MA 138 – Calculus 2 with Life Science Applications
Partial Derivatives
(Section 10.3)

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- Suppose that we want to know how the function $f(x, y)$ changes when x and y change.
- Instead of changing both variables simultaneously, we might get an idea of how $f(x, y)$ depends on x and y when we change one variable while keeping the other variable fixed.
- To illustrate this, we look at $f(x, y) = 0.3x^2 - 0.5y^2$.
- We want to know how $f(x, y)$ changes if we change, say, x and keep y fixed. So we fix $y = y_0$. Then the change in f with respect to x is simply the derivative of f with respect to x when $y = y_0$. That is,
$$\frac{d}{dx} f(x, y_0) = \frac{d}{dx}(0.3x^2 - 0.5y_0^2) = 0.6x.$$
- Similarly the change in f with respect to y is simply the derivative of f with respect to y when $x = x_0$. That is,
$$\frac{d}{dy} f(x_0, y) = \frac{d}{dy}(0.3x_0^2 - 0.5y^2) = -y.$$

Such derivatives are called *partial derivatives*.

Partial Derivatives

Definition

Suppose that f is a function of two independent variables x and y. The partial derivative of f with respect to x is defined by

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

The partial derivative of f with respect to y is defined by

$$\frac{\partial f(x,y)}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

To denote partial derivatives, we use "∂" instead of "d." We will also use

$$f_x(x,y) = \frac{\partial f(x,y)}{\partial x} \qquad f_y(x,y) = \frac{\partial f(x,y)}{\partial y}.$$

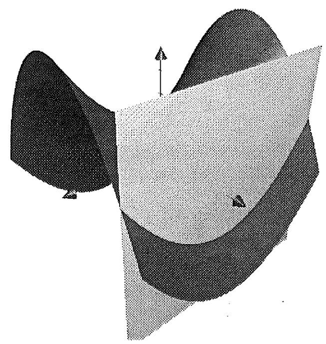
- In the definition of partial derivatives, one should recognize the formal definition of derivatives of Chapter 4.
- That is, to compute $\partial f/\partial x$, we look at the ratio of the difference in the f-values, $f(x+h, y) - f(x, y)$, and the difference in the x-values, $(x+h) - x = h$. The other variable, y, is not changed. We then let h tend to 0.
- To compute $\partial f/\partial x$, we differentiate f with respect to x while treating y as a constant. When we read $\partial f(x, y)/\partial x$, we can say "the partial derivative of f of x and y with respect to x." To read $f_x(x, y)$, we say "f sub x of x and y."
- Finding partial derivatives is no different from finding derivatives of functions of one variable, since, by keeping all but one variable fixed, computing a partial derivative is reduced to computing a derivative of a function of one variable.
- We just need to keep straight which of the variables we have fixed and which one we will vary.

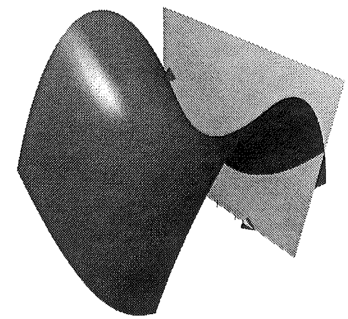
Geometric Interpretation

- The partial derivative $\partial f/\partial x$ evaluated at (x_0, y_0) is the slope of the tangent line to the curve $z = f(x, y_0)$ at the point (x_0, y_0, z_0), with $z_0 = f(x_0, y_0)$.

Ex.: $f(x, y) = 0.3x^2 - 0.5y^2$

$\partial f/\partial x = 0.6x$

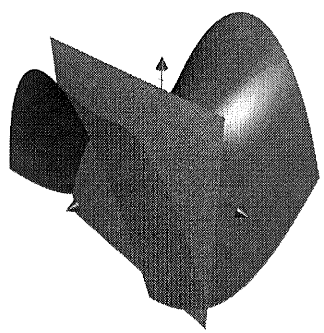


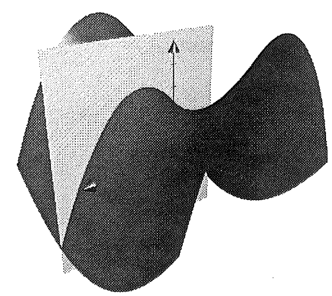


- The partial derivative $\partial f/\partial y$ evaluated at (x_0, y_0) is the slope of the tangent line to the curve $z = f(x_0, y)$ at the point (x_0, y_0, z_0), with $z_0 = f(x_0, y_0)$.

Ex.: $f(x, y) = 0.3x^2 - 0.5y^2$

$\partial f/\partial y = -y$





Example 1 (Problem #2, Section 10.3, p. 525)

Find $\partial f/\partial x$ and $\partial f/\partial y$ when

$$f(x,y) = 2x\sqrt{y} - \frac{3}{x^2 y}$$

$$f(x,y) = (2\sqrt{y})\,x - \frac{3}{y}\cdot x^{-2}$$

So $$\frac{\partial f}{\partial x} = 2\sqrt{y} - \frac{3}{y}\left(-2x^{-3}\right) = \boxed{2\sqrt{y} + \frac{6}{yx^{3}}}$$

Rewrite $f(x,y)$ as: $$f(x,y) = (2x)\,y^{1/2} - \frac{3}{x^2}\cdot y^{-1}$$

So $$\frac{\partial f}{\partial y} = (2x)\frac{1}{2}y^{\frac{1}{2}-1} - \left(\frac{3}{x^2}\right)\cdot\left[-1\cdot y^{-2}\right]$$

$$= \boxed{\frac{x}{\sqrt{y}} + \frac{3}{x^2y^2}}$$

Example 2 (Example #1, Section 10.3, p. 520)

Find $\partial f/\partial x$ and $\partial f/\partial y$ when

$$f(x, y) = ye^{xy}$$

$$f(x,y) = ye^{xy}$$

$$\frac{\partial f}{\partial x} = y \cdot \left[e^{xy}\right] \cdot \underbrace{(y)} = y^2 e^{xy}$$

derivative of the exponent w.r.t. x

$$\frac{\partial f}{\partial y} = \underbrace{1 \cdot e^{xy} + y\left(e^{xy} \cdot x\right)}$$

product rule

$$= e^{xy} + xy\,e^{xy} = e^{xy}(1 + xy)$$

Example 3

Let $f(x,y) = \dfrac{y^2}{x+y}$. Find $f_x(1,1)$ and $f_y(1,1)$.

$$f(x,y) = y^2 \cdot (x+y)^{-1}$$

$$\frac{\partial f}{\partial x} = y^2 \cdot (-1) \cdot (x+y)^{-2} \cdot (1) = \boxed{\frac{-y^2}{(x+y)^2}}$$

$$\frac{\partial f}{\partial x}(1,1) = \frac{-(1)^2}{(1+1)^2} = \boxed{-1/4}$$

$$f(x,y) = \frac{y^2}{x+y}$$

$$\frac{\partial f}{\partial y} = \frac{2y(x+y) - y^2(1)}{(x+y)^2}$$

$$= \boxed{\frac{2xy + y^2}{(x+y)^2}}$$

$$\frac{\partial f}{\partial y}(1,1) = \frac{2(1)(1)+1^2}{(1+1)^2} = \boxed{\frac{3}{4}}$$

Example 4 (Problem #4(d), Exam 3, Spring 2013)

The number of calories $C(w, a)$ a dog requires each day depends on both the weight w (in pounds) and activity level a (in minutes).

Using the data listed in the table on the right, estimate the partial derivative of the number of calories needed with respect to weight, that is $\frac{\partial C}{\partial w}$, for a dog that weighs 40 pounds and is active for 45 minutes a day.

		Activity level a (in min.)		
		15	45	90
weight w (in lbs)	10	234	303	441
	20	373	483	702
	30	489	633	921
	40	593	768	1,117
	50	689	892	1,297
	60	779	1,008	1,466
	70	863	1,117	1,625
	80	944	1,222	1,777
	90	1,022	1,322	1,923
	100	1,097	1,419	2,064

$$\frac{\partial C}{\partial W}(40,45) = \lim_{h \to 0} \frac{C(40+h,45) - C(40,45)}{h} \cong \boxed{12.95}$$

$h = (40+h) - 40$

Thus from the table we need to look at the ratios

$$\frac{C(40+h,45) - C(40,45)}{h}$$

take the average $\cong 12.95$

Thus:

$$\frac{C(20,45) - C(40,45)}{-20} \cong 14.25$$

$$\frac{C(30,45) - C(40,45)}{-10} \cong 13.5$$

$$\frac{C(50,45) - C(40,45)}{10} \cong 12.40$$

$$\frac{C(60,45) - C(40,45)}{20} \cong 12$$

$$\frac{C(70,45) - C(40,45)}{30} \cong 11.63$$

Example 5 (Problem #10, Online Homework)

The **gas law for** a fixed mass m of **an ideal gas** at absolute temperature T, pressure P, and volume V is

$$PV = mRT,$$

where R is the gas constant.

- Find $\dfrac{\partial P}{\partial V} \qquad \dfrac{\partial V}{\partial T} \qquad \dfrac{\partial T}{\partial P}$

- Show that $\dfrac{\partial P}{\partial V} \cdot \dfrac{\partial V}{\partial T} \cdot \dfrac{\partial T}{\partial P} = -1.$

- Show that $T \cdot \dfrac{\partial P}{\partial T} \cdot \dfrac{\partial V}{\partial T} = mR.$

(*) $PV = mRT \rightsquigarrow P = \frac{mRT}{V} = (mRT)V^{-1}$ so $\boxed{\frac{\partial P}{\partial V} = -\frac{mRT}{V^2}}$

(*) $PV = mRT \rightsquigarrow V = \frac{mRT}{P} = \left(\frac{mR}{P}\right) \cdot T$ so $\boxed{\frac{\partial V}{\partial T} = \frac{mR}{P}}$

(*) $PV = mRT \rightsquigarrow T = \frac{PV}{mR} = \left(\frac{V}{mR}\right) \cdot P$ so $\boxed{\frac{\partial T}{\partial P} = \frac{V}{mR}}$

Finally: $\frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial P} = \left(-\frac{mRT}{V^2}\right) \cdot \left(\frac{mR}{P}\right) \cdot \left(\frac{V}{mR}\right)$

Substitute

$$= -\frac{mRT}{PV} = -\frac{mRT}{(mRT)} = \boxed{-1}$$

(*) $PV = mRT \rightsquigarrow P = \left(\frac{mR}{V}\right) \cdot T \rightsquigarrow \boxed{\frac{\partial P}{\partial T} = \frac{mR}{V}}$

(*) $PV = mRT \rightsquigarrow V = \left(\frac{mR}{P}\right) \cdot T \rightsquigarrow \boxed{\frac{\partial V}{\partial T} = \frac{mR}{P}}$

Finally, substitute and get

$$T \cdot \frac{\partial P}{\partial T} \cdot \frac{\partial V}{\partial T} = T \cdot \left(\frac{mR}{V}\right) \cdot \left(\frac{mR}{P}\right) = \underbrace{\left(\frac{mRT}{PV}\right)}_{=1} \cdot mR$$

$$= mR \checkmark$$

Example 6 (Example #4, Section 10.3, p. 522)

Holling (1959) derived an expression for the number of prey items P_e eaten by a predator during an interval T as a function of prey density N and the handling time T_h of each prey item:

$$P_e = \frac{aNT}{1 + aT_hN}.$$

Here, a is a positive constant called the predator attack rate. The above equation is called Hollings disk equation[1]. We can consider P_e as a function of N and T_h.

- Use partial derivatives to determine how the prey density N influences the number of prey eaten per predator.
- Use partial derivative to determine how the handling time T_h influences the number of prey eaten per predator.

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[1]Holling came up with the equation when he measured how many sandpaper disks (representing prey) a blindfolded assistant (representing the predator) could pick up during a certain interval.

(*) We need to look at $\frac{\partial P_e}{\partial N}$ when $P_e = \frac{aNT}{1 + aT_h N}$

$$\frac{\partial P_e}{\partial N} = \text{Quotient rule} = \frac{(aT)(1 + aT_h N) - aNT\,(aT_h)}{(1 + aT_h N)^2}$$

$$= \frac{aT + a^2 T T_h N - a^2 N T T_h}{(1 + aT_h N)^2} = \frac{aT}{(1 + aT_h N)^2}$$

Notice that all quantities in the derivative are positive, so that

$$\boxed{\frac{\partial P_e}{\partial N} > 0}$$

Thus if N increases the function P_e also increases; i.e. if the prey density N increases then the prey item P_e eaten also increase when the other variables are fixed.

(*) We need to look at $\frac{\partial P_e}{\partial T_h}$, where $P_e = \frac{aNT}{1+aT_hN}$

$$\frac{\partial P_e}{\partial T_h} = \text{quotient rule} = \frac{0\cdot(1+aT_hN) - aNT\,(aN)}{(1+aT_hN)^2}$$

$$= \frac{-a^2N^2T}{(1+aT_hN)^2} < 0$$ it is always negative

Thus an increase in the handling time T_h of each prey produces a decrease of the prey item P_e eaten when the other variables are fixed.

Higher Order Partial Derivatives

As in the case of functions of one variable, we define higher-order partial derivatives for functions of more than one variable. For instance, to find the second partial derivative of $f(x, y)$ with respect to x, denoted by $\frac{\partial^2 f}{\partial x^2}$, we compute

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right).$$

We can write $\partial^2 f/\partial x^2$ as f_{xx}.

We can also compute **mixed derivatives**. For instance,

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

Note the order of yx in the subscript of f and the order of $\partial x \partial y$ in the denominator: Either notation means that we differentiate with respect to

Example 7

Let $f(x,y) = 4x^2y - 6xy^2$. Find f_{xx}, f_{yy}, f_{xy}, and f_{yx}.

$$f(x,y) = 4x^2y - 6xy^2$$

$$f_x = 8xy - 6y^2 \rightsquigarrow \boxed{f_{xx} = 8y}$$

$$\boxed{f_{xy} = 8x - 12y}$$

$$f_y = 4x^2 - 12xy \rightsquigarrow \boxed{f_{yy} = -12x}$$

$$\boxed{f_{yx} = 8x - 12y}$$

notice $f_{xy} = f_{yx}$

The Mixed-Derivative Theorem

In the preceding example, $f_{xy} = f_{yx}$, implying that the order of differentiation did not matter.

This is not always the case!

However, there are conditions under which the order of differentiation in mixed partial derivatives does not matter. More precisely

The Mixed-Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are continuous on an open disk centered at the point (x_0, y_0), then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

Example 8 (Partial Differential Equations)

Partial derivatives occur in *partial differential equations*, which describes certain physical phenomena. For example,

- Show that $u(x, y) = e^x \sin y$ is a solution of **Laplaces equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(solutions of this equation are called harmonic functions; they play a role in problems of heat conduction, fluid flow, and electric potential).

- Show that $u(x, t) = \sin(x - ct)$, where c is a fixed constant, satisfies the (one dimensional) **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(which describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string).

(*) Consider $u(x,y) = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cdot \sin y$; $\boxed{\frac{\partial^2 u}{\partial x^2} = e^x \cdot \sin y}$

$\frac{\partial u}{\partial y} = e^x \cdot \cos y$; $\boxed{\frac{\partial^2 u}{\partial y^2} = -e^x \sin y}$

Notice : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (e^x \cdot \sin y) + (-e^x \sin y)$

$= 0$ ✓

Hence $u(x,y) = e^x \sin y$ satisfies the Laplace equation

(**) Consider $u(x,t) = \sin(x - ct)$

$$\frac{\partial u}{\partial t} = \cos(x-ct)\cdot(-c) = -c\cdot\cos(x-ct)$$

$$\boxed{\frac{\partial^2 u}{\partial t^2}} = -c\cdot\left[-\sin(x-ct)(-c)\right] = \boxed{-c^2\sin(x-ct)}$$

$$\frac{\partial u}{\partial x} = \cos(x-ct)\cdot(1) = \cos(x-ct)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin(x-ct)\cdot(1) = -\sin(x-ct)$$

Thus

$$\frac{\partial^2 u}{\partial t^2} \stackrel{?}{=} c^2\cdot\frac{\partial^2 u}{\partial x^2} \quad \boxed{\text{Yes}}$$

$$\frac{\partial^2 u}{\partial t^2} = -c^2\sin(x-ct), \qquad c^2\cdot\frac{\partial^2 u}{\partial x^2} = c^2(-\sin(x-ct))$$

Functions of More Than Two Variables

The definition of partial derivatives extends in a straightforward way to functions of more than two variables. These are ordinary derivatives with respect to one variable while all other variables are treated as constants.

Example 9

Let f be a function of three independent variables x, y, and z:

$$f(x, y, z) = e^{yz}(x^2 + z^3).$$

Then

$$\frac{\partial f}{\partial x} = 2x\, e^{yx} \qquad \frac{\partial f}{\partial y} = z\, e^{yz}(x^2 + z^3)$$

$$\frac{\partial f}{\partial z} = y\, e^{yz}(x^2 + z^3) + 3z^2\, e^{yz}$$