MA 138 – Calculus 2 with Life Science Applications
Equilibria and Their Stability
(Section 8.2)

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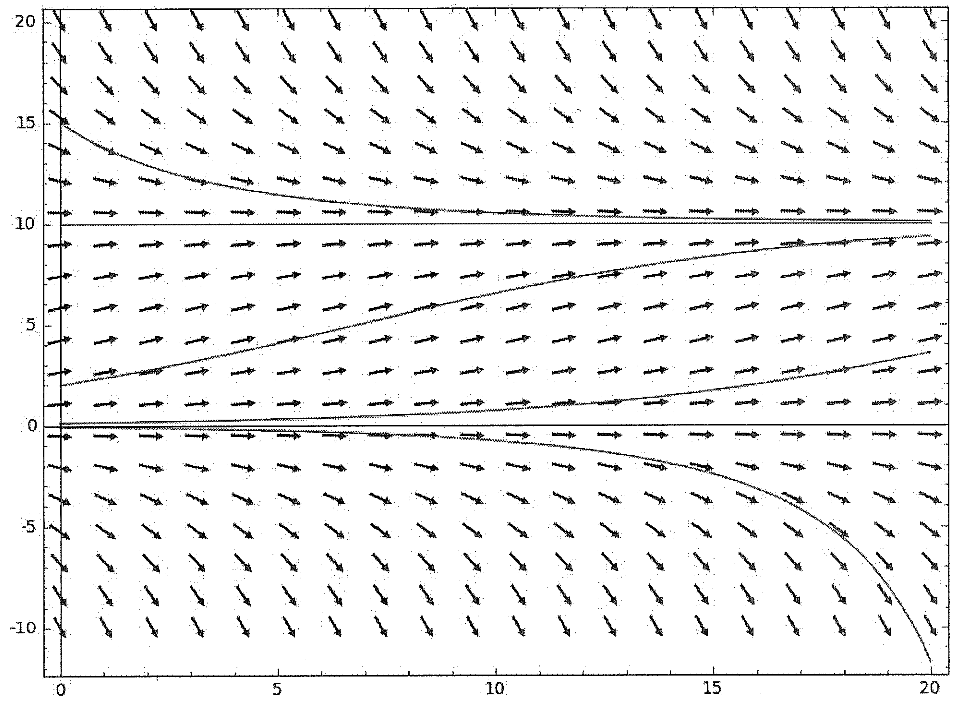
An explicit solution of a DE can inform us about long-term behavior.

What if it is hard to find the solutions?

We saw, for example, that a direction field gives us visual information about the solutions of a first order DE.

E.g.: $$\frac{dN}{dt} = 0.2N\left(1 - \frac{N}{10}\right)$$



Q.: What does the above direction field tell us about the solutions of the DE?

Equilibria of an Autonomous DE

We consider autonomous differential equations of the form

$$\frac{dy}{dx} = g(y)$$

where we will typically think of x as time.

Constant solutions form a special class of solutions of autonomous differential equations. These solutions are called (point) **equilibria**.

Example For example

$$N_1(t) = 0 \qquad \text{and} \qquad N_2(t) = K$$

are constant solutions to the logistic equation $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$.

Finding Equilibria

If $\widehat{y}$ (read "y hat") satisfies

$$g(\widehat{y}) = 0$$

then $\widehat{y}$ is an equilibrium of the autonomous differential equation

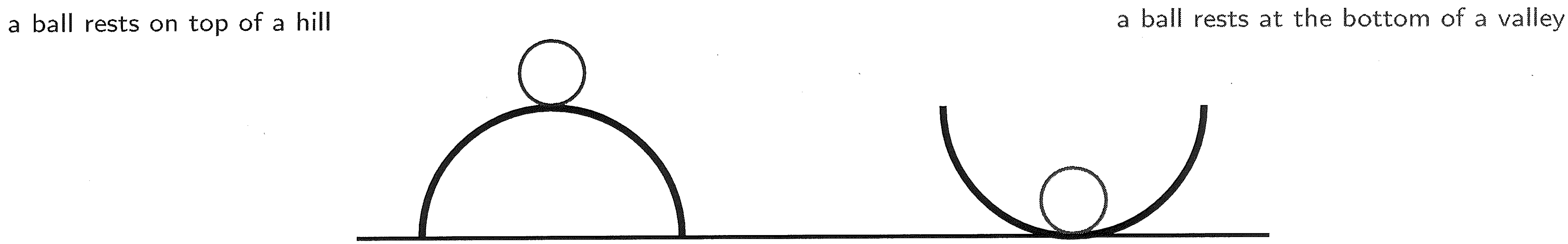
$$\frac{dy}{dx} = g(y).$$

Basic Property

The basic property of equilibria is that if, initially (say, at $x = 0$), $y(0) = \widehat{y}$ and $\widehat{y}$ is an equilibrium, then $y(x) = \widehat{y}$ for all $x \geq 0$.

Stability of Equilibria

Of great interest is the stability of equilibria of a differential equation. This is best explained by the example of a ball on a hill vs a ball in a valley:



In either case, the ball is in equilibrium because it does not move.

If we perturb the ball by a small amount (i.e., if we move it out of its equilibrium slightly) the ball on the left will roll down the hill and not return to the top, whereas the ball on the right will return to the bottom of the valley.

The ball on the **left** is **unstable** and the ball on the **right** is **stable**.

Stability for Equilibria of DE

Suppose that $\widehat{y}$ is an equilibrium of $\dfrac{dy}{dx} = g(y)$; that is, $g(\widehat{y}) = 0$.

We look at what happens to the solution when we start close to the equilibrium; that is, we consider the solution of the DE when we move away from the equilibrium by a small amount, called a *small perturbation*.

We say that $\widehat{y}$ is **locally stable** if the solution returns to the equilibrium $\widehat{y}$ after a small perturbation;

We say that $\widehat{y}$ is **unstable** if the solution does not return to the equilibrium $\widehat{y}$ after a small perturbation.

We will now discuss an **analytical** and a **graphical method** for analyzing stability of equilibria.

Analytical Approach to Stability

Stability Criterion

Consider the differential equation $\dfrac{dy}{dx} = g(y)$ where $g(y)$ is a differentiable function.

Assume that $\widehat{y}$ is an equilibrium; that is, $g(\widehat{y}) = 0$.

Then

- $\widehat{y}$ is **locally stable** if $g'(\widehat{y}) < 0$;
- $\widehat{y}$ is **unstable** if $g'(\widehat{y}) > 0$.

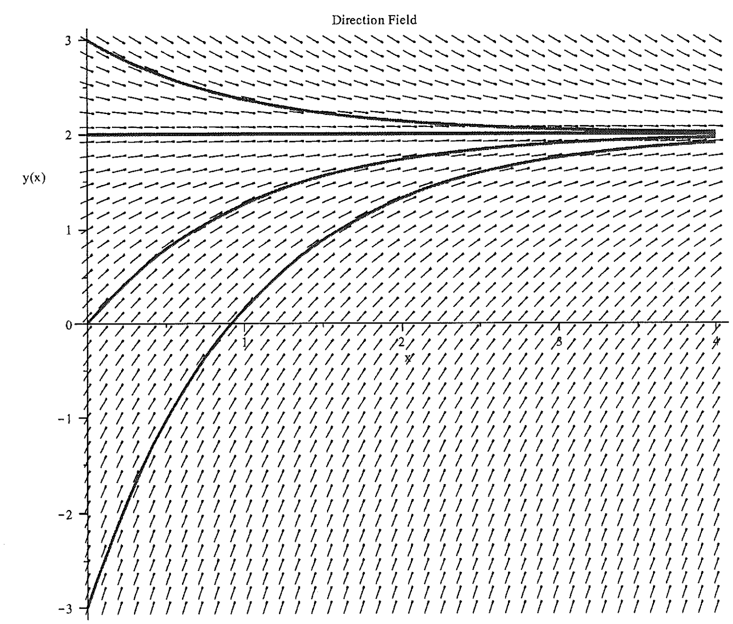
Note:

$g'(\widehat{y})$ is called an **eigenvalue**; it is the slope of the tg. line of $g(y)$ at $\widehat{y}$.

Example 1

Find the equilibria of this differential equation and discuss their stability using the analytical approach ($\equiv$ stability criterion)

$$\frac{dy}{dx} = 2 - y.$$



$$\frac{dy}{dx} = 2 - y \qquad \text{so} \qquad g(y) = 2 - y$$

$$g(y) = 0 \iff 2 - y = 0 \qquad \therefore \quad \boxed{\hat{y} = 2 \text{ is the unique equilibrium}}$$

Now $g'(y) = -1$

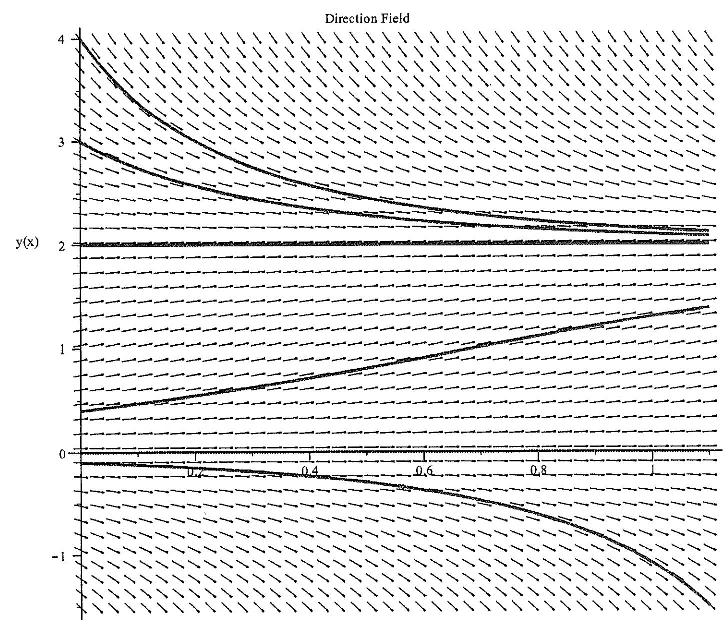
$$g'(2) = -1 < 0$$

$\therefore \quad \hat{y} = 2$ is a <u>locally stable equilibrium</u>

Example 2

Find the equilibria of this differential equation and discuss their stability using the analytical approach ($\equiv$ stability criterion)

$$\frac{dy}{dx} = y(2-y).$$



$$\frac{dy}{dx} = y(2-y) \qquad \text{so} \qquad g(y) = y(2-y)$$
$$= 2y - y^2$$

<u>Now</u>: $g(y) = 0 \iff y(2-y) = 0$

$\therefore$ $\hat{y} = 0$ and $\hat{y} = 2$ are the two equilibria

$$g(y) = 2y - y^2 \implies g'(y) = 2 - 2y$$

$g'(0) = 2 > 0$ $\therefore$ $\hat{y} = 0$ is an <u>unstable</u> equilibrium

$g'(2) = 2 - 2(2) = -2 < 0$ $\therefore$ $\hat{y} = 2$ is a locally stable equil.

Example 3

Find the equilibria of this differential equation and discuss their stability using the analytical approach ($\equiv$ stability criterion)

$$\frac{dy}{dx} = y^2 - 4.$$

$$\frac{dy}{dx} = y^2 - 4 \qquad \therefore \quad g(y) = y^2 - 4$$

$$g(y) = 0 \iff y^2 - 4 = 0 \iff$$

$$(y-2)(y+2)$$

$\therefore$ $\hat{y} = 2$ and $\hat{y} = -2$ are the two equilibria

About the stability

$$g'(y) = 2y$$

(*) $g'(2) = 2 \cdot 2 = 4 > 0$ $\quad\therefore$ $\hat{y} = 2$ is <u>unstable</u>

(*) $g'(-2) = 2(-2) = -4 < 0$ $\quad\therefore$ $\hat{y} = -2$ is <u>locally stable equil</u>

Example 4 (Problem # 5, Exam 2, Spring '13)

Suppose that a fish population evolves according to the logistic equation and that fish are harvested at a rate proportional to the population size. That is,

$$\frac{dN}{dt} = g(N) = 3N\left(1 - \frac{N}{6,000}\right) - 0.5N.$$

(*a*) Find all equilibria $\widehat{N}$ of the given differential equation.

(*b*) Use the eigenvalue approach, that is compute $g'(\widehat{N})$, to analyze the stability of the equilibria found in (*a*).

$$g(N) = 3N\left(1 - \frac{N}{6{,}000}\right) - 0.5N$$

(a)

$$g(N) = 0 \iff 3N\left(1 - \frac{N}{6{,}000}\right) - 0.5N = 0$$

$$\iff N\left[3 - \frac{N}{2{,}000} - 0.5\right] = 0$$

$$N\left[2.5 - \frac{N}{2{,}000}\right] = 0 \iff \hat{N} = 0, \quad \hat{N} = 5{,}000$$

are the two equilibria

(b) Rewrite $g(N)$ as:

$$g(N) = 2.5N - \frac{N^2}{2000}$$

$$g'(N) = 2.5 - \frac{2N}{2,000} = 2.5 - \frac{N}{1,000}$$

Thus:

$g'(0) = 2.5 > 0$ $\therefore$ $\hat{N} = 0$ is an unstable equilibrium

$$g'(5,000) = 2.5 - \frac{5,000}{1,000} = 2.5 - 5 = -2.5 < 0$$

$\therefore$ $\hat{N} = 5,000$ is a locally stable equilibrium

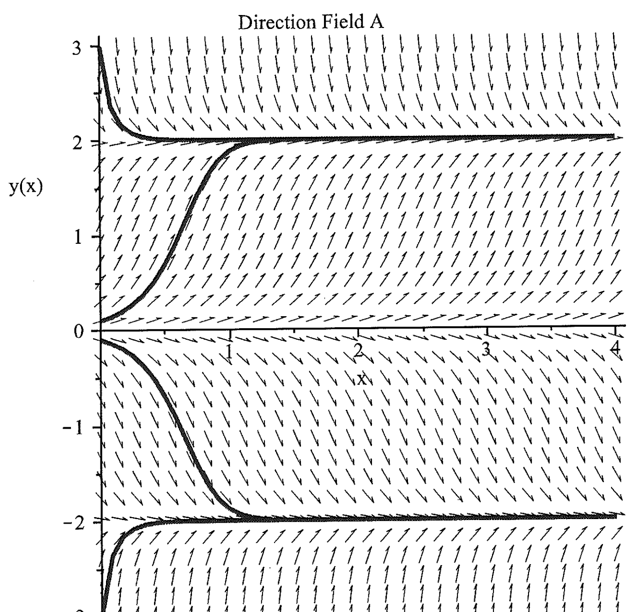
Example 5

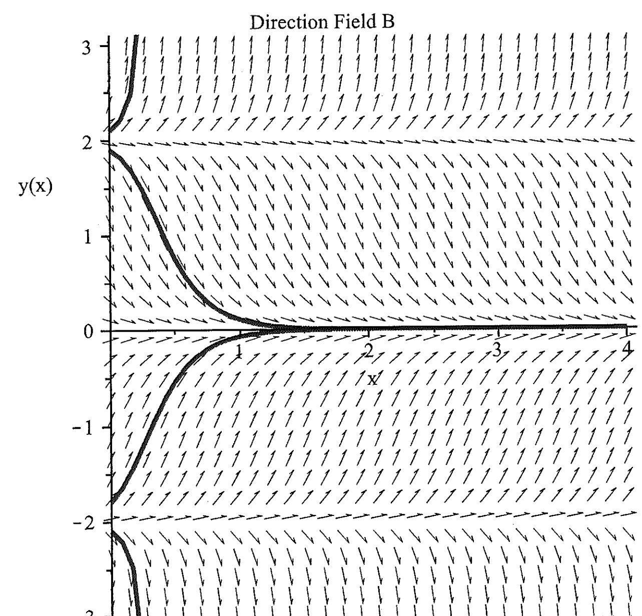
Consider the DE $$\frac{dy}{dx} = g(y) = y(y+2)(2-y).$$

(*a*) Find the equilibria $\widehat{y}$ of this differential equation.

(*b*) Compute the eigenvalues associated with each equilibrium, that is compute $g'(\widehat{y})$, and discuss the stability of each equilibrium.

(*c*) Using the information found in (*a*) and (*b*), which of the following phase portraits matches the given differential equation?





$g(y) = y(y+2)(2-y)$

(a) To find the equilibria we set: $g(y) = 0$

$$y(y+2)(2-y) = 0 \iff \boxed{\hat{y} = 0, -2, 2}$$

equilibria

(b) To classify the equilibria we need $g'(y)$.

$$g(y) = y(y+2)(2-y) = (y^2+2y)(2-y)$$

$$= \cancel{2y^2} - y^3 + 4y - \cancel{2y^2} = -y^3 + 4y$$

Thus $\boxed{g'(y) = -3y^2 + 4}$

$g'(0) = 4 > 0 \quad \therefore \quad \hat{y} = 0$ is an unstable equilibrium

$g'(-2) = -3(-2)^2 + 4 = -12 + 4 = -8 < 0$

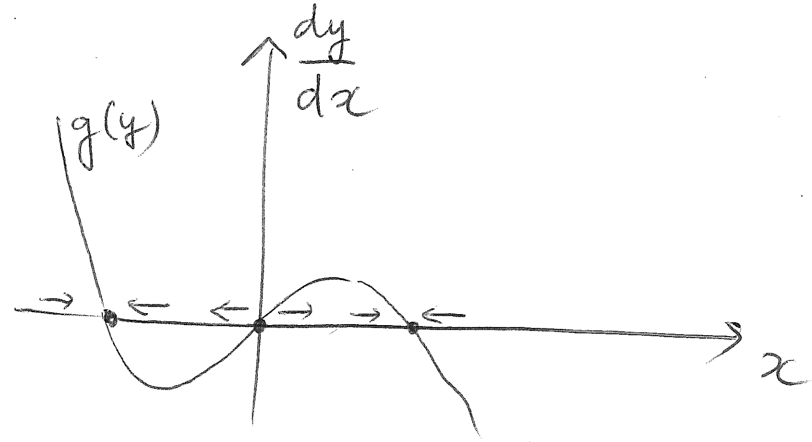
$\therefore \ \hat{y} = -2$ is a locally stable equilibrium

$g'(2) = -3(2)^2 + 4 = -12 + 4 = -8 < 0$

$\therefore \ \hat{y} = 2$ is also a locally stable equilibrium

(c) the direction field (A) is the correct one.

graph of $g(y)$



Allee Effect (Section 8.2.4, pp. 416-417)

A sexually reproducing species may experience a disproportionately low recruitment rate when the population density falls below a certain level, due to lack of suitable mates. This phenomenon is called an **Allee effect** (Allee, 1931).

A simple extension of the logistic equation incorporates the effect.

We denote the size of a population at time t by $N = N(t)$; then we have

$$\frac{dN}{dt} = rN(N-a)\left(1 - \frac{N}{K}\right)$$

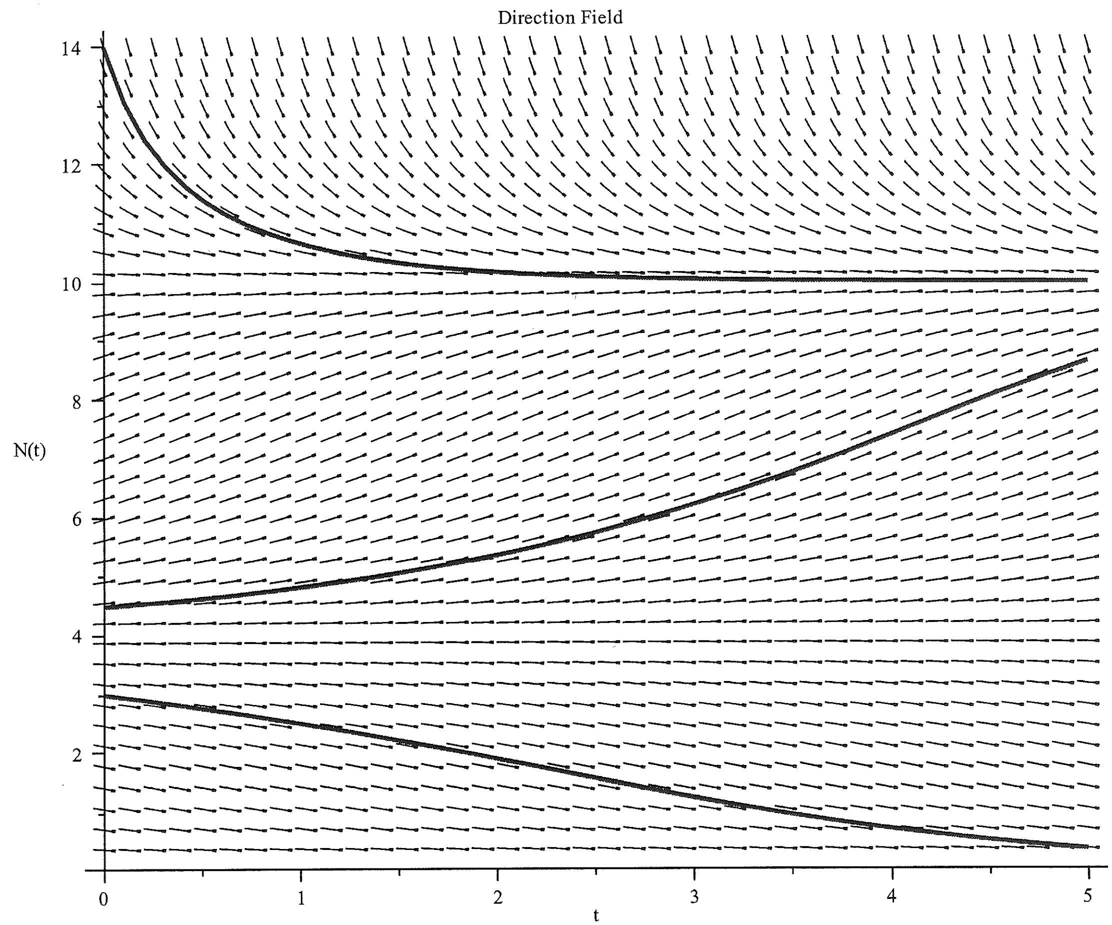
where r, a, and K are positive constants. We assume that $0 < a < K$.

As in the logistic equation, K denotes the carrying capacity.

The constant a is a **threshold population size** below which the recruitment rate is negative, meaning that the population will shrink and ultimately go to extinction. Analyze the equilibria $\widehat{N} = 0, a, K$.

Phase portrait of $\dfrac{dN}{dt} = 0.2N(N-4)\left(1 - \dfrac{N}{10}\right)$

equilibria: $\widehat{N} = 0, 4, 10$



$$\frac{dN}{dt} = g(N) = rN(N-a)\left(1-\frac{N}{K}\right)$$

the equilibria are $\boxed{\hat{N} = 0,\ a,\ K}$

To classify them we need $g'(N)$. We need to rewrite $g(N)$ in a better way.

$$g(N) = rN(N-a)\left(1-\frac{N}{K}\right) =$$

$$= \frac{r}{K} N(N-a)(K-N)$$

$$= \frac{r}{K}(N^2-aN)(K-N)$$

$$= \frac{r}{K}\left[KN^2 - N^3 - aKN + aN^2\right]$$

$$= -\frac{r}{K}N^3 + \frac{r}{K}(K+a)N^2 - raN$$

$$g'(N) = -\frac{3r}{K}N^2 + \frac{2r}{K}(K+a)N - ra$$

Now:

$$g'(0) = -ra < 0$$

$\therefore$ $\hat{N}=0$ is locally stable

$$g'(a) = -\frac{3r}{K}a^2 + \frac{2r}{K}(K+a)a - ra$$

$$= -\frac{3r}{K}a^2 + 2ra + \frac{2ra^2}{K} - ra$$

$$= -\frac{r}{K}a^2 + ra = \frac{ra}{K}\left[K-a\right] > 0$$

($\frac{ra}{K}$ positive, $[K-a]$ positive)

$\hat{N}=a$ is unstable

$$g'(K) = -\frac{3r}{K}(K)^2 + \frac{2r}{K}(K+a)(K) - ra$$

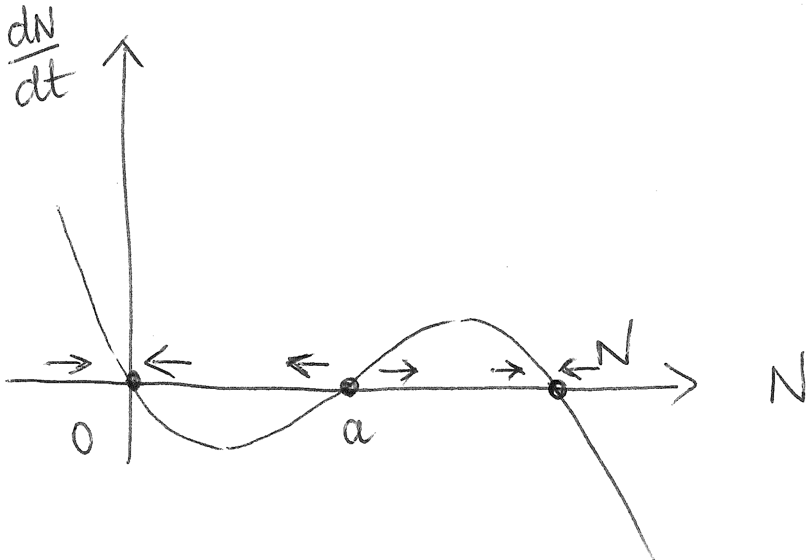
$$= -3rK + 2r(K+a) - ra$$

$$= -3rK + 2rK + 2ar - ra$$

$$= -rK + ar = \underbrace{-r}_{\text{negative}}\underbrace{[K-a]}_{\text{positive}} < 0$$

$\therefore$ $\hat{N} = K$ is locally stable

graph of $g(N)$



Example 6 (Bonus Problem (*b*), Exam 2, Spring '14)

A tumor can be modeled as a spherical collection of cells and that it acquires resources for growth only through its surface area. All cells in a tumor are also subject to a constant per capita death rate. With these assumptions, the dynamics of tumor mass M (in grams) is therefore modeled by the differential equation

$$\frac{dM}{dt} = \kappa M^{2/3} - \mu M,$$

where κ and μ are positive constants. The first term represents tumor growth via nutrients entering through the surface; the second term represents a constant per capita death rate.

Suppose $\kappa = 1$, that is the dynamics of tumor mass is modeled as

$$\frac{dM}{dt} = M^{2/3} - \mu M.$$

Which value does the tumor mass approach as time $t \to \infty$? Explain.

$$\frac{dM}{dt} = \boxed{M^{2/3} - \mu M = g(M)}$$

The equilibrium is given by $g(M) = 0$

$$M^{2/3} - \mu M = M^{2/3}\left(1 - \mu M^{1/3}\right) = 0$$

$$\Longleftrightarrow \boxed{\hat{M} = 0} \quad \text{or} \quad 1 - \mu M^{1/3} = 0$$

$$\Longleftrightarrow$$

$$\hat{M}^{1/3} = \frac{1}{\mu}$$

$$\Longleftrightarrow$$

$$\boxed{\hat{M} = \frac{1}{\mu^3}}$$

About their stability.

$$g'(M) = \frac{2}{3} M^{2/3 - 1} - \mu$$

$$\therefore \quad g'(M) = \frac{2}{3} M^{-1/3} - \mu$$

$$g'(0) = +\infty \qquad \therefore \quad \hat{M} = 0 \text{ is unstable}$$

$$\therefore \quad g'\left(\frac{1}{\mu^3}\right) = \frac{2}{3}\left(\frac{1}{\mu^3}\right)^{-1/3} - \mu$$

$$= \frac{2}{3}\left[\frac{1}{\mu}\right]^{-1} - \mu = \frac{2}{3}\mu - \mu$$

$$= -\frac{1}{3}\mu < 0$$

$$\therefore \quad \boxed{\hat{M} = \frac{1}{\mu^3} \text{ is locally stable}}$$