MA 138 – Calculus 2 with Life Science Applications
Equilibria and Their Stability
(Section 8.2)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

February 22, 2017

Analytical Approach to Stability

Stability Criterion

Consider the differential equation $\dfrac{dy}{dx} = g(y)$ where $g(y)$ is a differentiable function.

Assume that $\widehat{y}$ is an equilibrium; that is, $g(\widehat{y}) = 0$.

Then

- $\widehat{y}$ is **locally stable** if $g'(\widehat{y}) < 0$;
- $\widehat{y}$ is **unstable** if $g'(\widehat{y}) > 0$.

Note:
$g'(\widehat{y})$ is called an **eigenvalue**; it is the slope of the tg. line of $g(y)$ at $\widehat{y}$.

Proof of the Stability Criterion

- We assume that $\widehat{y}$ is an equilibrium of $\dfrac{dy}{dx} = g(y)$. [i.e., $g(\widehat{y}) = 0$.]
- We consider a small perturbation about the equilibrium $\widehat{y}$; we express it as $y = \widehat{y} + z$ where z is small and may be either positive or negative. Then

$$\frac{dy}{dx} = \frac{d}{dx}(\widehat{y} + z) = \frac{dz}{dx}$$

since $d\widehat{y}/dx = 0$ ($\widehat{y}$ is a constant). We find that $\dfrac{dz}{dx} = g(\widehat{y} + z)$.
- Since z is small, we can approximate $g(\widehat{y} + z)$ by its linear approximation about $\widehat{y}$.
- In general, the linear approximation of $g(\Box)$ about $\widehat{y}$ is given by

$$L(\Box) = g(\widehat{y}) + (\Box - \widehat{y})g'(\widehat{y}) = (\Box - \widehat{y})g'(\widehat{y}),$$

since $g(\widehat{y}) = 0$.

- Therefore, the linear approximation of $g(\widehat{y}+z)$ is given by

$$g(\widehat{y}+z) \approx L(\widehat{y}+z) = (\widehat{y}+z-\widehat{y})g'(\widehat{y}) = z\,g'(\widehat{y}).$$

- If we set $\lambda = g'(\widehat{y})$ then $\dfrac{dz}{dx} = \lambda z$ is the first-order approximation of the perturbation.
- This equation has the solution

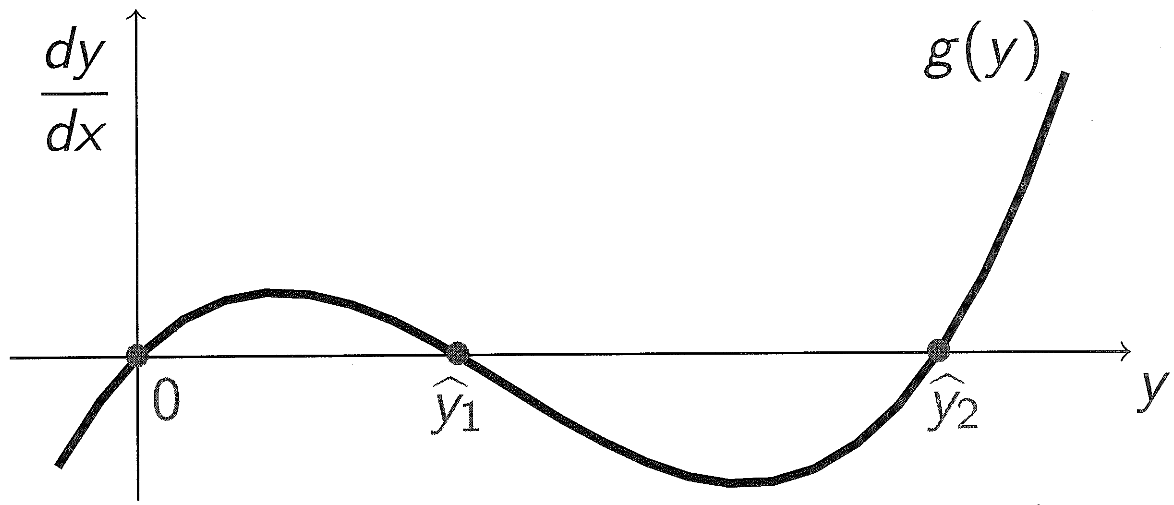
$$z(x) = z_0 e^{\lambda x} \quad \leftrightsquigarrow \quad y(x) = \widehat{y} + (y_0 - \widehat{y})e^{\lambda x}.$$

- This solution has the property that $\lim_{x\to\infty} y(x) = \widehat{y}$ if $\lambda < 0$. That is, the system returns to the equilibrium $\widehat{y}$ after a small perturbation. This means that $\widehat{y}$ is locally stable if $\lambda = g'(\widehat{y}) < 0$.
- On the other hand, if $\lambda > 0$, then $y(x)$ does not go to $\widehat{y}$ as $x \to \infty$, implying that the system will not return to the equilibrium $\widehat{y}$ after a small perturbation, and $\widehat{y}$ is unstable.

Graphical Approach to Stability

Consider the autonomous differential equation $\dfrac{dy}{dx} = g(y)$.

Suppose that $g(y)$ is of the form given in the figure below



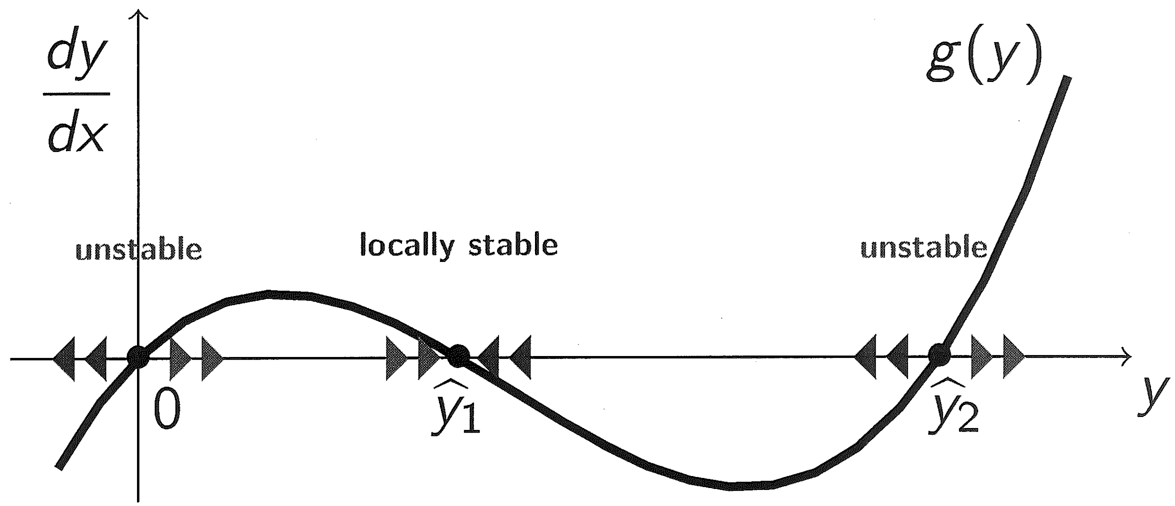
To find the equilibria of our DE, we set $g(y) = 0$.

Graphically, this means that if we graph $g(y)$ (i.e., the derivative of y with respect to x) as a function of y, then the equilibria are the points of intersection of $g(y)$ with the horizontal axis, which is the y-axis in this case, since y is the independent variable.

For our choice of $g(y)$, the equilibria are at $\hat{y} = 0$, $\hat{y}_1$, and $\hat{y}_2$.

We can then use the graph of g(y) to say the following about the fate of a solution on the basis of its current value:

- if the current value y is such that $g(y) > 0$ (i.e., $dy/dx > 0$), then y will increase as a function of x;
- if y is such that $g(y) < 0$ (i.e., $dy/dx < 0$), then y will decrease as a function of x;
- the points y where $g(y) = 0$ are the points where y will not change as a function of x [since $g(y) = dy/dx = 0$]. These are the equilibria.

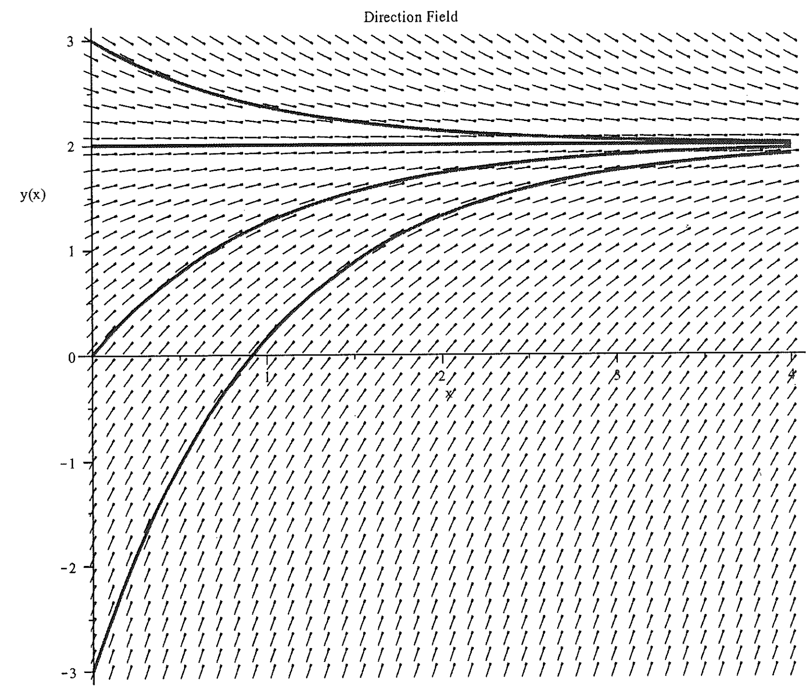


The arrows close to the equilibria indicate the type of stability.

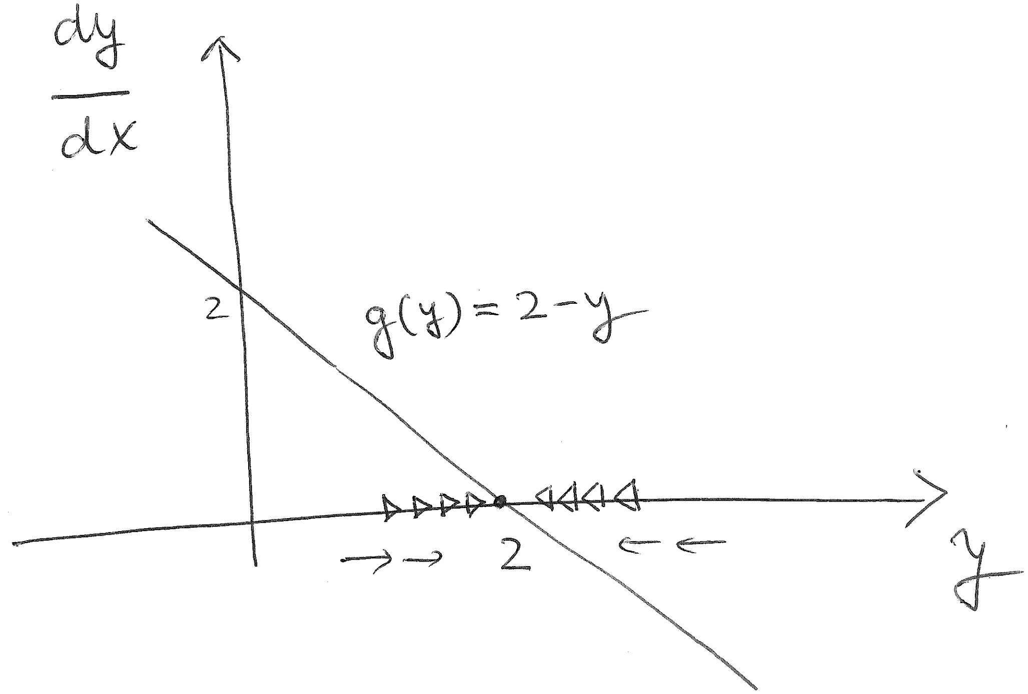
Example 1

Find the equilibria of this differential equation and discuss their stability using the graphical approach

$$\frac{dy}{dx} = 2 - y.$$



$$\frac{dy}{dx} = \underbrace{2-y}_{g(y)}$$

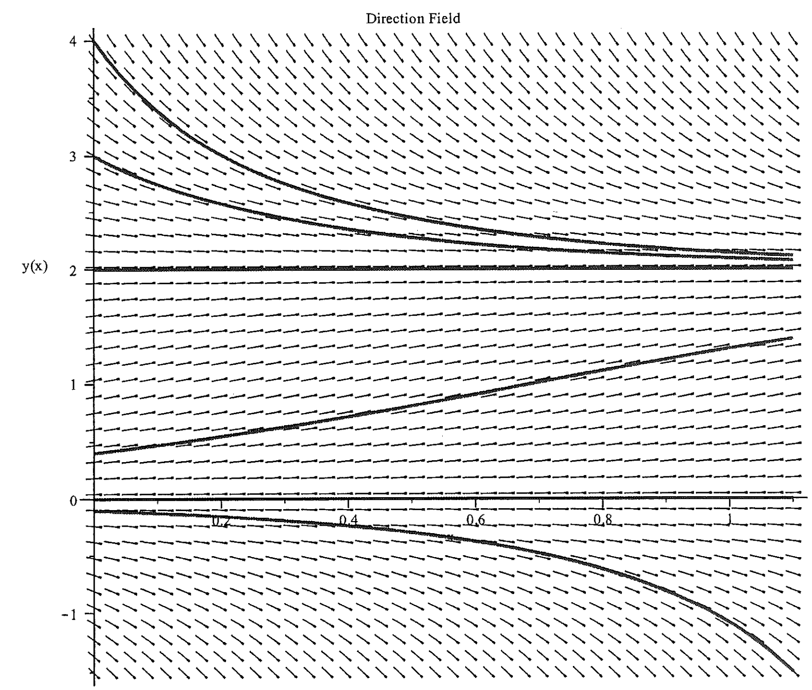


$\hat{y} = 2$ is locally stable

Example 2

Find the equilibria of this differential equation and discuss their stability using the graphical approach

$$\frac{dy}{dx} = y(2-y).$$

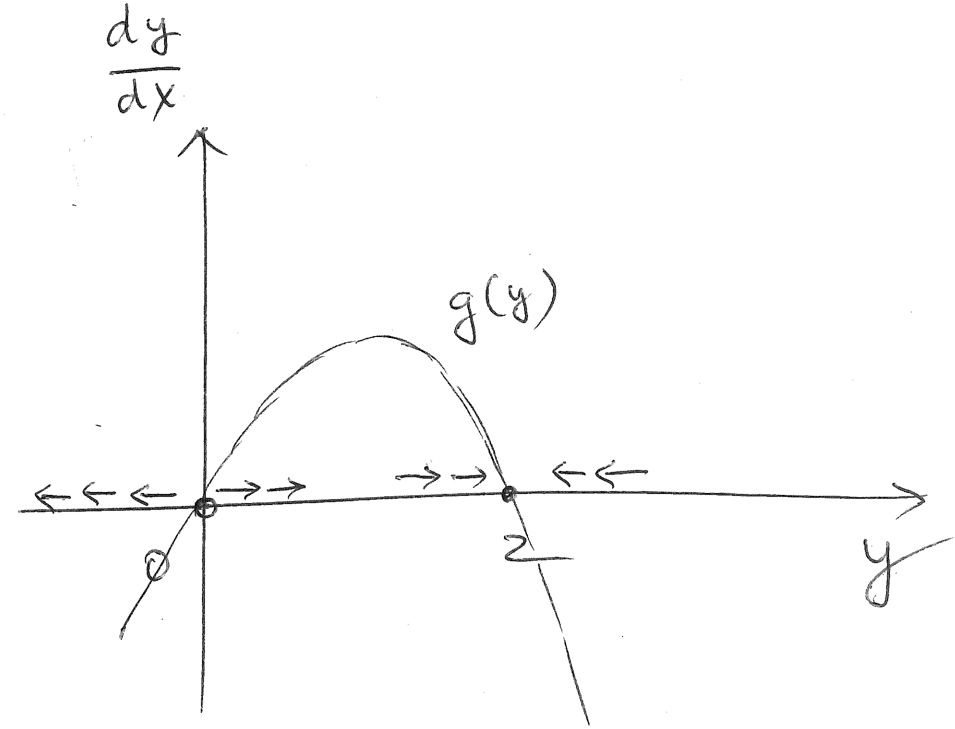


Note: $\dfrac{dy}{dx} = y(2-y) = 2y\left(1 - \dfrac{y}{2}\right)$ is a logistic equation with $r = K = 2$.

$$\frac{dy}{dx} = \underbrace{y(2-y)}_{g(y)}$$

$$g(y) = 2y - y^2$$

parabola that opens down



$\hat{y} = 0$ is unstable

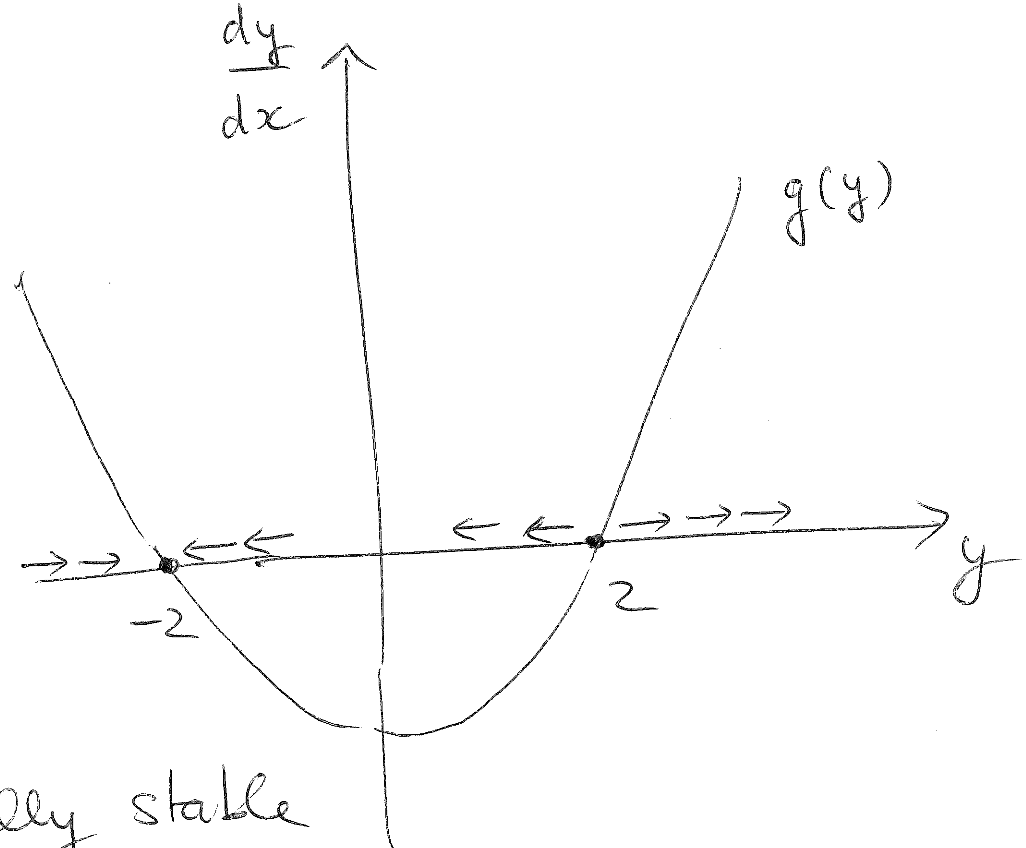
$\hat{y} = 2$ is locally stable

Example 3

Find the equilibria of this differential equation and discuss their stability using the graphical approach

$$\frac{dy}{dx} = y^2 - 4.$$

$$\frac{dy}{dx} = y^2 - 4$$



$\hat{y} = -2$ is locally stable

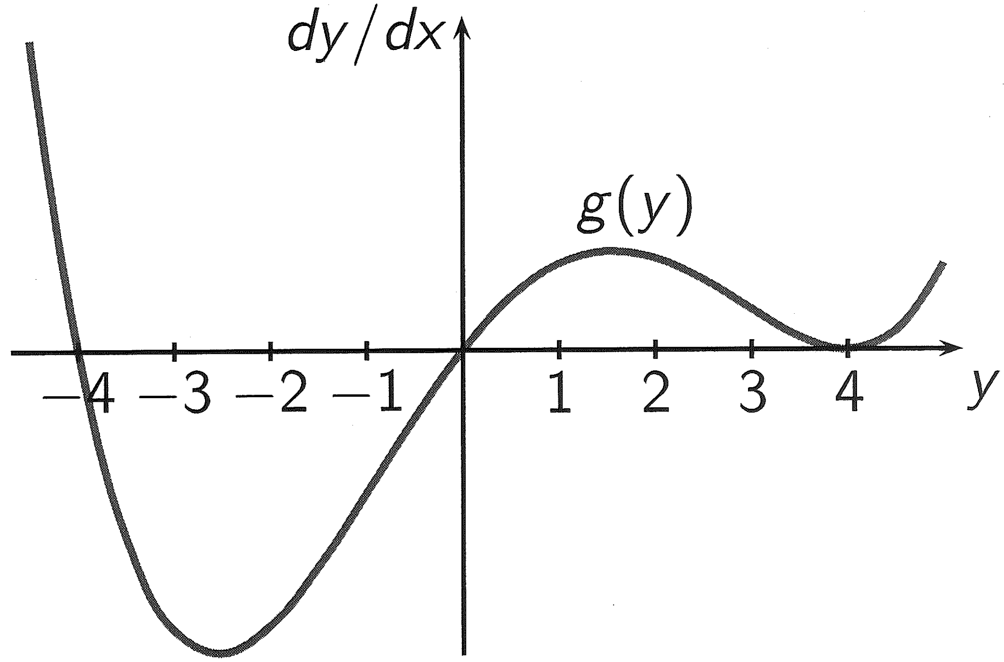
$\hat{y} = 2$ is unstable

Example 4 (Problem # 5, Exam 2, Spring '14)

Consider the autonomous differential equation $\dfrac{dy}{dx} = g(y)$.

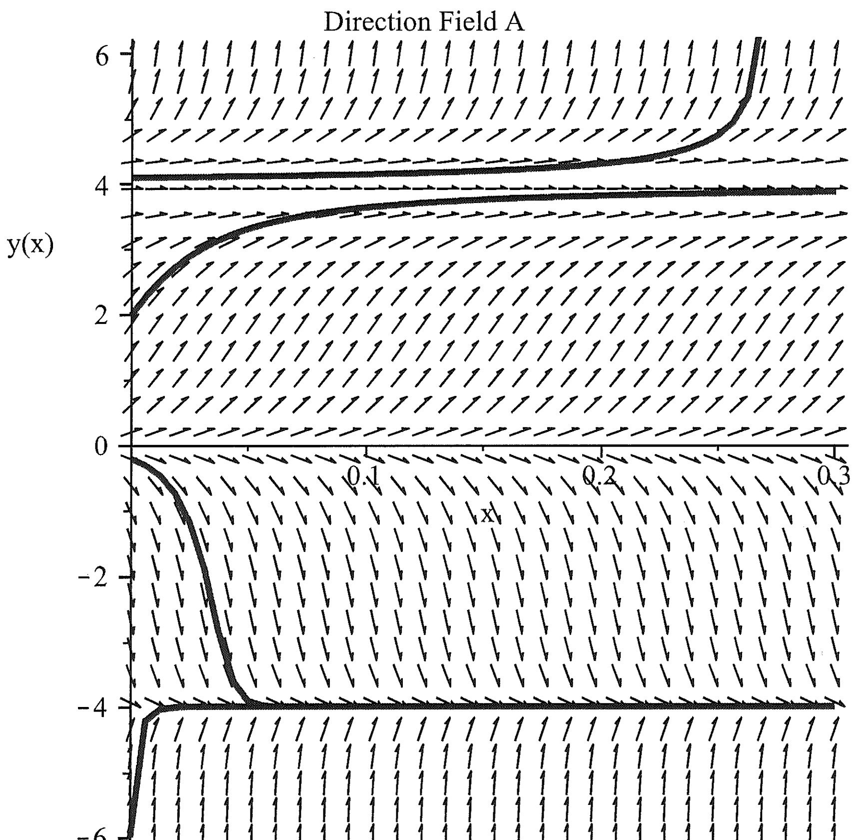
(*a*) Use the graph below to find the equilibria $\widehat{y}$ of the differential equation.

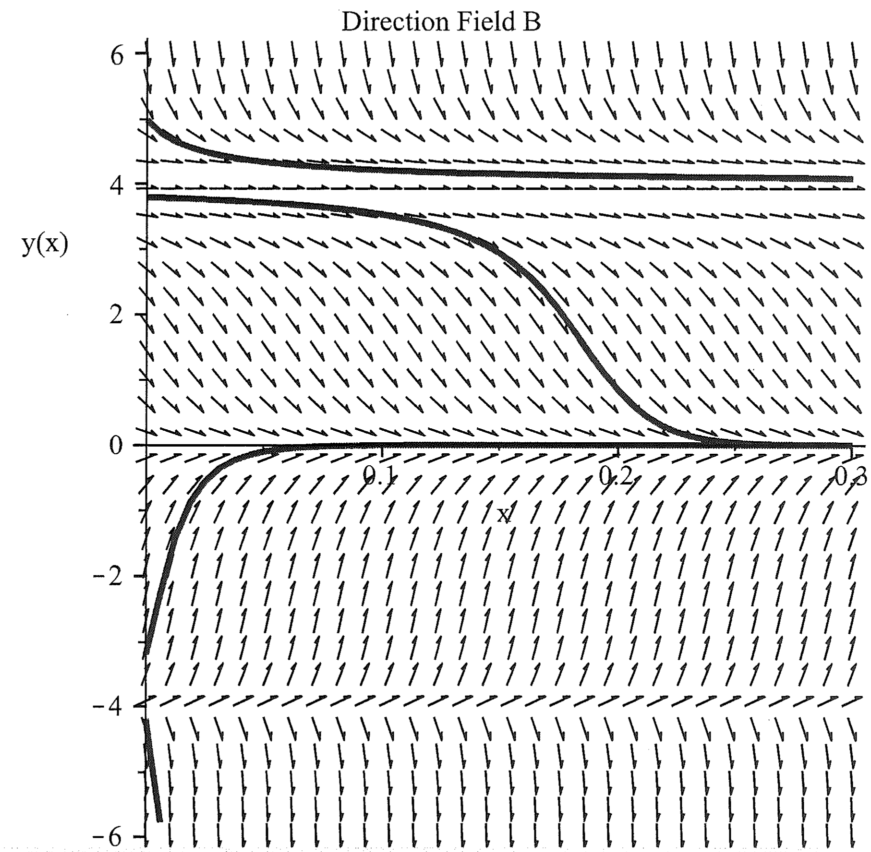
(*b*) Use the graph below and the geometric approach to discuss the stability of the equilibria you found in (*a*).



Example 4 (cont.ed)

(*c*) Using the information found in (*a*) and (*b*), which of the following direction fields (phase portraits) matches the given differential equation? Circle the correct one.

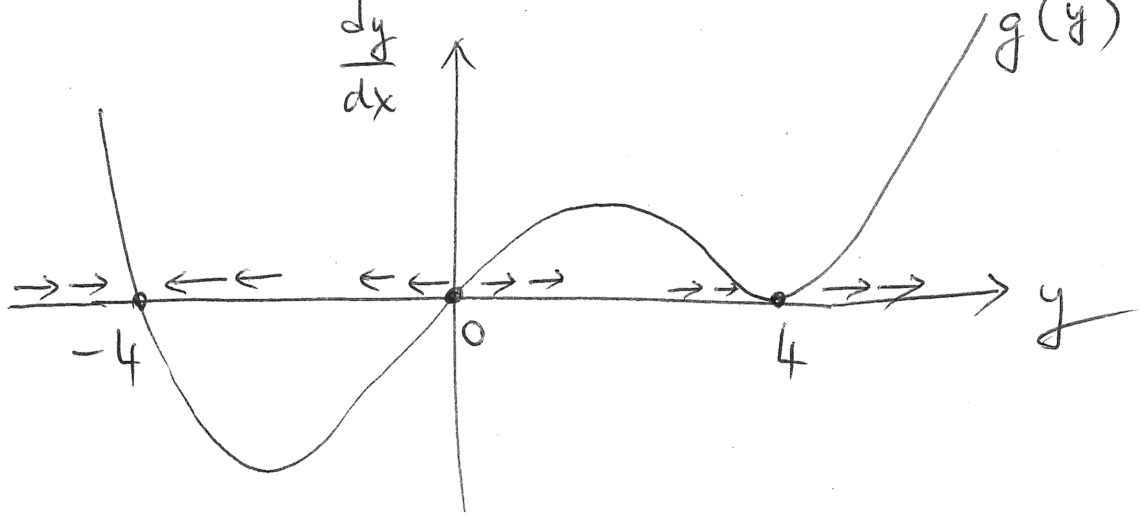




(a) from the graph the equilibria occur at

$$\hat{y} = -4 \,, 0 \,, 4$$

(b)



(c) Direction field
A describe
what found in
(a) and (b)

$\hat{y} = -4$ is locally stable

$\hat{y} = 0$ is unstable

$\hat{y} = 4$ is locally stable for values of y smaller than 4 but it is unstable for values of y bigger than 4

Example 5 (Problem # 9, Section 8.2, p. 418)

Suppose that a fish population evolves according to the logistic equation and that a fixed number H of fish per unit time are removed. That is,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H$$

Assume that $r = 2$ and $K = 1000$.

(a) Find possible equilibria, and discuss their stability when $H = 100$.

(b) What is the maximal harvesting rate that maintains a positive population size?

Note: This is a classical example of **bifurcation theory**. There is something silly about this model of fishery: the population can become negative! An improved model of a fishery is the following

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H\frac{N}{a + N},$$

where H and a are positive constants.

(a) $\frac{dN}{dt} = \underbrace{2N\left(1 - \frac{N}{1{,}000}\right) - 100}_{\text{with our data } r=2,\ K=1{,}000,\ H=100} = g(N)$

To find the equilibria we need to set: $g(N) = 0$

$\therefore\ 2N\left(1 - \frac{N}{1{,}000}\right) - 100 = 0 \iff 2N - \frac{N^2}{500} - 100 = 0$

$\iff 1{,}000N - N^2 - 50{,}000 = 0 \iff$

$N^2 - 1{,}000N + 50{,}000 = 0$ use the quadratic formula

to get $\hat{N}_{1,2} = \frac{1{,}000 \pm \sqrt{1{,}000^2 - 4 \cdot 50{,}000}}{2} = \frac{1{,}000 \pm \sqrt{800{,}000}}{2}$

$= \begin{cases} 947.21 \\ 52.79 \end{cases}$ $\therefore$ $\boxed{\hat{N}_1 = 52.79}$ $\boxed{\hat{N}_2 = 947.21}$

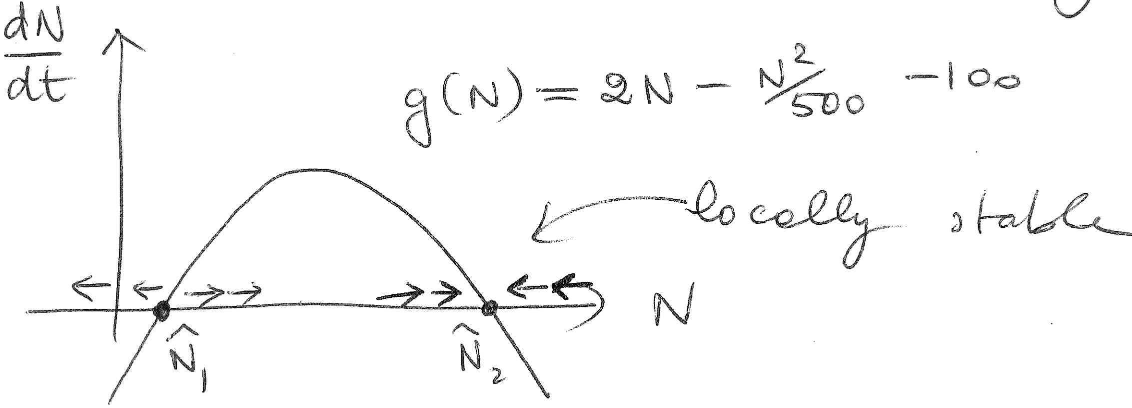
$g(N) = 2N - \frac{N^2}{500} - 100 \Longrightarrow g'(N) = 2 - \frac{N}{250}$

$g'(52.79) \cong 1.788 > 0$ $\qquad g'(947.21) \cong -1.788 < 0$

Hence by the stability criterion

$\hat{N}_1 = 52,79$ is unstable | $\hat{N}_2 = 947.21$ is locally stable

Alternatively we could have used the geometric approach:

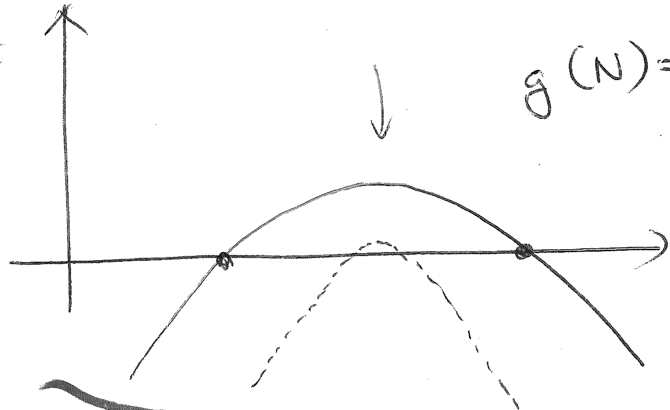


(b) Consider now the general case:

$$\boxed{\frac{dN}{dt} = rN\left(1 - N/K\right) - H} = g(N)$$

What we need is that the differential equation has 2 equilibria that are collapsing into each other.

$\frac{dN}{dt}$



$g(N) = rN(1 - N/K) - H$ ≡ negative vertical shift

$$g(N) = rN - \frac{r}{K}N^2 - H = 0 \iff rKN - rN^2 - KH = 0$$

$$\iff rN^2 - rKN + KH = 0 \iff N^2 - KN + \frac{KH}{r} = 0$$

The quadratic equation gives us the equilibria

$$\hat{N}_{1,2} = \frac{K \pm \sqrt{K^2 - 4KH/r}}{2}$$

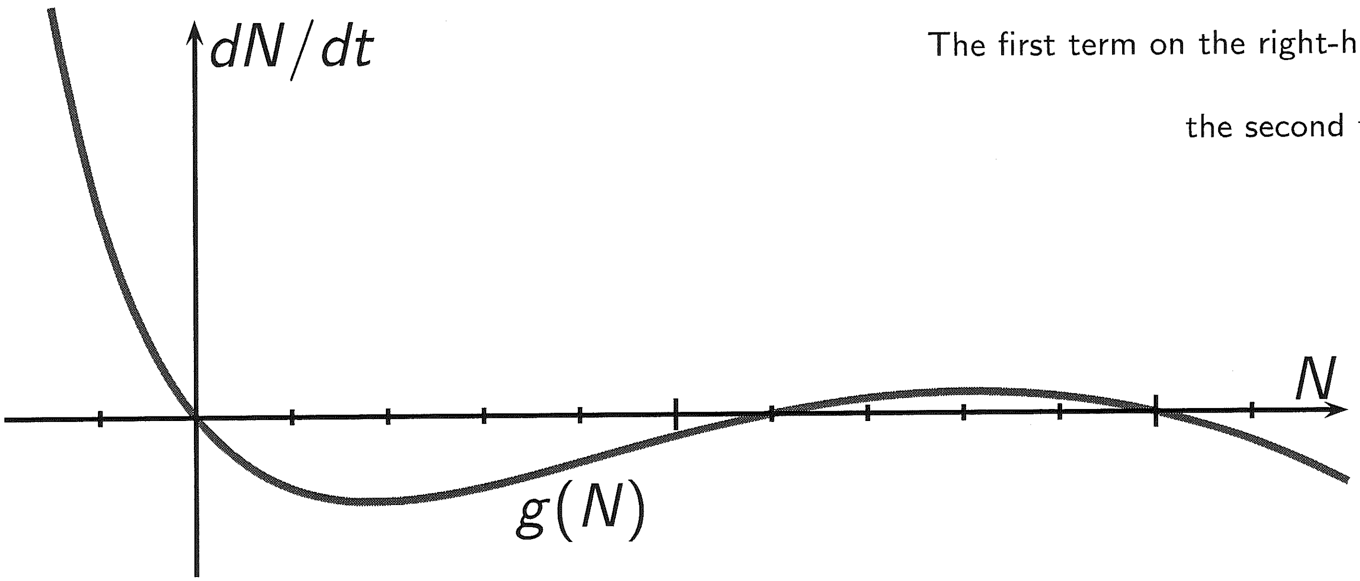
we need the discriminant to be positive:

$$K^2 - 4\frac{KH}{r} > 0 \quad \text{solve for } H \iff H < \frac{rK}{4}$$

Example 6 (Problem # 4, Exam 2, Spring '14)

A simple model of predation: Suppose that $N(t)$ denotes the size of a population at time t. The population evolves according to the logistic equation, but, in addition, predation reduces the size of the population so that the rate of change is given by

$$\frac{dN}{dt} = N\left(1 - \frac{N}{20}\right) - \frac{7N}{4+N}.$$



The first term on the right-hand side describes the logistic growth;

the second term describes the effect of predation.

Example 6 (cont.ed)

(a) Find (algebraically) all the equilibria $\widehat{N}$ of

$$\frac{dN}{dt} = N\left(1 - \frac{N}{20}\right) - \frac{7N}{4+N}.$$

(b) Use the graph of $g(N)$ to classify the stability of the equilibria $\widehat{N}$ found in (a).

(c) Find $g'(N)$, where $g(N) = N\left(1 - \frac{N}{20}\right) - \frac{7N}{4+N}$.

(d) Use the eigenvalues method (stability criterion) to classify the stability of the equilibria $\widehat{N}$ found in (a).

(a) $g(N) = N\left(1 - N/20\right) - \dfrac{7N}{4+N} = 0$

$\iff (4+N)\,N\left(\dfrac{20-N}{20}\right) - 7N = 0$

$\iff (4N+N^2)(20-N) - 140N = 0$

$\iff 80N - 4N^2 + 20N^2 - N^3 - 140N = 0$

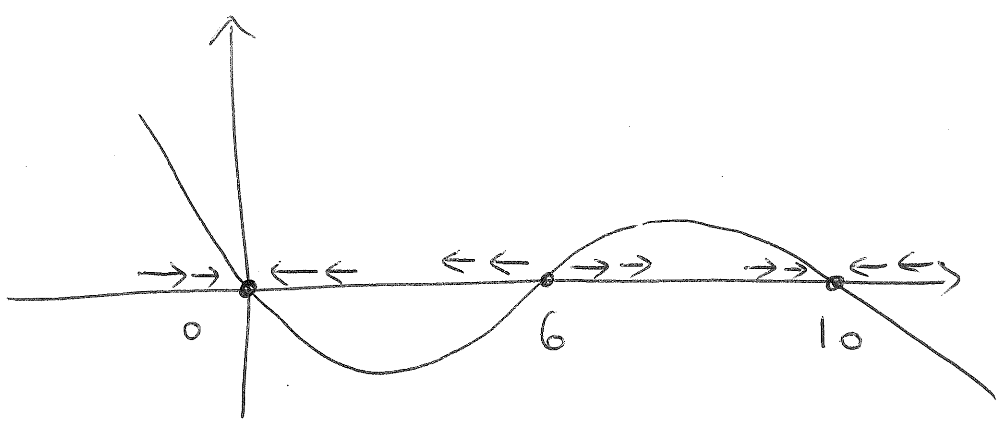
$\iff -N^3 + 16N^2 - 60N = 0$

$\iff -N(N^2 - 16N + 60) = 0$

$\iff N(N-6)(N-10) = 0$

$\therefore$ equilibria: $\hat{N}_1 = 0$, $\hat{N}_2 = 6$, $\hat{N}_3 = 10$

(b)



$0 = \hat{N}_1$ and $\hat{N}_3 = 10$ are locally stable

$\hat{N}_2 = 6$ is unstable

(c) $g(N) = N(1 - N/20) - \frac{7N}{4+N}$

$$g(N) = N - \frac{1}{20}N^2 - \frac{7N}{4+N}$$

$$g'(N) = 1 - \frac{1}{10}N - \frac{7(4+N) - 7N(1)}{(4+N)^2}$$

$$= 1 - \frac{N}{10} - \frac{28}{(4+N)^2}$$

$$= \frac{10(4+N)^2 - N(4+N)^2 - 280}{10(4+N)^2}$$

$$= \frac{10(16 + 8N + N^2) - N(16 + 8N + N^2) - 280}{10(4+N)^2}$$

$$= \frac{-N^3 + 2N^2 + 64N - 120}{10(4+N)^2}$$

$$g'(0) = -\frac{120}{10(16)} = -0.75$$

$\hat{N}_1 = 0$ is locally stable

$$g'(6) = \frac{120}{10(10)^2} = 0.12 > 0$$

$\hat{N}_2 = 6$ is unstable

$$g'(10) = \frac{-280}{10(14)^2} \cong -0.1428 < 0$$

$\hat{N}_3 = 10$ is locally stable

What is Bifurcation?

The dynamics of direction fields for first order autonomous differential equations is rather limited: all solutions either settle down to equilibrium or head out to $\pm\infty$.

Given the triviality of the dynamics, what's interesting about these DEs? The answer is: *dependence on parameters*.

The qualitative structure of the flow can change as parameters are varied. In particular, equilibria can be created or destroyed, or their stability can change.

These qualitative changes in the dynamics are called **bifurcations**, and the parameter values at which they occur are called **bifurcation points**.

Extrapolating from our simple model of fishery (Example 5), the **prototypical example of a bifurcation** is given by

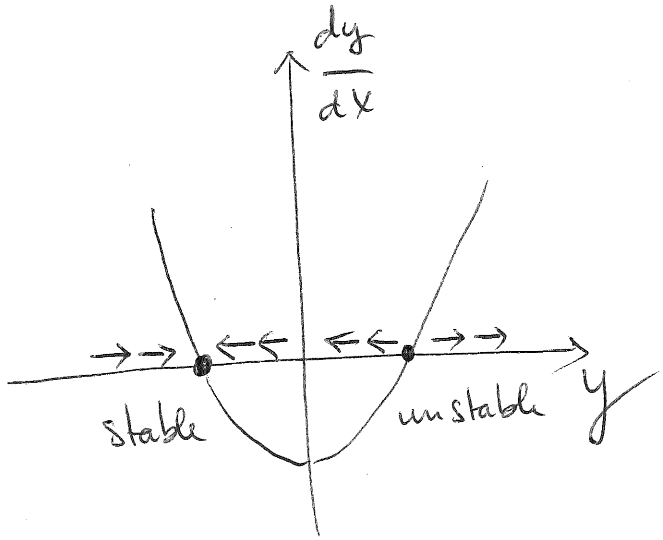
$$\frac{dy}{dx} = r + y^2$$

where r is a parameter, which may be positive, negative, or zero.

- When $r < 0$ there are two equilibria, one stable and one unstable;
- when $r = 0$, the two equilibria coalesce into a half-stable equilibrium at $\widehat{y} = 0$;
- as soon as $r > 0$ there are no fixed points.

We say that a bifurcation occurred at $r = 0$ since the direction fields for $r < 0$ and $r > 0$ are qualitatively different.

$$\frac{dy}{dx} = r + y^2$$



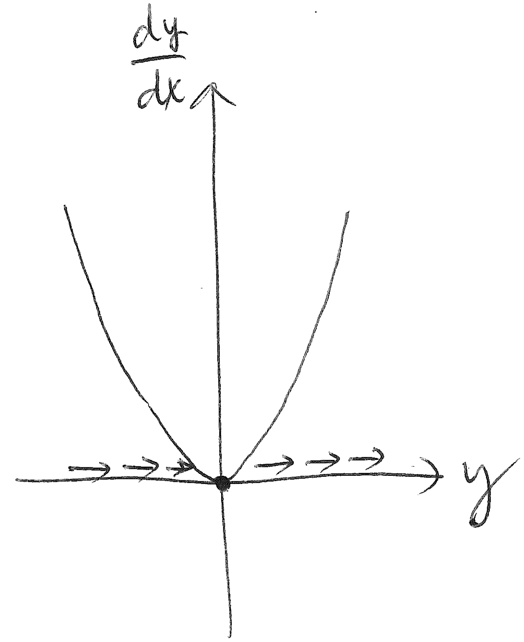
$r < 0$

two equilibria

one stable

and one

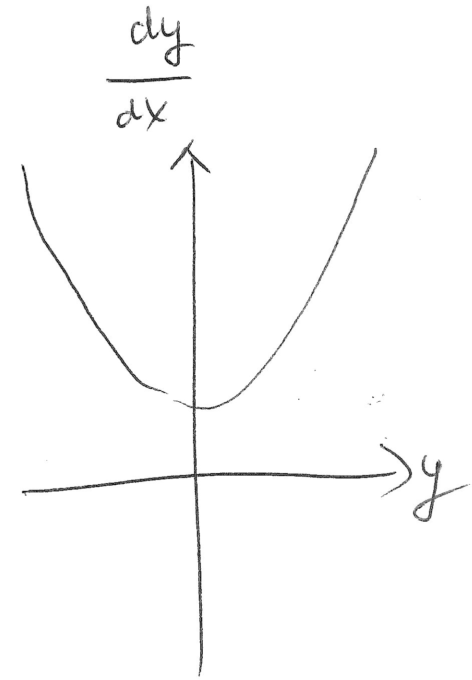
unstable



$r = 0$

half-stable

equilibrium



$r > 0$

no equilibria