN}(t) = M_{John}\left(1 - e^{-k_{John}t}\right) \quad \text{Thus}$$

$$\begin{cases} 12 = M_{John}\left(1 - e^{-k_{John}}\right) \\ 15 = M_{John}\left(1 - e^{-2k_{John}}\right) \end{cases}$$

$$\frac{12}{M_J} = 1 - e^{-k_J} \quad \text{and} \quad \frac{15}{M_J} = 1 - \left(e^{-k_J}\right)^2$$

Thus

$$e^{-k_{John}} = 1 - \frac{12}{M_{John}} \qquad \left(e^{-k_{John}}\right)^2 = 1 - \frac{15}{M_{John}}$$

Substitute

$$\left(1 - \frac{12}{M_{John}}\right)^2 = 1 - \frac{15}{M_{John}}$$

$$\rightsquigarrow \quad (M_{John} - 12)^2 = (M_{John} - 15) M_{John}$$

$$\rightsquigarrow \quad M_{John}^2 - 24 M_{John} + 144 = M_{John}^2 - 15 M_{John}$$

$$\therefore \quad 144 = (24 - 15) M_{John} \quad \longrightarrow \quad M_{John} = \frac{144}{9}$$

$$\therefore \quad \boxed{M_{John} = 16}$$ maximum level of performance for John.

Allometric Growth (pp. 401-403 of Section 8.1)

In biology, **allometry** is the study of the relationship between sizes of parts of an organism (e.g., skull length and body length, or leaf area and stem diameter).

We denote by $L_1(t)$ and $L_2(t)$ the respective sizes of two organs of an individual of age t. We say that L_1 and L_2 are related through an allometric law if their specific growth rates are proportional—that is, if

$$\frac{1}{L_1} \cdot \frac{dL_1}{dt} = k \frac{1}{L_2} \cdot \frac{dL_2}{dt}$$

for some constant k. If k is equal to 1, then the growth is called isometric; otherwise it is called allometric.

Integrating, we find that $\boxed{L_1 = C\, L_2^k}$ for some constant C.

$$\frac{1}{L_1}\frac{dL_1}{dt} = k\,\frac{1}{L_2}\frac{dL_2}{dt}$$

$$\Longrightarrow \int \frac{1}{L_1}\,dL_1 = \int k\,\frac{1}{L_2}\,dL_2$$

$$\Longrightarrow \ln L_1 = k \ln L_2 + \tilde{C}$$ ← constant

$$\ln L_1 = \ln\left(L_2^k\right) + \tilde{C}$$

take now exp

$$L_1 = e^{\ln\left(L_2^k\right) + \tilde{C}}$$

$$= L_2^k \cdot \underbrace{e^{\tilde{C}}}_{C}$$

$$\therefore \quad \boxed{L_1 = C L_2^k}$$

Keibler's Law

Kleiber's law, named after Max Kleiber's biological work in the early 1930s, is the observation that, for the vast majority of animals, an animal's metabolic rate scales to the 3/4 power of the animal's mass.

If q_0 is the animal's metabolic rate, and M the animal's mass, then Kleiber's law states that

$$q_0 \propto M^{3/4}.$$

In plants, the exponent is close to 1.

Note: The exponent for Kleiber's law was a matter of dispute for many decades. It is still contested by a diminishing number as being 2/3 rather than the more widely accepted 3/4.

Homeostasis (p. 403 of Section 8.1)

The nutrient content of a consumer can range from reflecting the nutrient content of its food to being constant. A model for homeostatic regulation is provided in Sterner and Elser (2002). It relates a consumers nutrient content (denoted by y) to its foods nutrient content (denoted by x) as

$$\frac{dy}{dx} = \frac{1}{\theta}\frac{y}{x}$$

where $\theta \geq 1$ is a constant.

Integrating, we find that $\boxed{y = C\,x^{1/\theta}}$ for some positive constant C.

Absence of homeostasis means that the consumer reflects the foods nutrient content. This occurs when $y = Cx$ and thus when $\theta = 1$.

Strict homeostasis means that the nutrient content of the consumer is independent of the nutrient content of the food; that is, $y = C$; this occurs in the limit as $\theta \longrightarrow \infty$.

$$\frac{dy}{dx} = \frac{1}{\theta}\,\frac{y}{x} \implies$$

$$\frac{1}{y}\,dy = \frac{1}{\theta}\cdot\frac{1}{x}\,dx \implies \int \frac{1}{y}\,dy = \int \frac{1}{\theta}\cdot\frac{1}{x}\,dx$$

$$\implies \ln y = \frac{1}{\theta}\cdot \ln x + \tilde{C}$$

$$\stackrel{!}{=} \ln\left(x^{\frac{1}{\theta}}\right) + \tilde{C} \qquad \text{take exp}$$

$$\implies y = e^{\ln(x^{1/\theta})} \cdot \underbrace{e^{\tilde{C}}}_{\text{constant}}$$

$$\boxed{y = C \cdot x^{1/\theta}}$$